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An Exposition on the Nonlinear Kinematics of Shells, Including Transverse Shearing Deformations

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February 2013

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National Aeronautics and
Space Administration

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SUMMARY

An in-depth exposition on the nonlinear deformations of engineering shell structures, or simply shells, is presented without the use of tensors. First, the mathematical description of an undeformed shell is given in general nonorthogonal coordinates, which includes the definition of a reference surface that is the basis for formulating a two-dimensional theory of deformation. Next, the geometry of the reference surface and associated vector fields are presented. After defining the required reference-surface attributes, mathematical descriptions of the undeformed shell and the corresponding vector fields are given in terms of the reference-surface attributes. Then, a mathematical description of the deformed image of the reference surface and its corresponding geometry are presented. The reference-surface deformations are then characterized by introducing the primitive concepts of elongation and shearing, and by relating these concepts to the corresponding Green-Lagrange strains of continuum mechanics. After this step, the strains are simplified for the important practical case of “small” strains, which are typically exhibited by engineered load-bearing shell structures. To add physical clarity, the linearized reference-surface strains and rotations are derived, followed by the linearized curvatures and torsions of the corresponding deformed reference

SUMMARY - CONTINUED

surface. These quantities are then used to obtain the “small” Green-Lagrange strains and the deformed-reference-surface geometric parameters in terms of linearized deformation measures.

In the next major part of the present study, the geometry of the deformed shell is described mathematically, along with associated vector fields. Then, the shell deformations are characterized by using the primitive concepts of elongation and shearing, and by relating these concepts to the corresponding three-dimensional Green-Lagrange strains of continuum mechanics. The strains are then simplified for the case of “small” strains. Next, the deformations of the shell and its reference surface are related by introducing a kinematic hypothesis. The kinematic hypothesis used in the present study includes transverse shearing deformations and contains the classical Love-Kirchhoff kinematic hypothesis as a proper, well-defined subset. To add more physical clarity, an alternate formulation of the shell strains is presented next that emphasizes the geometric parameters of the deformed reference surface.

SUMMARY - CONCLUDED

After completing the fundamental derivations, resume's of the essential equations are given for general nonorthogonal and orthogonal reference-surface Gaussian coordinates. Then, the basis for further simplification of the essential equations is discussed, and the equations for the important case of “small” strains and “moderate” rotations are given. In addition, resume's of the essential linearized equations are given for general nonorthogonal and orthogonal reference-surface Gaussian coordinates. Moreover, several special cases of the linearized shell strain are discussed. Finally, strains are derived for shells with “small” initial geometric imperfections. The corresponding equations are also given for the practical case of “small” strains and “moderate” rotations.

PURPOSE AND SCOPE

Nonlinear shell theories generally involve the use of nonorthogonal coordinate systems and elements of differential geometry. As a result, almost all treatments of this subject are presented using generalized tensors. This approach provides a concise way of representing lengthy expressions, but tends to obfuscate the physical interpretation of certain mathematical quantities and manipulations. In addition, the use of generalized tensor analysis typically requires a higher level of mathematical maturity than most practicing engineers possess or have time to acquire. Thus, these drawbacks greatly reduce the size of the audience that can benefit from a detailed knowledge of nonlinear shell theories. This point is particularly important to the aerospace community because the demand for shell structures that are lightweight and exhibit high performance leads to thin-walled members that exhibit at least some nonlinear behavior.

The purpose of the present study is to give a detailed exposition of the nonlinear kinematics of shell structures. Toward this purpose, the basic ideas are presented and the corresponding equations are derived from first principles. In addition, many intermediate steps are presented and many visual representations of mathematical concepts are given.

PURPOSE AND SCOPE - CONCLUDED

The scope of the presently study is limited to the nonlinear kinematics of shell structures, but the formulation presented is based on general nonorthogonal coordinates and avoids the use of tensors. The presentation is physics based and includes details of the differential geometry of a shell in its undeformed and deformed states. Additionally, the basic concepts of deformation are presented and related to the changes in shell geometry that occur during deformation. The kinematics include the effects of transverse shearing deformations and retain the equations of classical Love-Kirchhoff shell theory as a well-defined, proper subset. Moreover, simplifications of the key equations that are based on the size of the strains and the rotations of material line elements within a shell are given and related to several well-known published theories. Several parts of the presentation are redundant, by design, to enhance comprehension.

NOTATION

INDICIAL NOTATION

- **Indicial notation** is used in the present study because it provides a compact means for representing lengthy expressions
- For example, the expression $a_1b_1 + a_2b_2 + a_3b_3$ is written concisely as a_kb_k , where the **repeated index** implies a **summation** over the values $k = 1, 2, \text{ and } 3$
- Similarly, the expression $a^{jk}b_k = c^j$ is used to represent the system of linear equations

$$\begin{aligned}a^{11}b_1 + a^{12}b_2 + a^{13}b_3 &= c^1 \\a^{21}b_1 + a^{22}b_2 + a^{23}b_3 &= c^2 \\a^{31}b_1 + a^{32}b_2 + a^{33}b_3 &= c^3\end{aligned}$$

- Coordinates such as the usual **Cartesian coordinates** (x, y, z) are written as (x_1, x_2, x_3) herein to facilitate the use of indicial notation

INDICIAL NOTATION - CONTINUED

- Herein, the following convention is used for indicial notation:
 - **Repeated indices** that imply a summation can appear as superscripts and subscripts
 - A **Latin index** such as “k” takes on the values 1, 2, and 3 unless noted otherwise; e.g., $a_k b_k = a_1 b_1 + a_2 b_2 + a_3 b_3$
 - A **Greek index** such as “ α ” takes on the values 1 and 2 unless noted otherwise; e.g., $a^\alpha b_\alpha = a^1 b_1 + a^2 b_2$
 - Summation implied by two repeated indices is **suspended if one index is enclosed in parenthesis**; e.g., $a^{(\alpha)} b_\alpha$ implies $a^1 b_1$ or $a^2 b_2$
 - Any index that is not a proper repeated index is called a **free index**
 - Free indices take on all the values of the given index; e.g., in the expression $a^{(\alpha)} b_\alpha$, “ α ” is a free index (because of the parenthesis) and can take on the values 1 and 2

SET-THEORY NOTATION

- In the following discussion of set notation, **sets** are denoted by boldface upper-case letters and the elements of a set are denoted by boldface lower-case letters, unless indicated otherwise
 - $\mathbf{x} \in \mathbf{S}$ indicates that $\mathbf{x}$ is an element or member of the set $\mathbf{S}$
 - $\mathbf{x} \notin \mathbf{S}$ indicates that $\mathbf{x}$ is not an element or member of the set $\mathbf{S}$
- A set may be specified explicitly by using braces and listing all of the elements
 - e.g., $\{2, 4, 6, 8\}$ indicates that the numbers 2, 4, 6, and 8 form a set
 - This type of set specification is called **roster or tabular notation**

SET-THEORY NOTATION - CONTINUED

- In giving a complete description of the elements of a set, one often uses a statement that specifies some **conditions or restrictions**
- A shorthand notation used to indicate a condition in set theory is

$$\{ \mathbf{a} \in \mathbf{S} \mid \text{condition} \}$$

- This notation reads, “**a** is a member of the set **S** such that the specific condition is fulfilled”
- This method is often called the **set-builder** or **property** method
- For example, $\{ \mathbf{x} \in \mathfrak{R} \mid \mathbf{x} > 0 \}$ indicates that **x** consists of the set of all **real numbers** $\mathfrak{R}$ that are greater than zero
- An **empty set** is a set without any elements and is denoted by $\emptyset$

SET-THEORY NOTATION - CONTINUED

- A set **A** is a **subset** of a set **B** whenever every element of the set **A** is also an element of the set **B**
- This relationship is denoted by $\mathbf{A \subseteq B}$
- In general, the subset **A** may contain all the elements of the set **B**; thus, if $\mathbf{A \subseteq B}$ and $\mathbf{B \subseteq A}$ then $\mathbf{A = B}$
- When the sets **A** and **B** are restricted to be unequal, the set **A** is described as a **proper subset** of the set **B** - this relationship is denoted by $\mathbf{A \subset B}$
- That is, $\mathbf{A \subset B}$ is used to indicate that set **A** **is included in** set **B**

SET-THEORY NOTATION - CONTINUED

- The **common elements** of two sets **A** and **B** are denoted by **$A \cap B$**
- When there are no common elements **$A \cap B = \emptyset$** and the sets are described as **disjoint**
- **$A \cap B$** is the **“largest”** set that is a subset of both **A** and **B**
- **$A \cap \emptyset = \emptyset$** for any set **A**
- The intersection of two sets, **A** and **B**, is defined as the set

$$\mathbf{A \cap B = \left\{ x \mid x \in A \text{ and } x \in B \right\}}$$

SET-THEORY NOTATION - CONTINUED

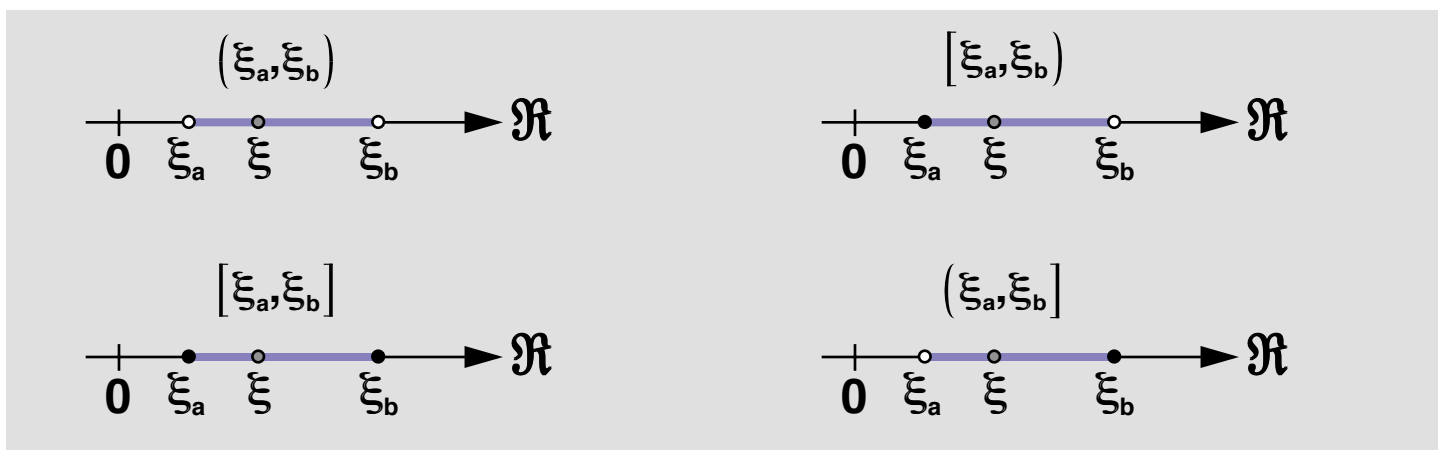
- The union of two sets **A** and **B** is another set denoted by **$A \cup B$** and consists of **all elements of both sets A and B**; i.e.,

$$\mathbf{A \cup B = \left\{ x \mid x \in A \text{ or } x \in B \right\} = B \cup A}$$

- **$A \cup B$** can be thought of as **"addition"** of the sets **A** and **B**
- **$A \cup \emptyset = A$** for any set **A**, and **$A \cup B$** is the "smallest" set for which **A** and **B** are subsets
- The **Cartesian product** of two sets **A** and **B** is denoted by **$A \times B$** and consists of the set of all ordered pairs of elements **(a, b)** such that **$a \in A$** and **$b \in B$**

NOTATION FOR INTERVALS OF REAL NUMBERS

- Let $\Re$ denote the set of real numbers and let $\xi \in \Re$
- The **open interval** of $\Re$ given by $\xi_a < \xi < \xi_b$ is denoted by (ξ_a, ξ_b)
- The **closed interval** of $\Re$ given by $\xi_a \leq \xi \leq \xi_b$ is denoted by $[\xi_a, \xi_b]$
- The **half-open interval** of $\Re$ given by $\xi_a \leq \xi < \xi_b$ is denoted by $[\xi_a, \xi_b)$
- The **half-open interval** of $\Re$ given by $\xi_a < \xi \leq \xi_b$ is denoted by $(\xi_a, \xi_b]$



NOTATION FOR VECTORS

- Vectors are indicated by an arrow above a letter; e.g., $\vec{a}$
- Unit-magnitude vectors are indicated by a circumflex; e.g., $\hat{a}$
- The symbolism $\vec{a} \perp \vec{b}$ is used herein to indicate that the vector $\vec{a}$ is perpendicular to the vector $\vec{b}$
- Similarly, $\vec{a} \parallel \vec{b}$ is used herein to indicate that the vector $\vec{a}$ is parallel to the vector $\vec{b}$

MATHEMATICAL DESCRIPTION OF AN UNDEFORMED SHELL

THE PREMISE OF CONTINUUM MODELING

- In order to obtain solutions to practical engineering problems, that involve load-carrying structures made of solid materials, the discrete atomic structure of matter is replaced with a mathematical construct known as **continuum modeling**
- In continuum modeling, a body of material is idealized as contiguous collection of material particles that are placed into a unique correspondence with geometric points of **three-dimensional Euclidean space** $\mathcal{E}^3$, referred to herein as **material points**
- Thus, one-dimensional chains of material particles can be envisioned that are placed into unique correspondence with geometric curves in $\mathcal{E}^3$, referred to herein as **material curves**
- Likewise, two-dimensional collections of material particles are placed into unique correspondence with geometric surfaces in $\mathcal{E}^3$, referred to herein as **material surfaces**

THE PREMISE OF CONTINUUM MODELING

CONTINUED

- Furthermore, three-dimensional collections of material particles are envisioned and placed into unique correspondence with geometric regions in $\mathcal{E}^3$, referred to herein as **material regions or material bodies**
- With these correspondences, each point of $\mathcal{E}^3$ can be endowed with the physical attributes of the corresponding material particle, which represents a homogenization of the properties of some corresponding finite collection of discrete atoms
- These “pointwise” properties include **scalar quantities** such as mass and temperature, which are characterized by magnitudes
- Pointwise properties also include **vector quantities** such as force and momentum, which are characterized by magnitude and direction, and **tensor quantities**, such as stress, that are linear functions of vectors

THE PREMISE OF CONTINUUM MODELING

CONCLUDED

- Thus, the physical behavior of the continuum is fully characterized by a set of points in **three-dimensional Euclidean space** $\mathcal{E}^3$ that are endowed with the algebraic structures of scalar, vector, and tensor fields
- The set of points, and corresponding algebraic structures, used to represent a curved, relatively thin-walled surface-like material body is referred to herein as the **shell space**
- Some particular information regarding the shell space is presented subsequently

THE SHELL SPACE, $\mathcal{R}_0$

- A shell structure is described mathematically as a set of material points, $\mathbf{B}$, that occupy a region, $\mathcal{R}_t$, of **three-dimensional Euclidean space** $\mathcal{E}^3$ at time t
- Herein, the time $t = 0$ is defined as the **reference configuration or reference state** of the shell, which occupies $\mathcal{R}_0 \subset \mathcal{E}^3$
 - In the **reference configuration**, the shell is presumed to be **undeformed and unstressed**
- The subset $\mathcal{R}_0 \subset \mathcal{E}^3$ is also referred to herein as the **shell space**
- Likewise, the subset $\mathcal{R}_t \subset \mathcal{E}^3$ is referred to herein as the **deformed image of the shell space** at time t

THE SHELL SPACE, $\mathcal{R}_0$ - CONTINUED

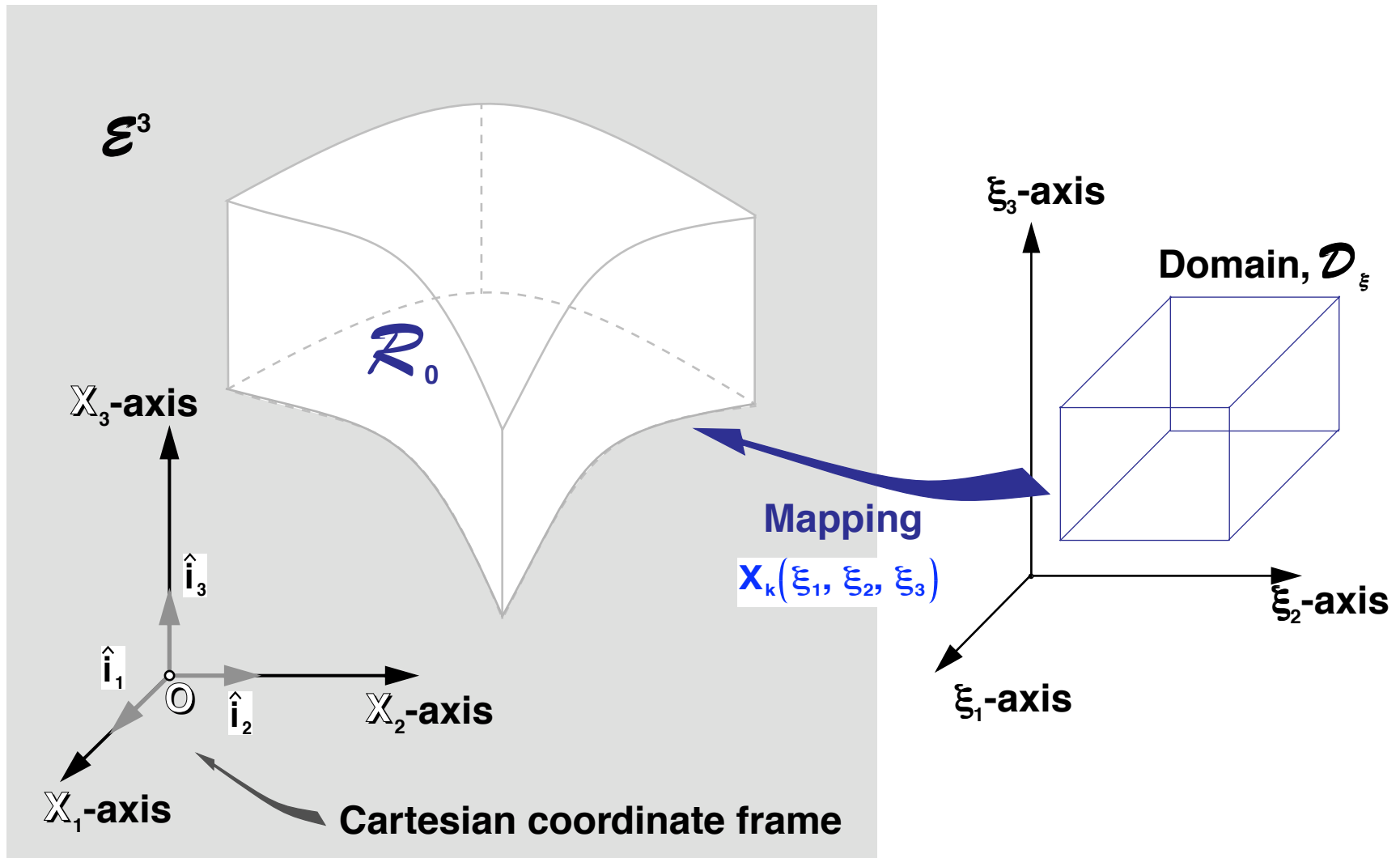
- Points of the shell space are described **parametrically** by the Cartesian coordinates $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$, with respect to the frame $\mathbf{O} - \mathbf{X}_1 - \mathbf{X}_2 - \mathbf{X}_3$, where the Cartesian coordinates are given by

$$\mathbf{X}_1 = \mathbf{X}_1(\xi_1, \xi_2, \xi_3) \quad \mathbf{X}_2 = \mathbf{X}_2(\xi_1, \xi_2, \xi_3) \quad \mathbf{X}_3 = \mathbf{X}_3(\xi_1, \xi_2, \xi_3)$$

- The corresponding basis of the Cartesian coordinate system is denoted herein by $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$
- The parameters ξ_1, ξ_2 , and ξ_3 are interpreted as a set of points (ξ_1, ξ_2, ξ_3) that span a **cuboid region** of a three-dimensional space that gets mapped onto $\mathcal{R}_0 \subset \mathcal{E}^3$ in a unique, one-to-one manner
- The cuboid region is defined as the **domain** $\mathcal{D}_\xi$ of the mapping specified by

$$\mathbf{X}_1 = \mathbf{X}_1(\xi_1, \xi_2, \xi_3), \quad \mathbf{X}_2 = \mathbf{X}_2(\xi_1, \xi_2, \xi_3), \quad \text{and} \quad \mathbf{X}_3 = \mathbf{X}_3(\xi_1, \xi_2, \xi_3)$$

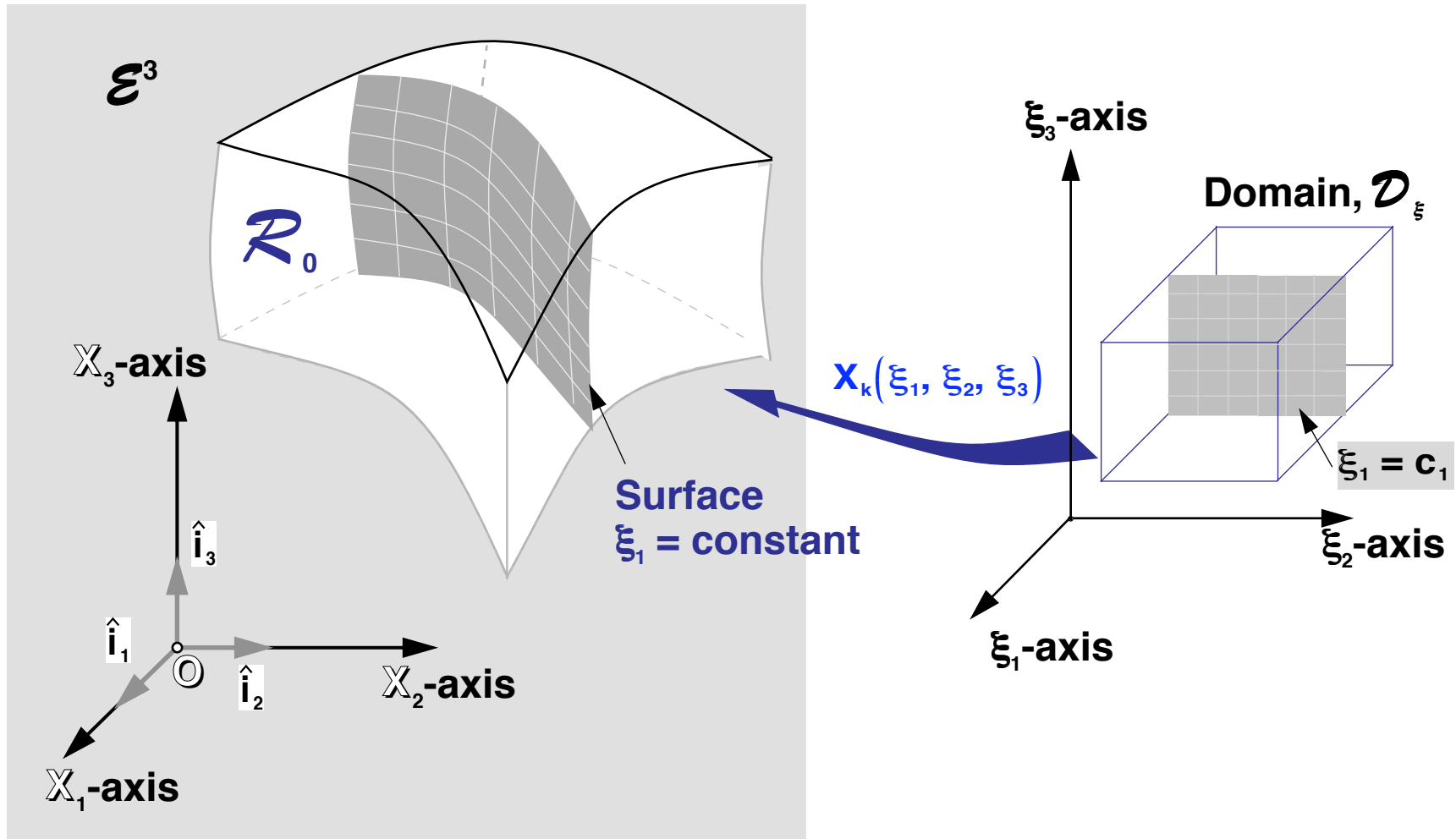
THE SHELL SPACE, $\mathcal{R}_0$ - CONTINUED



THE SHELL SPACE, $\mathcal{R}_0$ - CONTINUED

- Consider the plane $\xi_1 = c_1$ in the domain $\mathcal{D}_\xi$, where c_1 is a specified constant value
- For this case, the three mapping functions $\mathbf{X}_k = \mathbf{X}_k(c_1, \xi_2, \xi_3)$, defined on the **implied domain** for the points (ξ_2, ξ_3) , yield a parametric representation for a surface within the shell space $\mathcal{R}_0 \subset \mathcal{E}^3$
 - Thus, the plane $\xi_1 = c_1$ in the domain is mapped onto a surface in $\mathcal{R}_0$
- Likewise, the planes $\xi_2 = \text{constant}$ and $\xi_3 = \text{constant}$ in the domain $\mathcal{D}_\xi$ are mapped onto surfaces in $\mathcal{R}_0$

THE SHELL SPACE, $\mathcal{R}_0$ - CONTINUED



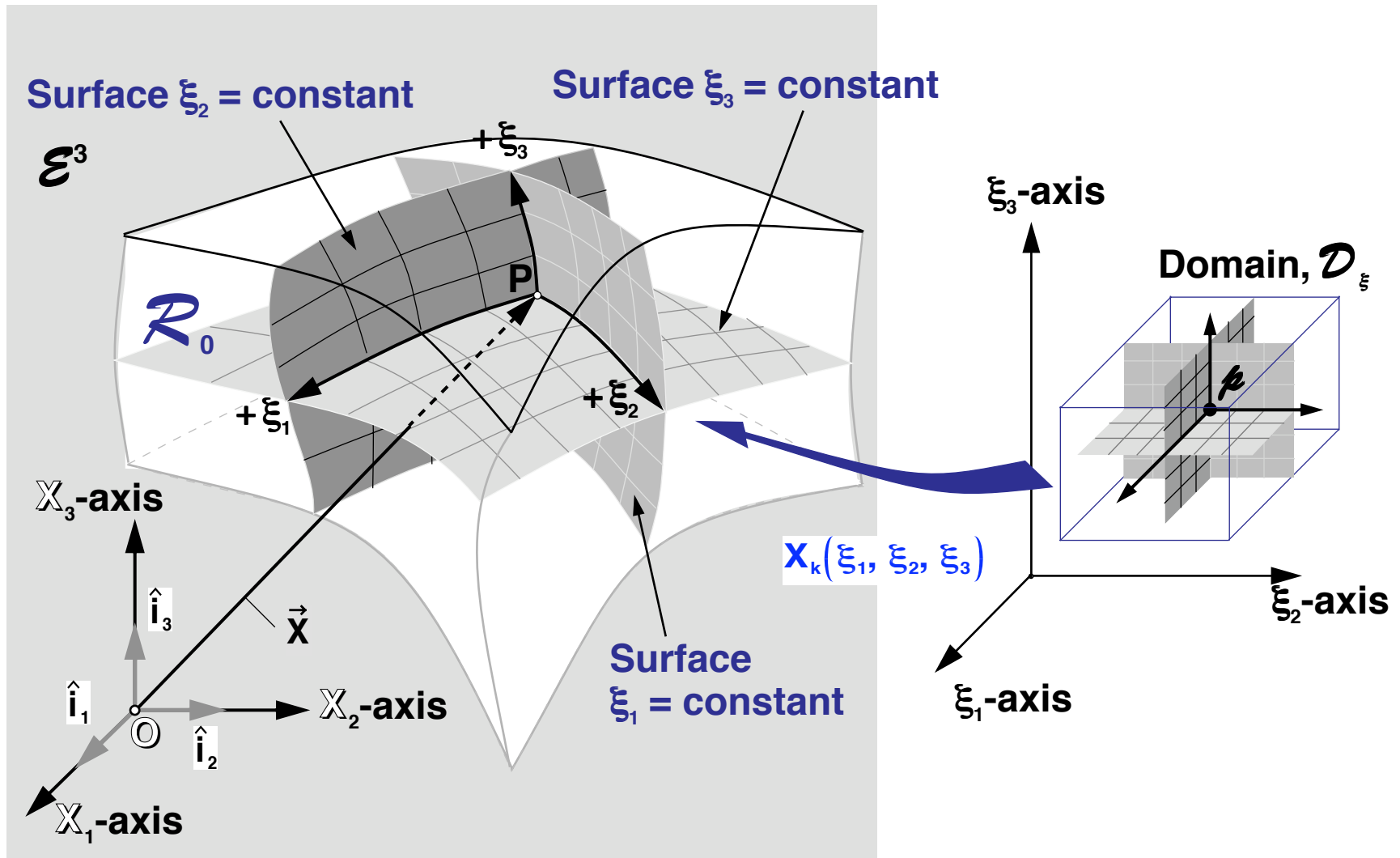
THE SHELL SPACE, $\mathcal{R}_0$ - CONTINUED

- Now consider the line in the domain $\mathcal{D}_\xi$, given by $\xi_1 = c_1$ and $\xi_2 = c_2$, where c_1 and c_2 are specified constant values
- For this case, the three mapping functions $\mathbf{X}_k = \mathbf{X}_k(c_1, c_2, \xi_3)$, defined on the **implied domain** for the points given by ξ_3 , yield a parametric representation for a space curve within the shell space
 - Thus, the line given by $\xi_1 = c_1$ and $\xi_2 = c_2$ in the domain is mapped onto a curve in $\mathcal{R}_0$
 - Moreover, as the numerical values of ξ_3 increase, a **positive direction of traversal** of the curve is implied by the specific functional form of the three mapping functions
- Likewise, the line given by $\xi_1 = \text{constant}$ and $\xi_3 = \text{constant}$, and the line given by $\xi_2 = \text{constant}$ and $\xi_3 = \text{constant}$ in the domain are mapped onto space curves in $\mathcal{R}_0$ with an implied direction of traversal

THE SHELL SPACE, $\mathcal{R}_0$ - CONTINUED

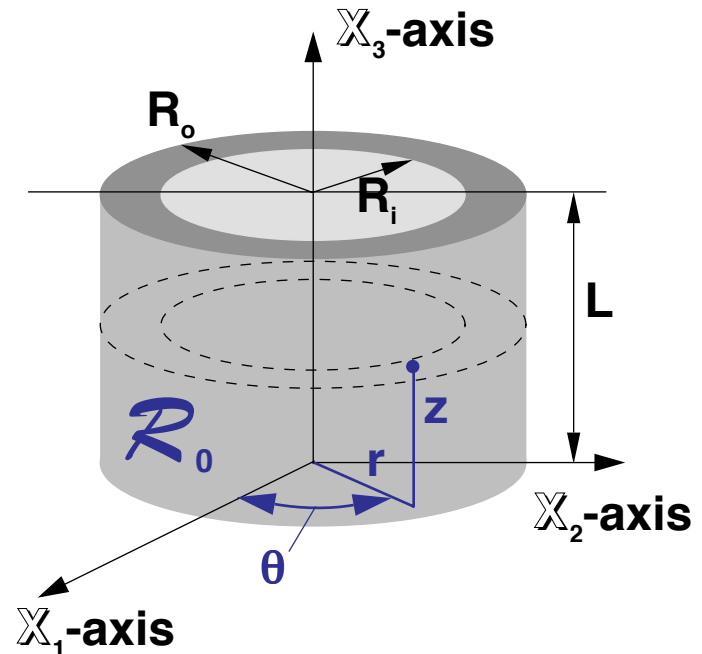
- Therefore, by requiring the mapping functions $\mathbf{X}_k = \mathbf{X}_k(\xi_1, \xi_2, \xi_3)$ to map distinct lines in the domain onto distinct curves in the shell space, it follows that the parameters ξ_1, ξ_2 , and ξ_3 define a system of **curvilinear coordinates** for the shell space $\mathcal{R}_0 \subset \mathcal{E}^3$
- In the figure that follows, the symbols $+\xi_1$, $+\xi_2$, and $+\xi_3$ are used to indicate **positive directions of curvilinear-coordinate traversal** associated with positive increments in the corresponding parameters
- In addition, the **position vector** to point P shown in the following figure, with coordinates (X_1, X_2, X_3) , defined relative to the Cartesian coordinate frame and basis $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ is given by $\vec{\mathbf{X}} = \mathbf{X}_k(\xi_1, \xi_2, \xi_3)\hat{\mathbf{i}}_k$

THE SHELL SPACE, $\mathcal{R}_0$ - CONTINUED



THE SHELL SPACE, $\mathcal{R}_0$ - CONCLUDED

- As an example of a shell space, consider the set of points given by the cylindrical coordinates (r, θ, z) , where $R_i \leq r \leq R_o$, $0 \leq \theta < 2\pi$, and $0 \leq z \leq L$
- These points fill a **right circular cylindrical shell** with an inner radius R_i , an outer radius R_o , and length L
- Let $z \rightarrow \xi_1$, $\theta \rightarrow \xi_2$, and $r \rightarrow \xi_3$
- The domain $\mathcal{D}_\xi$ with points (ξ_1, ξ_2, ξ_3) is given by $[0, L] \times [0, 2\pi) \times [R_i, R_o]$
- The mapping functions are $X_1 = \xi_3 \cos \xi_2$, $X_2 = \xi_3 \sin \xi_2$, and $X_3 = \xi_1$



NATURAL BASE-VECTOR FIELDS OF $\mathcal{R}_0$

- To model pointwise physical attributes of a shell that have **magnitude and direction**, the ability to define vector fields for points of the shell space is needed
- In courses on differential geometry, it is shown that a space curve is given parametrically by $\vec{X} = X_k(\xi)\hat{i}_k$
- Moreover, it is shown that a vector **tangent to the space curve** is given by $\frac{d\vec{X}}{d\xi} = \lim_{\Delta\xi \rightarrow 0} \left[\frac{X_k(\xi + \Delta\xi) - X_k(\xi)}{\Delta\xi} \right] \hat{i}_k$, provided that the limit exists
- For this limit to exist, $\vec{X} = X_k(\xi)\hat{i}_k$ must be continuous and its derivative must also be continuous; that is, it must be **smooth**
- Thus, a unique set of vectors that are tangent to the curvilinear-coordinate curves can be defined provided that mapping functions $X_k = X_k(\xi_1, \xi_2, \xi_3)$ and their partial derivatives are continuous

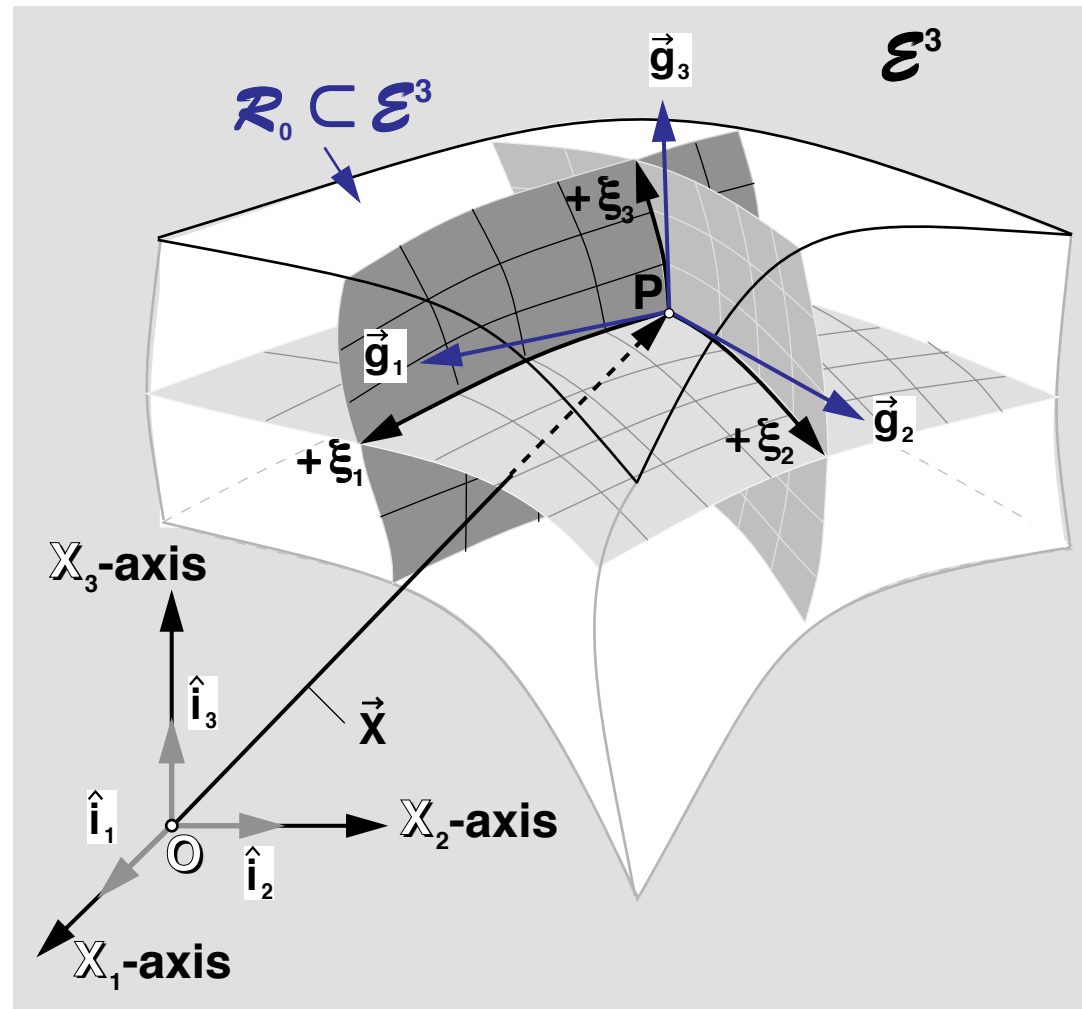
NATURAL BASE-VECTOR FIELDS OF $\mathcal{R}_0$ - CONTINUED

- Therefore, **base-vector fields** of the curvilinear coordinates (ξ_1, ξ_2, ξ_3) are defined by

$$\vec{g}_k(\xi_1, \xi_2, \xi_3) \equiv \frac{\partial \vec{X}}{\partial \xi_k}$$

for $k \in \{1, 2, 3\}$

- Each base vector at point P is **tangent** to the corresponding curvilinear-coordinate curve and **points in the direction of positive coordinate-curve traversal**, as shown in the figure



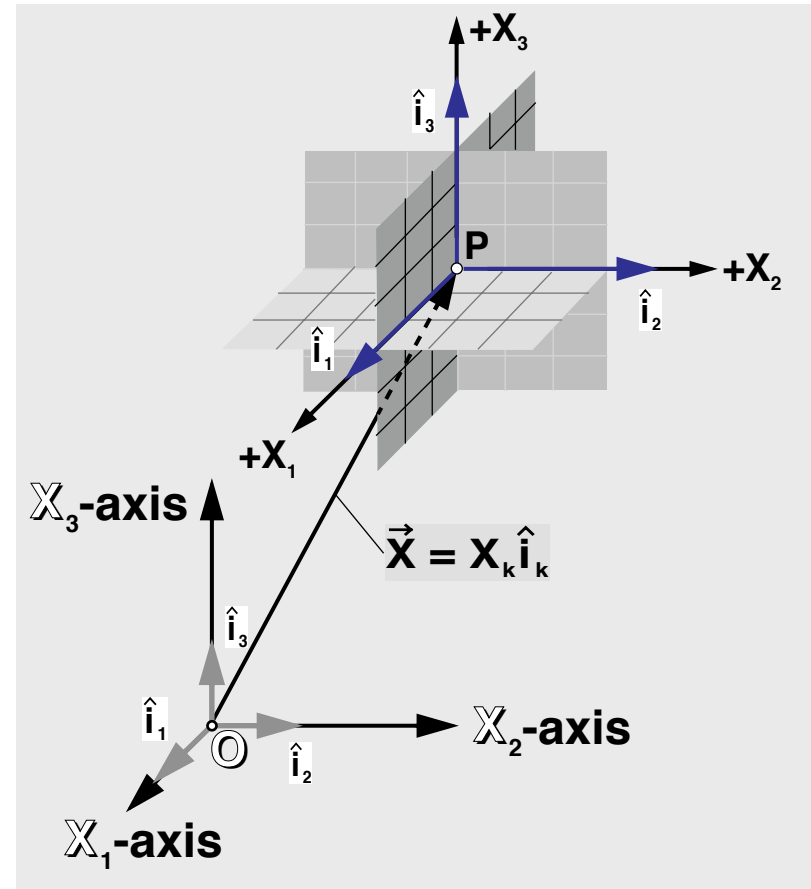
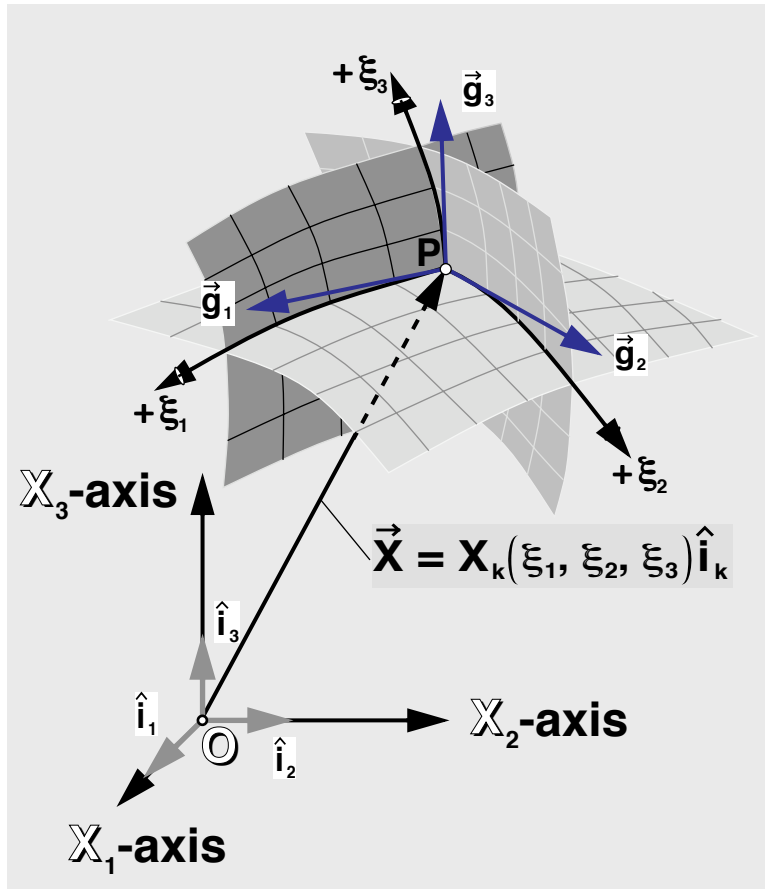
NATURAL BASE-VECTOR FIELDS OF $\mathcal{R}_0$ - CONTINUED

- For any legitimate curvilinear coordinate system, the three coordinate surfaces that correspond to constant values of ξ_1, ξ_2 , and ξ_3 are **noncoincident** at every point of $\mathcal{R}_0 \subset \mathcal{E}^3$, and as a result, the three coordinate curves are distinct
- Thus, the vector fields $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ are **linearly independent** at every point of $\mathcal{R}_0$, and as a result, **they provide a basis for quantitatively representing vector fields** associated with the points of $\mathcal{R}_0$
 - In addition, the three base-vector fields are **mutually orthogonal** at every point of $\mathcal{R}_0$ when the curvilinear coordinates are orthogonal
- In general, any three linearly independent vector fields provide a basis for representing all other related vector fields; as a result, the set of three linearly independent vector fields is given the name **“basis”**

NATURAL BASE-VECTOR FIELDS OF $\mathcal{R}_0$ - CONTINUED

- Because the set $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ appears naturally, or inherently, as partial derivatives of the of shell-space parametric representation, this particular basis is called the **natural basis** of the shell space
- An important subset of $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ is the set $\{\hat{i}_1, \hat{i}_2, \hat{i}_3\}$, which corresponds to planar coordinate surfaces and Cartesian coordinates
- For this special case, the **orientation and magnitude** of $\{\hat{i}_1, \hat{i}_2, \hat{i}_3\}$ are constant throughout the region $\mathcal{R}_0$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{R}_0$ - CONTINUED



NATURAL BASE-VECTOR FIELDS OF $\mathcal{R}_0$ - CONTINUED

- Another convenient form of the natural curvilinear-coordinate basis is the set of **unit-magnitude vector fields** $\{\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \hat{\mathbf{g}}_3\}$ defined by

$$\hat{\mathbf{g}}_1(\xi_1, \xi_2, \xi_3) \equiv \frac{\vec{\mathbf{g}}_1}{H_1}$$

$$\hat{\mathbf{g}}_2(\xi_1, \xi_2, \xi_3) \equiv \frac{\vec{\mathbf{g}}_2}{H_2}$$

$$\hat{\mathbf{g}}_3(\xi_1, \xi_2, \xi_3) \equiv \frac{\vec{\mathbf{g}}_3}{H_3}$$

where

$$H_1(\xi_1, \xi_2, \xi_3) \equiv |\vec{\mathbf{g}}_1| = \sqrt{\vec{\mathbf{g}}_1 \cdot \vec{\mathbf{g}}_1}$$

$$H_2(\xi_1, \xi_2, \xi_3) \equiv |\vec{\mathbf{g}}_2| = \sqrt{\vec{\mathbf{g}}_2 \cdot \vec{\mathbf{g}}_2}$$

$$H_3(\xi_1, \xi_2, \xi_3) \equiv |\vec{\mathbf{g}}_3| = \sqrt{\vec{\mathbf{g}}_3 \cdot \vec{\mathbf{g}}_3}$$

or $H_k(\xi_1, \xi_2, \xi_3) \equiv |\vec{\mathbf{g}}_k| = \sqrt{\vec{\mathbf{g}}_k \cdot \vec{\mathbf{g}}_{(k)}}$, where the index enclosed by parentheses indicates suspension of the summation convention for repeated indices

NATURAL BASE-VECTOR FIELDS OF $\mathcal{R}_0$ - CONTINUED

- Once a specific parametrization $\vec{X} = X_k(\xi_1, \xi_2, \xi_3) \hat{i}_k$ is given, $\vec{g}_k = \frac{\partial \vec{X}}{\partial \xi_k}$ is expressed in Cartesian-component form as $\vec{g}_k = \frac{\partial X_p}{\partial \xi_k} \hat{i}_p$

- Then, $H_k = \sqrt{\vec{g}_k \cdot \vec{g}_{(k)}}$ becomes $H_k = \sqrt{\frac{\partial X_p}{\partial \xi_k} \hat{i}_p \cdot \frac{\partial X_q}{\partial \xi_{(k)}} \hat{i}_q} = \sqrt{\frac{\partial X_p}{\partial \xi_k} \frac{\partial X_p}{\partial \xi_{(k)}}}$

- In particular,

$$H_1 = \sqrt{\left(\frac{\partial X_1}{\partial \xi_1}\right)^2 + \left(\frac{\partial X_2}{\partial \xi_1}\right)^2 + \left(\frac{\partial X_3}{\partial \xi_1}\right)^2}$$

$$H_2 = \sqrt{\left(\frac{\partial X_1}{\partial \xi_2}\right)^2 + \left(\frac{\partial X_2}{\partial \xi_2}\right)^2 + \left(\frac{\partial X_3}{\partial \xi_2}\right)^2}$$

$$H_3 = \sqrt{\left(\frac{\partial X_1}{\partial \xi_3}\right)^2 + \left(\frac{\partial X_2}{\partial \xi_3}\right)^2 + \left(\frac{\partial X_3}{\partial \xi_3}\right)^2}$$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{R}_0$ - CONTINUED

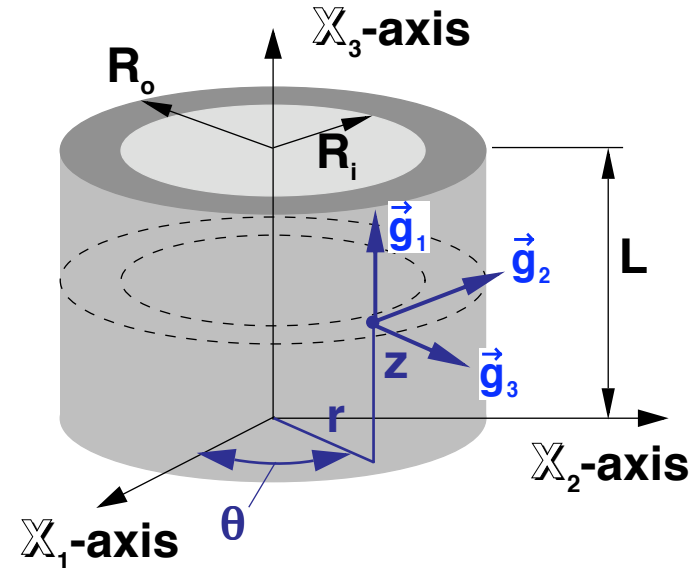
- Consider the previous example of a **cylindrical shell space** defined by the mapping functions $X_1 = \xi_3 \cos \xi_2$, $X_2 = \xi_3 \sin \xi_2$, and $X_3 = \xi_1$; where $0 \leq \xi_1 \leq L$, $0 \leq \xi_2 < 2\pi$, and $R_i \leq \xi_3 \leq R_o$

- The partial derivatives are

$$\frac{\partial X_1}{\partial \xi_1} = 0 \quad \frac{\partial X_1}{\partial \xi_2} = -\xi_3 \sin \xi_2 \quad \frac{\partial X_1}{\partial \xi_3} = \cos \xi_2$$

$$\frac{\partial X_2}{\partial \xi_1} = 0 \quad \frac{\partial X_2}{\partial \xi_2} = \xi_3 \cos \xi_2 \quad \frac{\partial X_2}{\partial \xi_3} = \sin \xi_2$$

$$\frac{\partial X_3}{\partial \xi_1} = 1 \quad \frac{\partial X_3}{\partial \xi_2} = 0 \quad \frac{\partial X_3}{\partial \xi_3} = 0$$



- The natural base-vector fields are found to be

$$\vec{g}_1 = \hat{i}_3, \quad \vec{g}_2 = \xi_3 (\cos \xi_2 \hat{i}_2 - \sin \xi_2 \hat{i}_1), \quad \text{and} \quad \vec{g}_3 = \cos \xi_2 \hat{i}_1 + \sin \xi_2 \hat{i}_2$$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{R}_0$ - CONCLUDED

- The magnitudes of the natural base-vector fields are computed as

$$H_1 = |\vec{g}_1| = 1, \quad H_2 = |\vec{g}_2| = \xi_3, \quad \text{and} \quad H_3 = |\vec{g}_3| = 1$$

- With these quantities, the **unit-magnitude natural base-vector fields** are given by

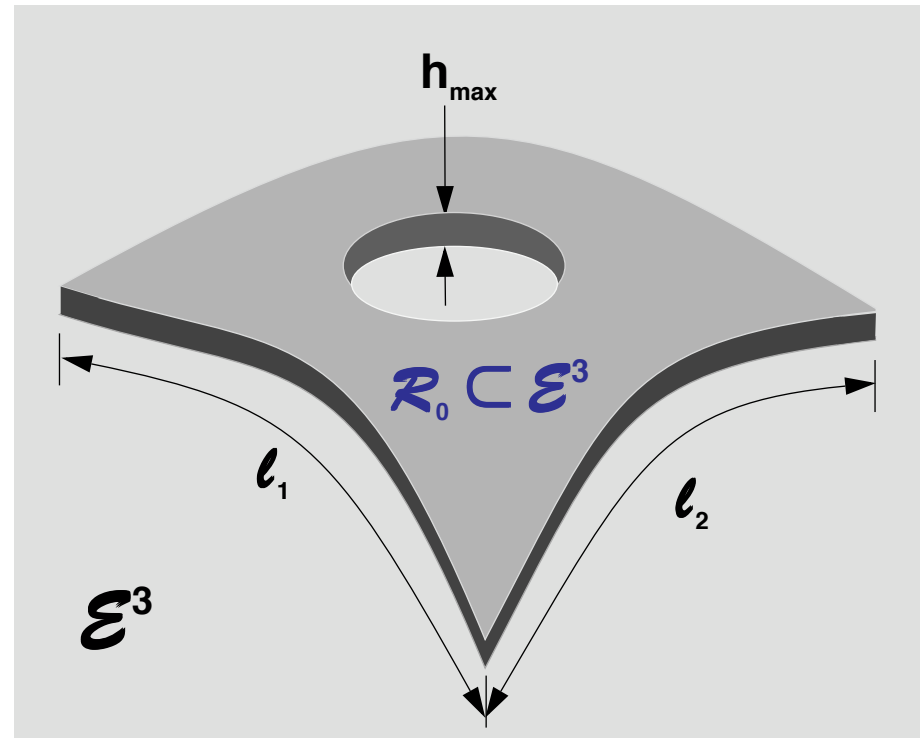
$$\hat{g}_1 = \frac{|\vec{g}_1|}{H_1} = \hat{i}_3,$$

$$\hat{g}_2 = \frac{|\vec{g}_2|}{H_2} = \cos \xi_2 \hat{i}_2 - \sin \xi_2 \hat{i}_1, \quad \text{and}$$

$$\hat{g}_3 = \frac{|\vec{g}_3|}{H_3} = \cos \xi_2 \hat{i}_1 + \sin \xi_2 \hat{i}_2$$

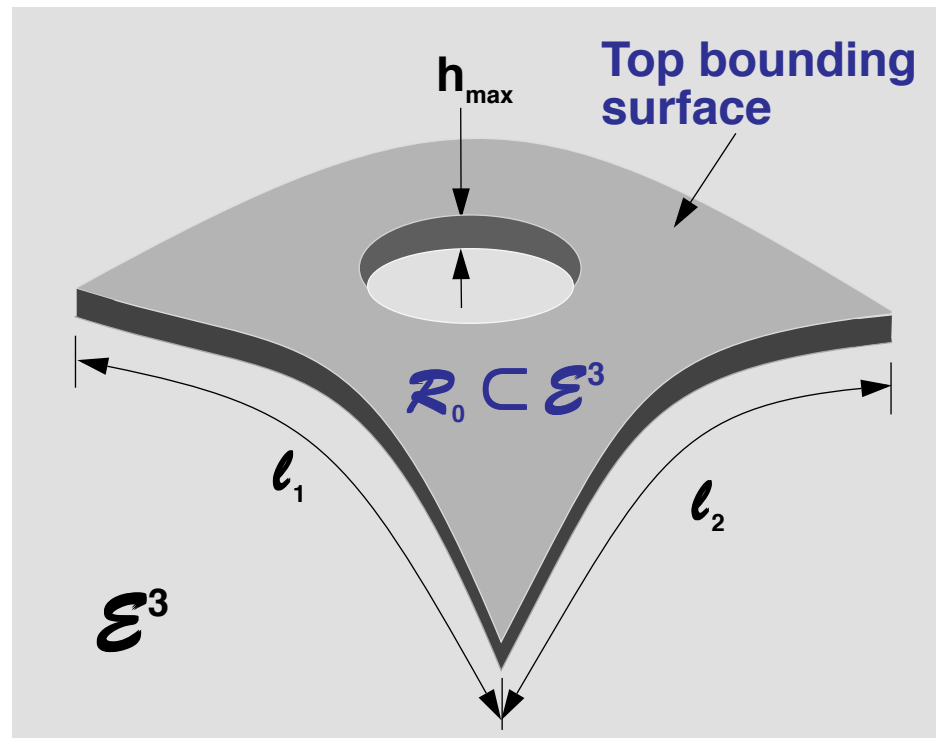
PARALLEL SURFACES OF $\mathcal{R}_0$

- In formulating a shell theory, it is useful to envision a shell as a **two-dimensional reference surface**, defined by $\mathbf{x}_k(\xi_1, \xi_2) = \mathbf{X}_k(\xi_1, \xi_2, 0)$, with **characteristic areal dimensions** ℓ_1 and ℓ_2 , and with a **finite thickness distribution** $h(\xi_1, \xi_2)$ that has a maximum value $h_{\max}$
- Typically, the characteristic dimensions ℓ_1 and ℓ_2 are much larger than the maximum shell thickness; that is, $\ell_1, \ell_2 \gg h_{\max}$
- To facilitate this approach, the ξ_3 coordinate is defined as the distance along a line **perpendicular to the plane tangent** to the reference surface at the point (ξ_1, ξ_2)



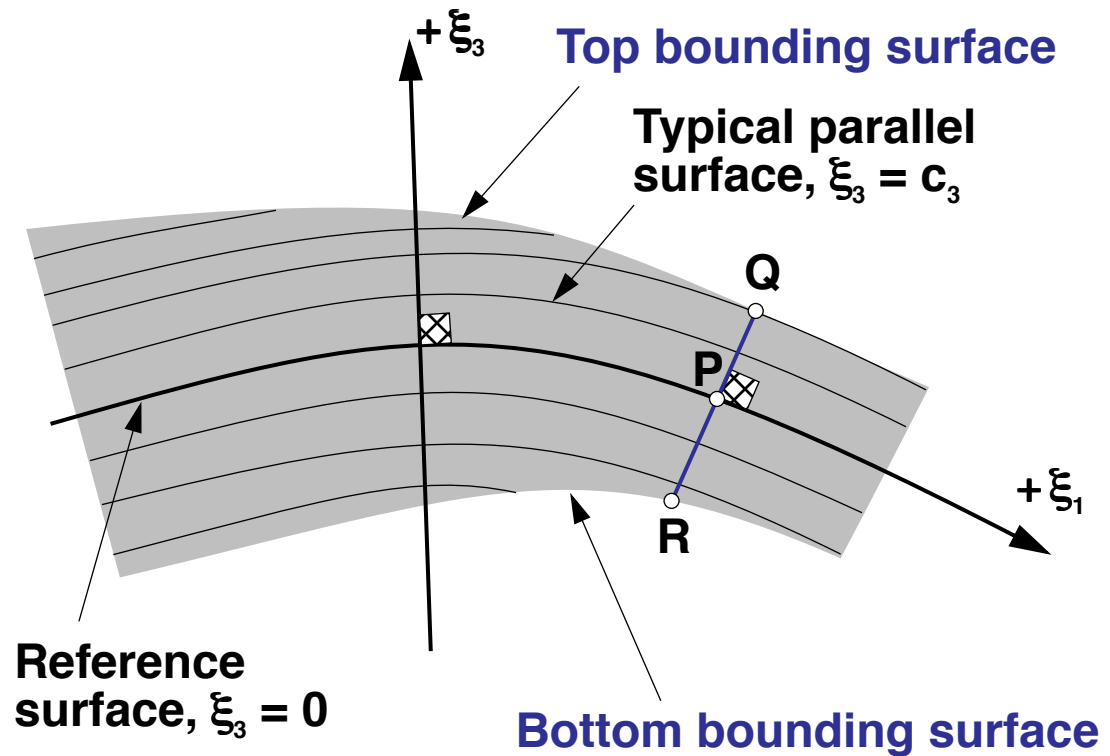
PARALLEL SURFACES OF $\mathcal{R}_0$ - CONTINUED

- For a constant value of the ξ_3 coordinate, given by $\xi_3 = c_3$, the mapping functions $X_k(\xi_1, \xi_2, c_3)$ define a surface that is located a distance c_3 above the reference surface at each of its points
 - A surface of this type is defined herein as a **parallel surface**
- Thus, in this approach a shell is modelled mathematically as a set of **contiguous parallel surfaces** that fill the three-dimensional region of space occupied by the shell, $\mathcal{R}_0$
- Note that the particles of a shell are contained within and on a top and a bottom **bounding surface**, that are generally not parallel surfaces



PARALLEL SURFACES OF $\mathcal{R}_0$ - CONTINUED

- For example, consider the cross section of a shell shown in the figure for a constant value of ξ_2
- The reference surface corresponds to $\xi_3 = 0$ and several parallel surface are shown that correspond to constant values of the coordinate ξ_3
- For this particular shell, the top and bottom surfaces are not parallel surfaces

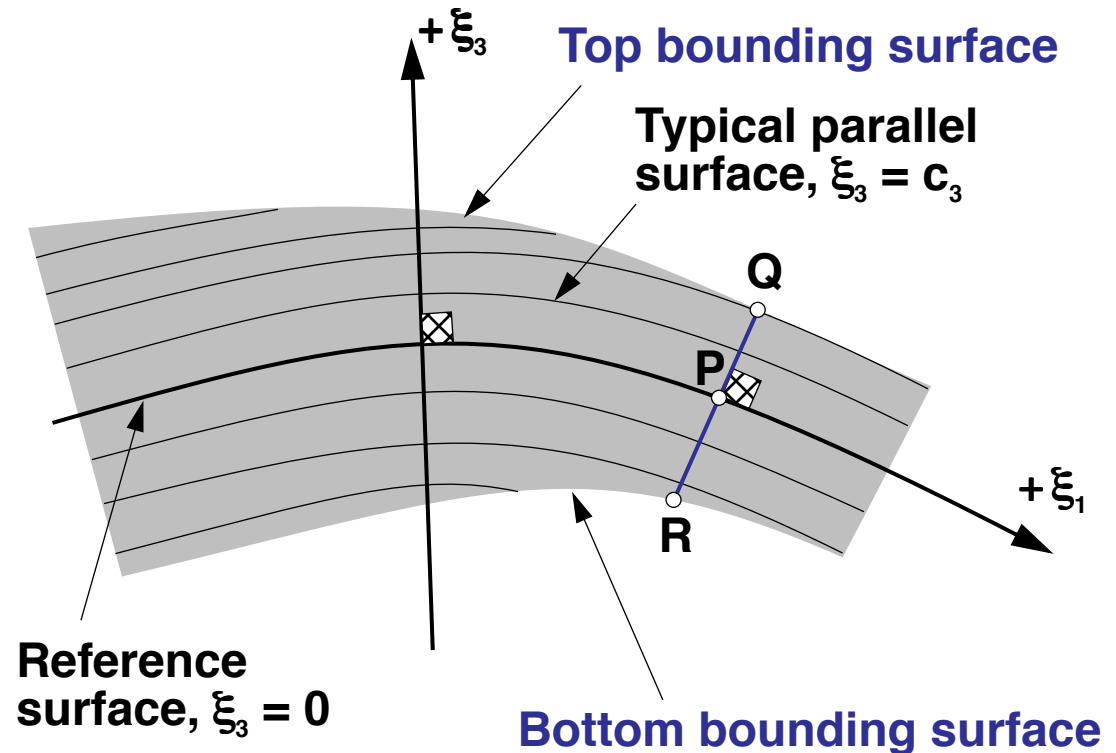


PARALLEL SURFACES OF $\mathcal{R}_0$ - CONTINUED

- Let P denote a generic point of the reference surface with coordinates

$$(\xi_1, \xi_2, 0)$$

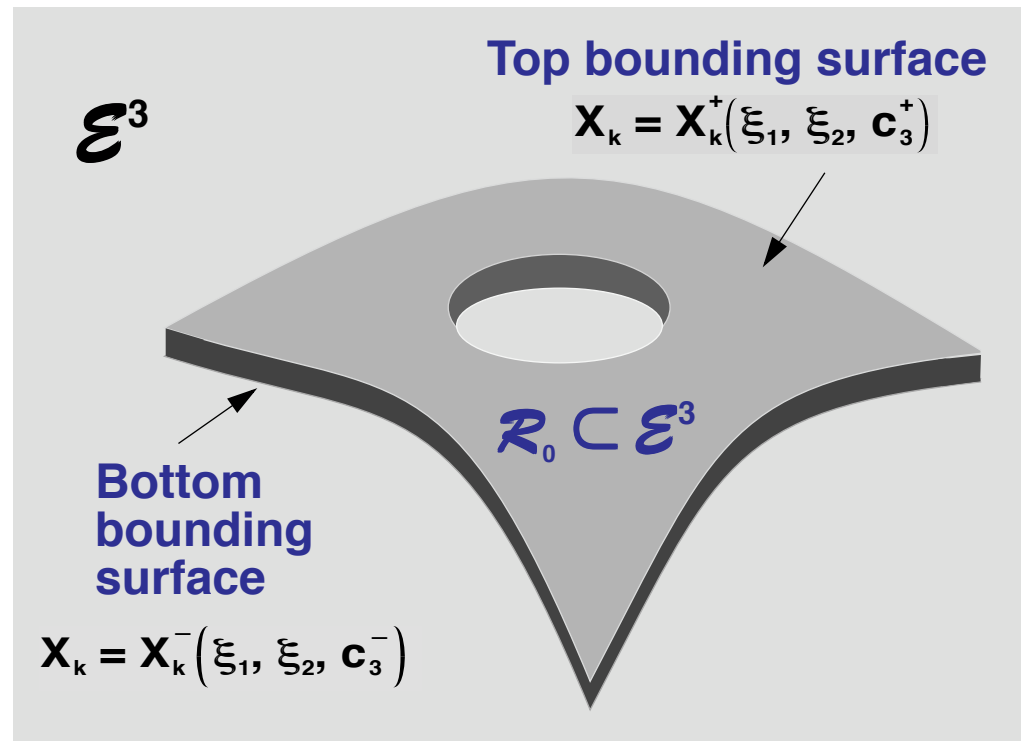
- Likewise, let Q and R denote the corresponding points of the top and bottom bounding surface, respectively



- The ξ_3 coordinates of points Q and R are denoted by $c_3^+(\xi_1, \xi_2) > 0$ and $c_3^-(\xi_1, \xi_2) < 0$, respectively, and the thickness is $h(\xi_1, \xi_2) = c_3^+ - c_3^- > 0$

PARALLEL SURFACES OF $\mathcal{R}_0$ - CONTINUED

- The **bounding surfaces** are, in general, defined by two additional mappings $\mathbf{X}_k = \mathbf{X}_k^+(\xi_1, \xi_2, \mathbf{c}_3^+)$ and $\mathbf{X}_k = \mathbf{X}_k^-(\xi_1, \xi_2, \mathbf{c}_3^-)$, where $\mathbf{c}_3^+ = \mathbf{c}_3^+(\xi_1, \xi_2)$ and $\mathbf{c}_3^- = \mathbf{c}_3^-(\xi_1, \xi_2)$ are, in general, specified functions of the reference-surface coordinates
- Typically, the **middle surface** of the shell is used as the reference surface, but it is not a necessary condition
- In general, the **reference surface** can be any surface within the shell or a convenient fictitious surface outside the actual shell



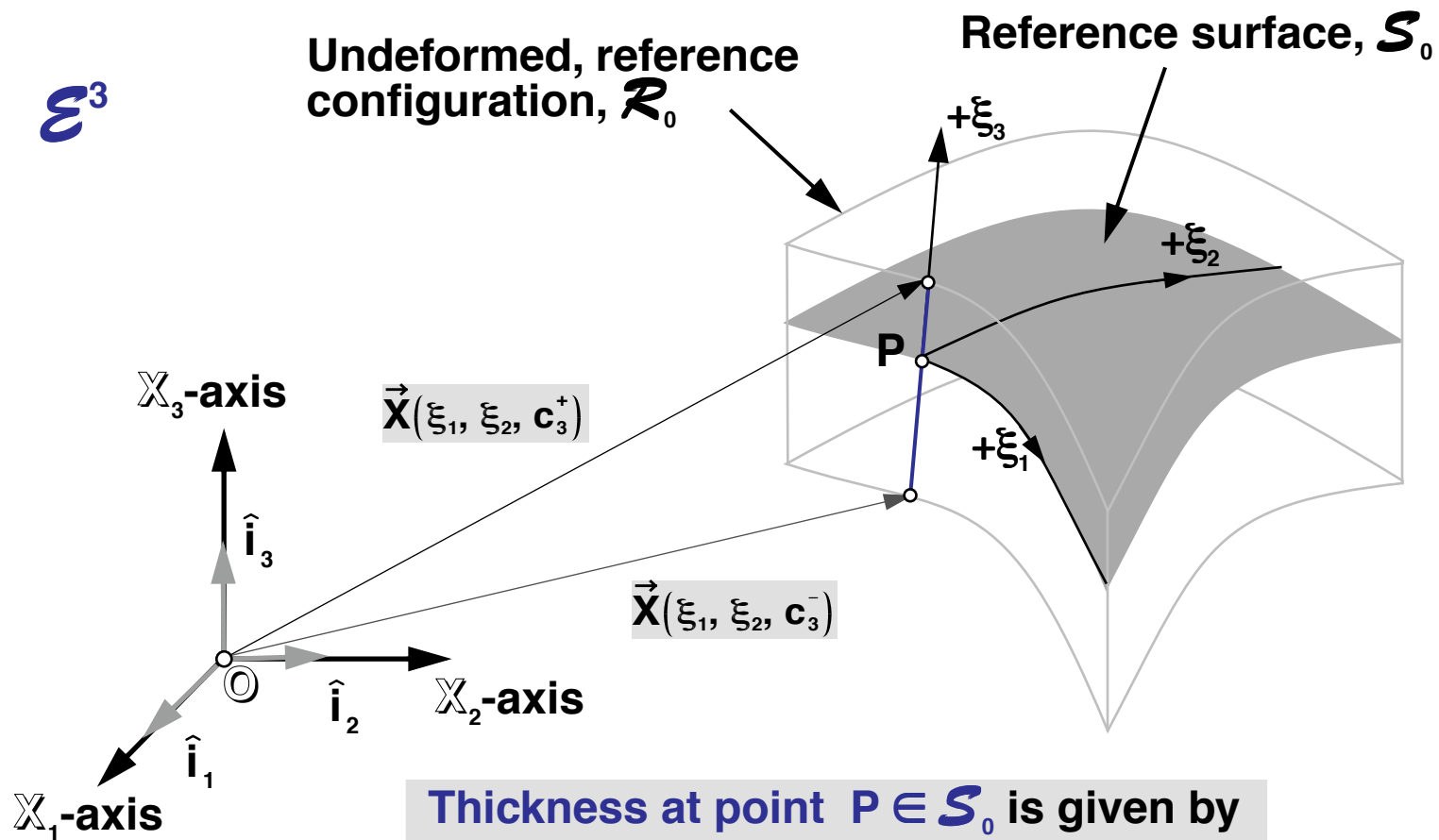
PARALLEL SURFACES OF $\mathcal{R}_0$ - CONTINUED

- The utility of defining a shell in terms of a corresponding reference surface is that it **permits the pointwise behavior of the shell to be described completely in terms of the reference surface attributes**
- As shown previously, the coordinates $(\xi_1, \xi_2, 0)$ locate points of the reference surface, and values of the coordinate ξ_3 are measured perpendicular to the reference-surface tangent plane, at the given point of the reference surface
- For this case, the shell thickness distribution is also given by

$$h(\xi_1, \xi_2) = \left| \vec{X}(\xi_1, \xi_2, c_3^+) - \vec{X}(\xi_1, \xi_2, c_3^-) \right| \quad \text{and} \quad c_3^- \leq \xi_3 \leq c_3^+$$

with $c_3^+ = c_3^+(\xi_1, \xi_2)$ and $c_3^- = c_3^-(\xi_1, \xi_2)$

PARALLEL SURFACES OF $\mathcal{R}_0$ - CONCLUDED



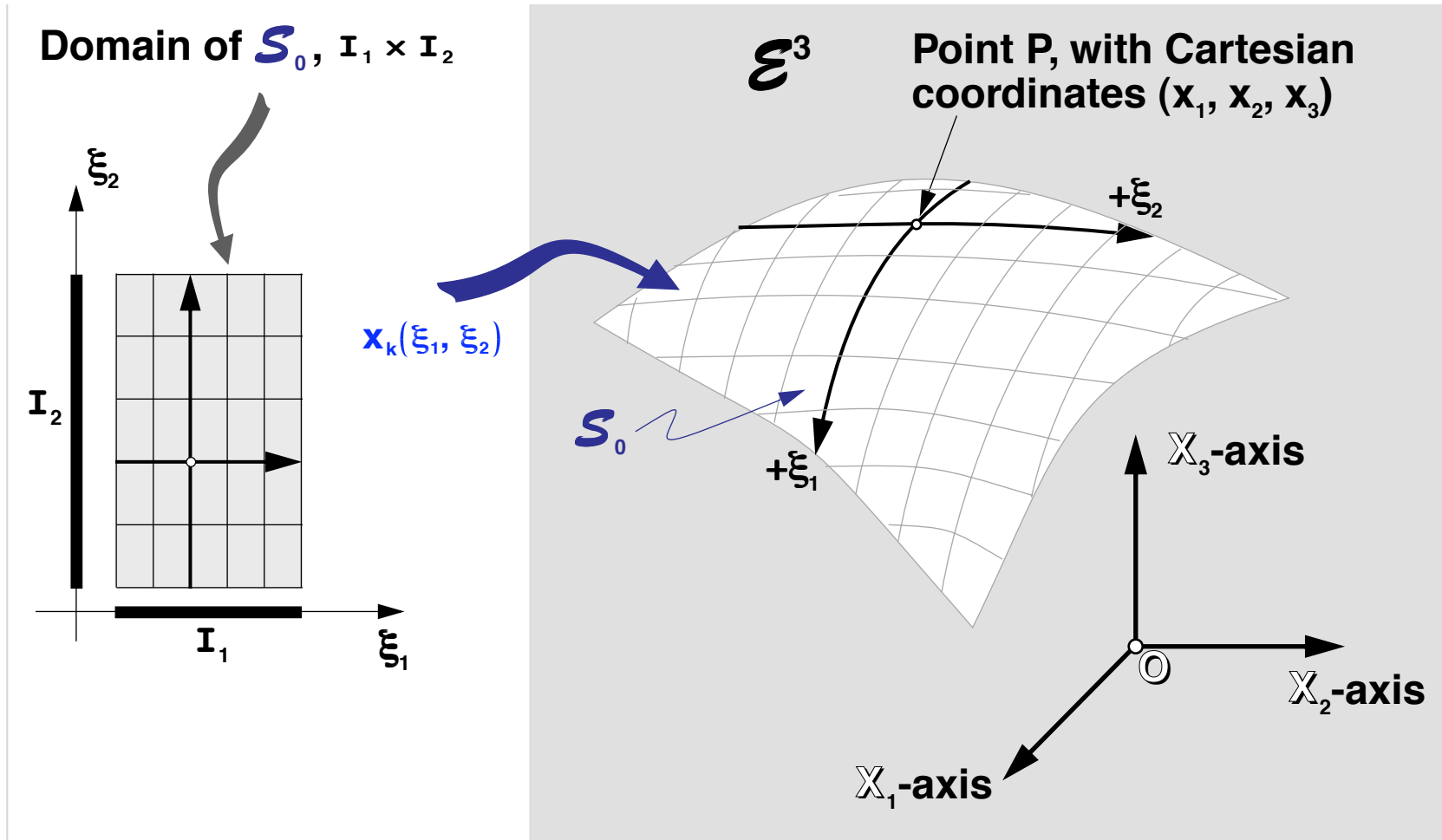
$$h(\xi_1, \xi_2) = \left| \vec{\mathbf{X}}(\xi_1, \xi_2, \mathbf{c}_3^+) - \vec{\mathbf{X}}(\xi_1, \xi_2, \mathbf{c}_3^-) \right|$$

MATHEMATICAL DESCRIPTION OF THE SHELL REFERENCE SURFACE, $\mathcal{S}_0$

MATHEMATICAL DESCRIPTION OF $\mathcal{S}_0$

- The Cartesian coordinates of the two-dimensional set of points, $\mathcal{S}_0 \subset \mathcal{R}_0 \subset \mathcal{E}^3$, forming the reference surface are specified parametrically by $\mathbf{x}_1 = \mathbf{x}_1(\xi_1, \xi_2)$, $\mathbf{x}_2 = \mathbf{x}_2(\xi_1, \xi_2)$, and $\mathbf{x}_3 = \mathbf{x}_3(\xi_1, \xi_2)$
- The parameters ξ_1 and ξ_2 are typically specified over two generally different intervals of the real line, $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively
- The domain of the mapping $\mathbf{x}_1 = \mathbf{x}_1(\xi_1, \xi_2)$, $\mathbf{x}_2 = \mathbf{x}_2(\xi_1, \xi_2)$, and $\mathbf{x}_3 = \mathbf{x}_3(\xi_1, \xi_2)$ is given by the **Cartesian product** $\mathcal{I}_1 \times \mathcal{I}_2$
- The set of ordered parameters (ξ_1, ξ_2) form a coordinate “net” within the surface, as shown in the next figure, and are called **Gaussian coordinates** of the surface

MATHEMATICAL DESCRIPTION OF $\mathcal{S}_0$ - CONTINUED



MATHEMATICAL DESCRIPTION OF $\mathcal{S}_0$ - CONCLUDED

- The specific form of the functions $\mathbf{x}_k = \mathbf{x}_k(\xi_1, \xi_2)$ define implicitly the positive traversal directions of the corresponding curvilinear Gaussian-coordinate curves
- To characterize the deformation of a shell, it is convenient to use a **vector description** of the reference surface and its deformed image at an arbitrary time $t > 0$
- Therefore, let $\mathcal{S}_0$ be an **arbitrary smooth surface** in $\mathcal{E}^3$, in proximity of the shell space $\mathcal{R}_0$, that is defined by the *position vector* $\vec{\mathbf{x}} = \mathbf{x}_k(\xi_1, \xi_2) \hat{\mathbf{i}}_k$
 - $\mathcal{S}_0$ is referred to herein as the **undeformed reference surface**
 - Points of $\mathcal{S}_0$ are presumed to have a unique correspondance with material particles of the shell only when $\mathcal{S}_0 \subset \mathcal{R}_0 \subset \mathcal{E}^3$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_0$

- In formulating a two-dimensional shell theory, the properties of a three-dimensional shell are represented in terms of the attributes of a corresponding reference surface $\mathcal{S}_0$
- Thus, the need arises to define vector fields associated with the points of $\mathcal{S}_0$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_0$ - CONTINUED

- Previously, it was shown that the natural base-vector fields for the shell space are defined by $\vec{g}_k(\xi_1, \xi_2, \xi_3) \equiv \frac{\partial \vec{X}}{\partial \xi_k}$ and that the reference surface is defined by $\mathbf{x}_k(\xi_1, \xi_2) \equiv \mathbf{X}_k(\xi_1, \xi_2, 0)$
- When $\mathcal{S}_0$ is not a subset of $\mathcal{R}_0$, $\mathcal{S}_0$ is defined by specifying the mapping functions $\mathbf{X}_k(\xi_1, \xi_2, \xi_3)$ to include points outside of $\mathcal{R}_0$
- In addition, the vectors $\vec{g}_k(\xi_1, \xi_2, \xi_3)$ are tangent to the coordinate curves at every point of the shell space
- Furthermore, the ξ_3 coordinate has been designated as a rectilinear coordinate that is measured perpendicular to the tangent plane $T_p(\mathcal{S}_0)$, at every point $P \in \mathcal{S}_0$ of the reference surface with coordinates $(\xi_1, \xi_2, 0)$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_0$ - CONTINUED

- Therefore, the vector fields

$$\vec{g}_\alpha(\xi_1, \xi_2, 0) = \left. \frac{\partial \vec{X}}{\partial \xi_\alpha} \right|_{\xi_3=0} = \frac{\partial \vec{X}}{\partial \xi_\alpha} \equiv \vec{a}_\alpha(\xi_1, \xi_2) \quad \text{and} \quad \vec{g}_3(\xi_1, \xi_2, 0) = \left. \frac{\partial \vec{X}}{\partial \xi_3} \right|_{\xi_3=0}$$

form natural base-vector fields for vectors associated with points of the shell reference surface $\mathcal{S}_0$

- The natural base-vector fields of $\mathcal{S}_0$, that span the tangent plane,

$$T_p(\mathcal{S}_0), \text{ at every point } P \in \mathcal{S}_0, \text{ are } \vec{a}_1(\xi_1, \xi_2) \equiv \frac{\partial \vec{X}}{\partial \xi_1} \quad \text{and} \quad \vec{a}_2(\xi_1, \xi_2) \equiv \frac{\partial \vec{X}}{\partial \xi_2}$$

- To complete the basis for points of $\mathcal{S}_0$, it is also convenient to

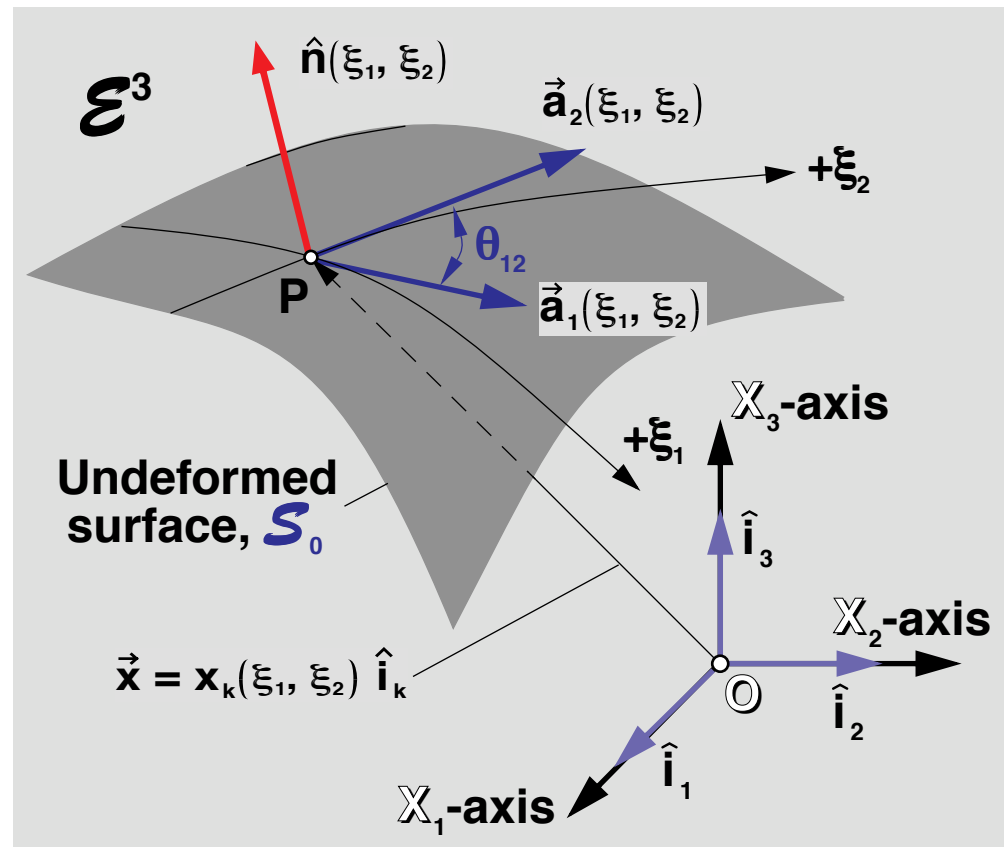
introduce the unit-magnitude vector field $\hat{n}(\xi_1, \xi_2) \equiv \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|}$ that is

perpendicular to the tangent plane at every point $P \in \mathcal{S}_0$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_0$ - CONTINUED

- At a given point of $\mathcal{S}_0$, the base vectors $\vec{a}_1$ and $\vec{a}_2$ are tangent to the Gaussian-coordinate curves, as shown in the figure
- In general, the base vectors $\vec{a}_1$ and $\vec{a}_2$ are not orthogonal
- The angle between $\vec{a}_1$ and $\vec{a}_2$ is denoted by θ_{12} , as shown in the figure
- Note that generally

$$\theta_{12} = \theta_{12}(\xi_1, \xi_2)$$



NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_0$ - CONTINUED

- The magnitudes of $\vec{a}_1$ and $\vec{a}_2$ are defined by

$$|\vec{a}_\alpha| \equiv A_\alpha(\xi_1, \xi_2) = \sqrt{\vec{a}_\alpha \cdot \vec{a}_{(\alpha)}}$$

- Noting that $|\vec{a}_\alpha| = |\vec{g}_\alpha(\xi_1, \xi_2, 0)|$, it follows that $A_\alpha(\xi_1, \xi_2) = H_\alpha(\xi_1, \xi_2, 0)$

- With these definitions, the **unit-magnitude natural base vector fields** of the undeformed reference surface $\mathcal{S}_0$ are defined by

$$\hat{a}_1 \equiv \frac{\vec{a}_1}{|\vec{a}_1|} = \frac{\vec{a}_1}{A_1} \quad \text{and} \quad \hat{a}_2 \equiv \frac{\vec{a}_2}{|\vec{a}_2|} = \frac{\vec{a}_2}{A_2}$$

- It is important to remember that although the magnitudes of these vector fields are constant, the directions are not - thus, $\hat{a}_\alpha = \hat{a}_\alpha(\xi_1, \xi_2)$
- The angle $\theta_{12}(\xi_1, \xi_2)$ is computed from $\cos\theta_{12} = \hat{a}_1 \cdot \hat{a}_2$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_0$ - CONTINUED

- The Cartesian-component forms of $\vec{a}_1(\xi_1, \xi_2) = \frac{\partial \vec{x}}{\partial \xi_1}$ and $\vec{a}_2(\xi_1, \xi_2) = \frac{\partial \vec{x}}{\partial \xi_2}$ are given by

$$\vec{a}_1(\xi_1, \xi_2) = \frac{\partial x_1}{\partial \xi_1} \hat{i}_1 + \frac{\partial x_2}{\partial \xi_1} \hat{i}_2 + \frac{\partial x_3}{\partial \xi_1} \hat{i}_3 \quad \text{and} \quad \vec{a}_2(\xi_1, \xi_2) = \frac{\partial x_1}{\partial \xi_2} \hat{i}_1 + \frac{\partial x_2}{\partial \xi_2} \hat{i}_2 + \frac{\partial x_3}{\partial \xi_2} \hat{i}_3$$

or in indicial form by $\vec{a}_\alpha(\xi_1, \xi_2) = \frac{\partial x_k}{\partial \xi_\alpha} \hat{i}_k$, where $x_k(\xi_1, \xi_2)$ are known

- Thus, $A_\alpha = \sqrt{\vec{a}_\alpha \cdot \vec{a}_{(\alpha)}}$ yields

$$A_1 = \sqrt{\left(\frac{\partial x_1}{\partial \xi_1}\right)^2 + \left(\frac{\partial x_2}{\partial \xi_1}\right)^2 + \left(\frac{\partial x_3}{\partial \xi_1}\right)^2} \quad A_2 = \sqrt{\left(\frac{\partial x_1}{\partial \xi_2}\right)^2 + \left(\frac{\partial x_2}{\partial \xi_2}\right)^2 + \left(\frac{\partial x_3}{\partial \xi_2}\right)^2}$$

or in indicial form $A_\alpha = \sqrt{\frac{\partial x_k}{\partial \xi_\alpha} \frac{\partial x_k}{\partial \xi_{(\alpha)}}}$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_0$ - CONTINUED

- Similarly,

$$\begin{aligned}\vec{a}_1 \times \vec{a}_2 = & \left(\frac{\partial \mathbf{x}_2}{\partial \xi_1} \frac{\partial \mathbf{x}_3}{\partial \xi_2} - \frac{\partial \mathbf{x}_3}{\partial \xi_1} \frac{\partial \mathbf{x}_2}{\partial \xi_2} \right) \hat{\mathbf{i}}_1 + \left(\frac{\partial \mathbf{x}_3}{\partial \xi_1} \frac{\partial \mathbf{x}_1}{\partial \xi_2} - \frac{\partial \mathbf{x}_1}{\partial \xi_1} \frac{\partial \mathbf{x}_3}{\partial \xi_2} \right) \hat{\mathbf{i}}_2 \\ & + \left(\frac{\partial \mathbf{x}_1}{\partial \xi_1} \frac{\partial \mathbf{x}_2}{\partial \xi_2} - \frac{\partial \mathbf{x}_2}{\partial \xi_1} \frac{\partial \mathbf{x}_1}{\partial \xi_2} \right) \hat{\mathbf{i}}_3\end{aligned}$$

- The components of the **unit-magnitude vector field** $\hat{\mathbf{n}} = n_k \hat{\mathbf{i}}_k$ are given in terms of the reference-surface parametrization $\mathbf{x}_k(\xi_1, \xi_2)$ by

$$n_1 = \frac{1}{|\vec{a}_1 \times \vec{a}_2|} \left[\frac{\partial \mathbf{x}_2}{\partial \xi_1} \frac{\partial \mathbf{x}_3}{\partial \xi_2} - \frac{\partial \mathbf{x}_3}{\partial \xi_1} \frac{\partial \mathbf{x}_2}{\partial \xi_2} \right], \quad n_2 = \frac{1}{|\vec{a}_1 \times \vec{a}_2|} \left[\frac{\partial \mathbf{x}_1}{\partial \xi_2} \frac{\partial \mathbf{x}_3}{\partial \xi_1} - \frac{\partial \mathbf{x}_1}{\partial \xi_1} \frac{\partial \mathbf{x}_3}{\partial \xi_2} \right], \quad \text{and}$$

$$n_3 = \frac{1}{|\vec{a}_1 \times \vec{a}_2|} \left[\frac{\partial \mathbf{x}_1}{\partial \xi_1} \frac{\partial \mathbf{x}_2}{\partial \xi_2} - \frac{\partial \mathbf{x}_2}{\partial \xi_1} \frac{\partial \mathbf{x}_1}{\partial \xi_2} \right] \quad \text{where}$$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_0$ - CONTINUED

$$\begin{aligned} \left| \vec{a}_1 \times \vec{a}_2 \right|^2 = & \left(\frac{\partial \mathbf{x}_2}{\partial \xi_1} \frac{\partial \mathbf{x}_3}{\partial \xi_2} - \frac{\partial \mathbf{x}_3}{\partial \xi_1} \frac{\partial \mathbf{x}_2}{\partial \xi_2} \right)^2 + \left(\frac{\partial \mathbf{x}_3}{\partial \xi_1} \frac{\partial \mathbf{x}_1}{\partial \xi_2} - \frac{\partial \mathbf{x}_1}{\partial \xi_1} \frac{\partial \mathbf{x}_3}{\partial \xi_2} \right)^2 \\ & + \left(\frac{\partial \mathbf{x}_1}{\partial \xi_1} \frac{\partial \mathbf{x}_2}{\partial \xi_2} - \frac{\partial \mathbf{x}_2}{\partial \xi_1} \frac{\partial \mathbf{x}_1}{\partial \xi_2} \right)^2 \end{aligned}$$

- Note that once $\hat{\mathbf{n}}(\xi_1, \xi_2) = \mathbf{n}_k(\xi_1, \xi_2) \hat{\mathbf{i}}_k$ is known, points of the shell space are located by using $\vec{\mathbf{X}}(\xi_1, \xi_2, \xi_3) = \vec{\mathbf{x}}(\xi_1, \xi_2) + \xi_3 \hat{\mathbf{n}}(\xi_1, \xi_2)$

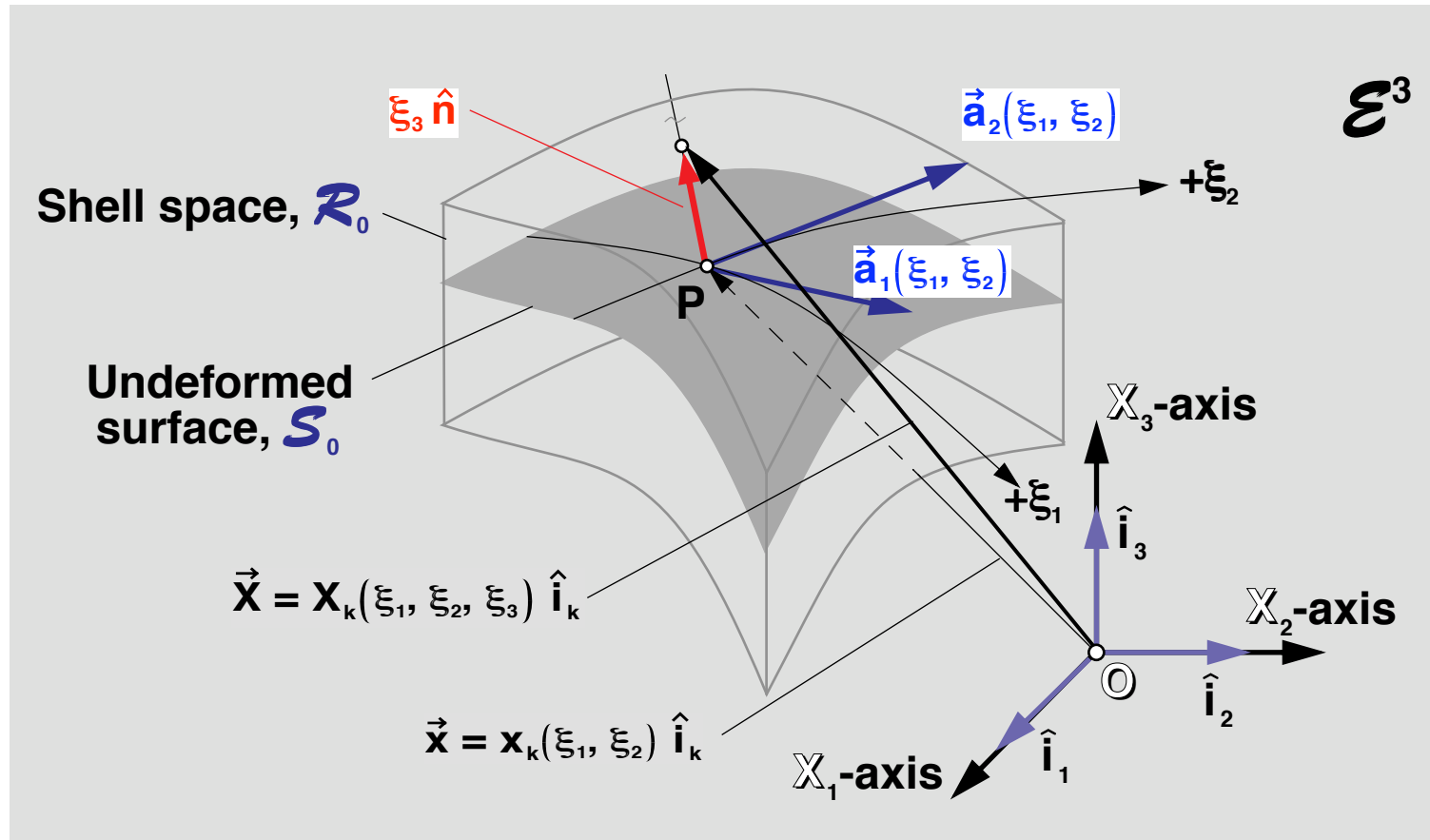
- Moreover, the bounding surfaces of the shell space are given by

$$\vec{\mathbf{X}}(\xi_1, \xi_2, \mathbf{c}_3^+) = \vec{\mathbf{x}}(\xi_1, \xi_2) + \mathbf{c}_3^+(\xi_1, \xi_2) \hat{\mathbf{n}}(\xi_1, \xi_2) \quad \text{and}$$

$$\vec{\mathbf{X}}(\xi_1, \xi_2, \mathbf{c}_3^-) = \vec{\mathbf{x}}(\xi_1, \xi_2) + \mathbf{c}_3^-(\xi_1, \xi_2) \hat{\mathbf{n}}(\xi_1, \xi_2) ,$$

and the shell thickness is given by $h(\xi_1, \xi_2) = \mathbf{c}_3^+ - \mathbf{c}_3^- > 0$

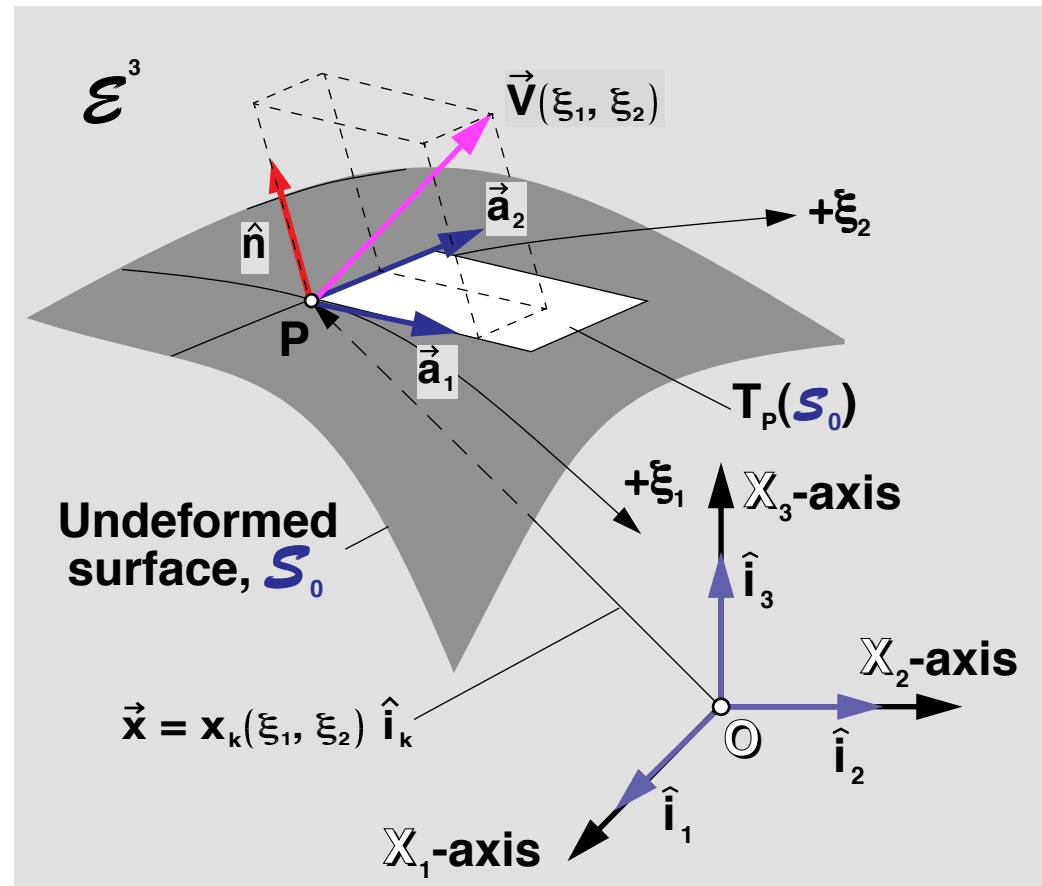
NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_0$ - CONCLUDED



- In this figure, $\mathcal{S}_0 \subset \mathcal{R}_0 \subset \mathcal{E}^3$

VECTOR FIELDS DEFINED ON $\mathcal{S}_0$

- The set $\{\vec{a}_1, \vec{a}_2, \hat{n}\}$ forms a **natural basis** for vector fields in $\mathcal{E}^3$ that are associated with points of the reference surface $\mathcal{S}_0$
- Thus, any vector $\vec{V}(\xi_1, \xi_2)$ in $\mathcal{E}^3$ that is associated with the point $P \in \mathcal{S}_0$, can be expressed as a **linear combination** of the three base vectors
- In addition, the set of **unit vectors** $\{\hat{a}_1, \hat{a}_2, \hat{n}\}$ forms a basis for vector fields associated with any point $P \in \mathcal{S}_0$

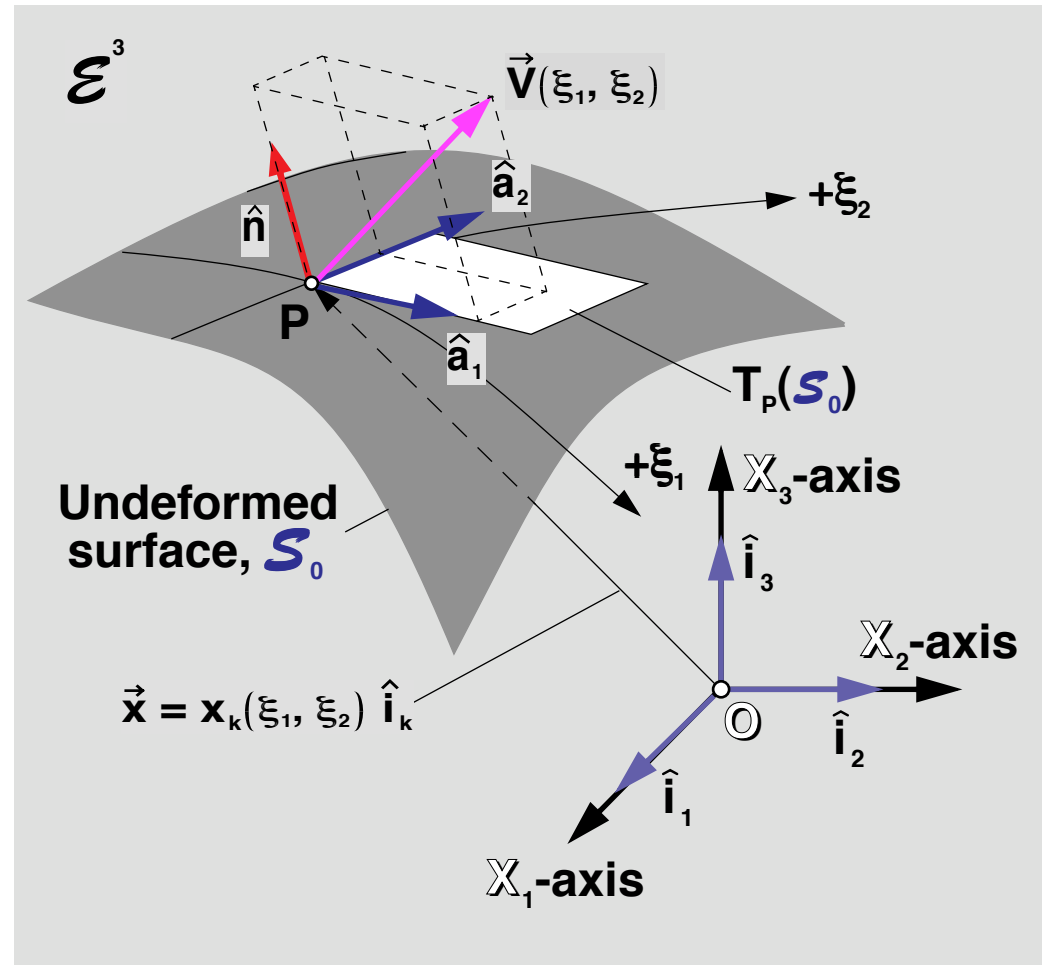


VECTOR FIELDS DEFINED ON $\mathcal{S}_0$ - CONCLUDED

- Thus, any vector $\vec{V}(\xi_1, \xi_2)$ in $\mathcal{E}^3$ that is associated with the point $P \in \mathcal{S}_0$, can also be expressed as a linear combination of the vectors $\{\hat{a}_1, \hat{a}_2, \hat{n}\}$
- In the present study, vectors associated with points of $\mathcal{S}_0$ are expressed as

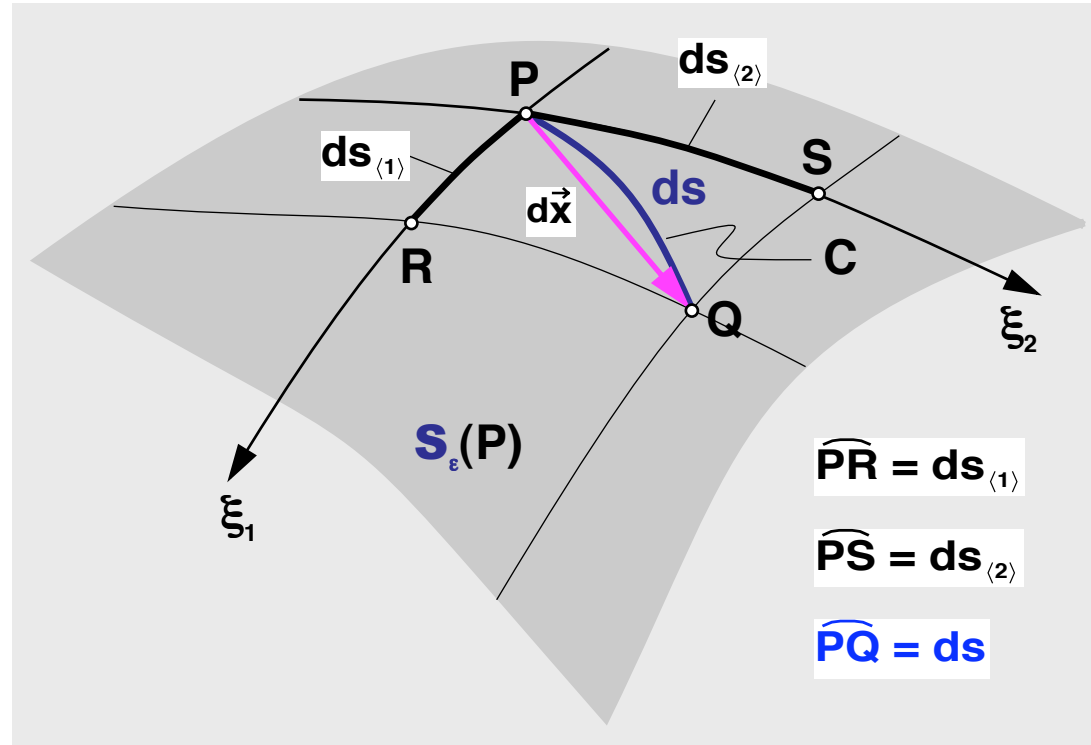
$$\vec{V} = V_\alpha \hat{a}_\alpha + V_3 \hat{n}$$

unless indicated otherwise



METRIC COEFFICIENTS OF $\mathcal{S}_0$

- Let $\mathbf{S}_\varepsilon(P)$ denote a small, *infinitesimal neighborhood* of an arbitrary point P of the undeformed reference surface, $\mathcal{S}_0$
- Likewise, let point Q be in $\mathbf{S}_\varepsilon(P)$ and let C denote a smooth **surface curve** that connects points P and Q
- The surface curve C is defined generally by a parametrization of the form $\xi_\alpha = \xi_\alpha(\mu)$, where μ is a parameter
- For convenience, let the parametrization of C be given by $\xi_\alpha = \xi_\alpha(s)$, where $0 \leq s \leq L$ is the arc-length coordinate of C and L is its length



METRIC COEFFICIENTS OF $\mathcal{S}_0$ - CONTINUED

- As the curve is traversed by an infinitesimal amount ds , from point P to point Q in the previous figure, a differential increment in the surface coordinates $\xi_\alpha = \xi_\alpha(s)$ is induced
- The position vector to point Q is $\vec{x}(\xi_1 + d\xi_1, \xi_2 + d\xi_2) = \vec{x}(\xi_1, \xi_2) + d\vec{x}$, where $d\vec{x}$ is the vector from point P to point Q shown in the previous figure
- The length of surface arc between points P and Q shown in the previous figure is given by $ds = \widehat{PQ}$
- The differential arc lengths $\widehat{PR}$ and $\widehat{PS}$ of the coordinate curves, shown in the previous figure, are denoted by $\widehat{PR} = ds_{\langle 1 \rangle}$ and $\widehat{PS} = ds_{\langle 2 \rangle}$

METRIC COEFFICIENTS OF $\mathcal{S}_0$ - CONTINUED

- The arc length ds of the two infinitesimally close points, P and Q, is given by the dot product $ds^2 = d\vec{x} \cdot d\vec{x}$

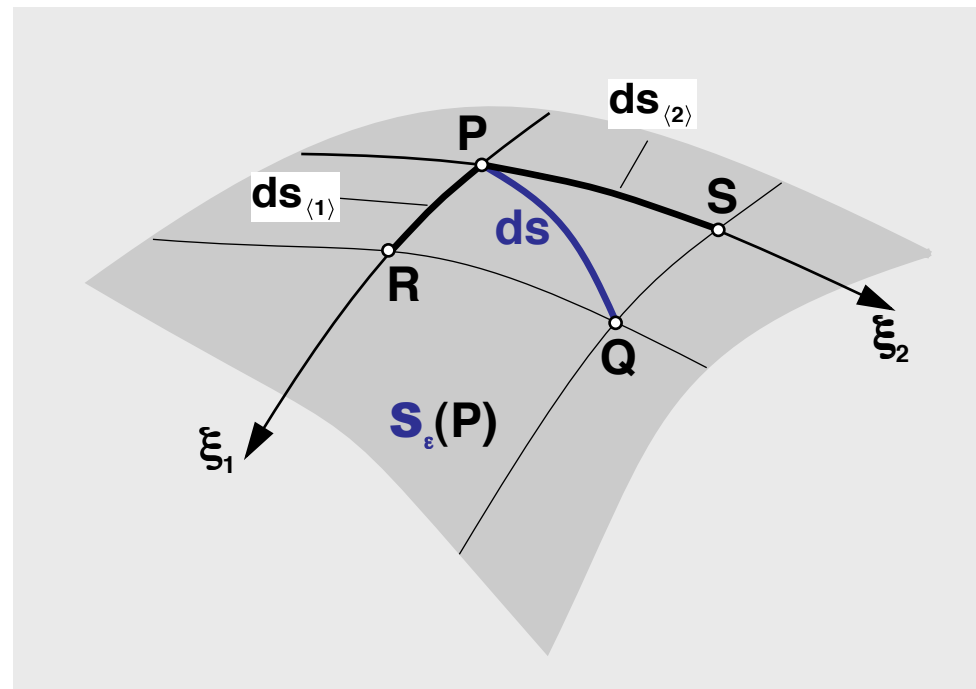
- Noting that $d\vec{x} = \frac{\partial \vec{x}}{\partial \xi_\alpha} d\xi_\alpha$

it follows that

$$ds^2 = (\vec{a}_\alpha \cdot \vec{a}_\beta) d\xi_\alpha d\xi_\beta$$

- The quantities $\vec{a}_\alpha \cdot \vec{a}_\beta \equiv a_{\alpha\beta}$ are called the **components of the surface metric tensor**

- These metric quantities enable the measurement of surface-curve lengths, surface areas, and angles between the tangent lines of intersecting surface curves



METRIC COEFFICIENTS OF $\mathcal{S}_0$ - CONTINUED

- Thus, the arc length ds is expressed as $ds^2 = a_{\alpha\beta} d\xi_\alpha d\xi_\beta$, or in expanded form as $ds^2 = a_{11}(d\xi_1)^2 + 2a_{12} d\xi_1 d\xi_2 + a_{22}(d\xi_2)^2$
- An alternative representation of the arc length ds used herein is given by

$$ds^2 = (A_1 d\xi_1)^2 + 2A_1 A_2 \cos\theta_{12} d\xi_1 d\xi_2 + (A_2 d\xi_2)^2$$

where

$$A_1(\xi_1, \xi_2) = \sqrt{\vec{a}_1 \cdot \vec{a}_1} = |\vec{a}_1| = \sqrt{a_{11}}, \quad A_2(\xi_1, \xi_2) = \sqrt{\vec{a}_2 \cdot \vec{a}_2} = |\vec{a}_2| = \sqrt{a_{22}}, \text{ and}$$

$$\cos\theta_{12} = \frac{\vec{a}_1}{|\vec{a}_1|} \cdot \frac{\vec{a}_2}{|\vec{a}_2|} = \frac{a_{12}}{\sqrt{a_{11}a_{22}}}$$

METRIC COEFFICIENTS OF $\mathcal{S}_0$ - CONCLUDED

- The functions A_1 and A_2 are known as the **Lame' parameters** of the undeformed reference surface, but are often referred to as metric coefficients or metric parameters
- For the special case of **orthogonal Gaussian reference-surface** coordinates,

$$\theta_{12} = \frac{\pi}{2} \quad \text{and} \quad ds^2 = (A_1 d\xi_1)^2 + (A_2 d\xi_2)^2$$

RECIPROCAL BASIS OF $\mathcal{S}_0$

- When dealing with general nonorthogonal Gaussian coordinates, it is also convenient to introduce the unit-magnitude vector fields

$$\hat{\mathbf{a}}^1 \equiv \hat{\mathbf{a}}_2 \times \hat{\mathbf{n}} \quad \text{and} \quad \hat{\mathbf{a}}^2 \equiv \hat{\mathbf{n}} \times \hat{\mathbf{a}}_1$$

for the reference surface such that

$$\hat{\mathbf{a}}^1 \cdot \hat{\mathbf{n}} = 0, \quad \hat{\mathbf{a}}^2 \cdot \hat{\mathbf{n}} = 0, \quad \hat{\mathbf{a}}^1 \cdot \hat{\mathbf{a}}_2 = \hat{\mathbf{a}}^2 \cdot \hat{\mathbf{a}}_1 = 0, \quad \text{and} \quad \hat{\mathbf{a}}^1 \times \hat{\mathbf{a}}^2 = \hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_2$$

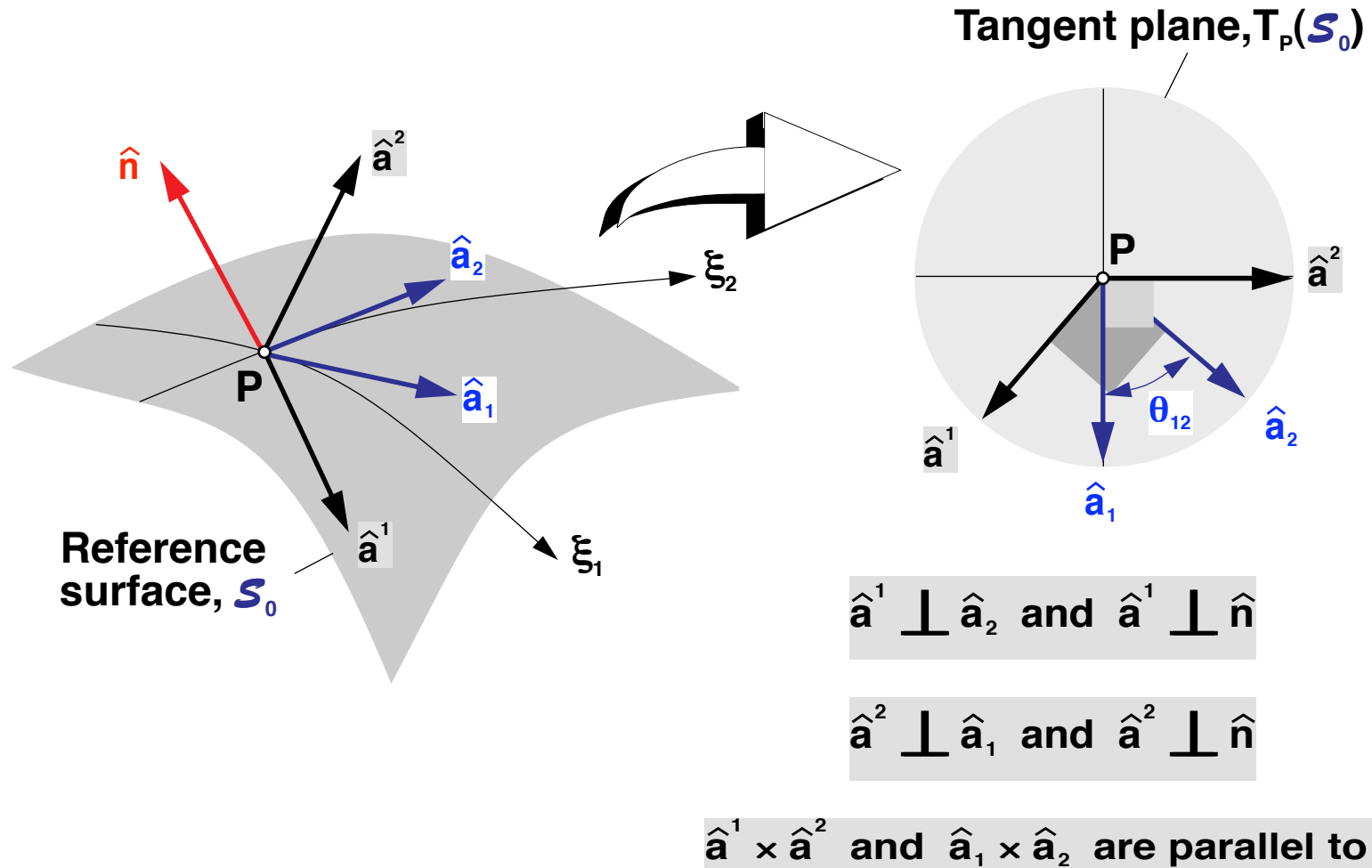
- Noting that these conditions on $\hat{\mathbf{a}}^1$ and $\hat{\mathbf{a}}^2$ imply that the angle between $\hat{\mathbf{a}}^1$ and $\hat{\mathbf{a}}_1$, and between $\hat{\mathbf{a}}^2$ and $\hat{\mathbf{a}}_2$, is $\frac{\pi}{2} - \theta_{12}$ it follows that

$$\hat{\mathbf{a}}^1 \cdot \hat{\mathbf{a}}_1 = \sin\theta_{12} \quad \text{and} \quad \hat{\mathbf{a}}^2 \cdot \hat{\mathbf{a}}_2 = \sin\theta_{12}; \quad \text{thus,} \quad \hat{\mathbf{a}}^\alpha \cdot \hat{\mathbf{a}}_\beta = \delta^\alpha_\beta \sin\theta_{12}$$

- Moreover, it follows that

$$\begin{aligned} \hat{\mathbf{a}}^1 \times \hat{\mathbf{a}}_1 &= \cos\theta_{12} \hat{\mathbf{n}} & \hat{\mathbf{a}}^1 \times \hat{\mathbf{a}}_2 &= \hat{\mathbf{n}} & \hat{\mathbf{a}}^1 \times \hat{\mathbf{n}} &= -\hat{\mathbf{a}}_2 \\ \hat{\mathbf{a}}^2 \times \hat{\mathbf{a}}_1 &= -\hat{\mathbf{n}} & \hat{\mathbf{a}}^2 \times \hat{\mathbf{a}}_2 &= -\cos\theta_{12} \hat{\mathbf{n}} & \hat{\mathbf{a}}^2 \times \hat{\mathbf{n}} &= \hat{\mathbf{a}}_1 \end{aligned}$$

RECIPROCAL BASIS OF $\mathcal{S}_0$ - CONTINUED



RECIPROCAL BASIS OF $\mathcal{S}_0$ - CONCLUDED

- Expressing $\hat{\mathbf{a}}^1$ and $\hat{\mathbf{a}}^2$ as linear combinations of $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ and applying these additional relationships yields the results

$$\hat{\mathbf{a}}^1 = \hat{\mathbf{a}}_1 \csc\theta_{12} - \hat{\mathbf{a}}_2 \cot\theta_{12} = \hat{\mathbf{a}}_2 \times \hat{\mathbf{n}} \quad \text{and} \quad \hat{\mathbf{a}}^2 = \hat{\mathbf{a}}_2 \csc\theta_{12} - \hat{\mathbf{a}}_1 \cot\theta_{12} = \hat{\mathbf{n}} \times \hat{\mathbf{a}}_1$$

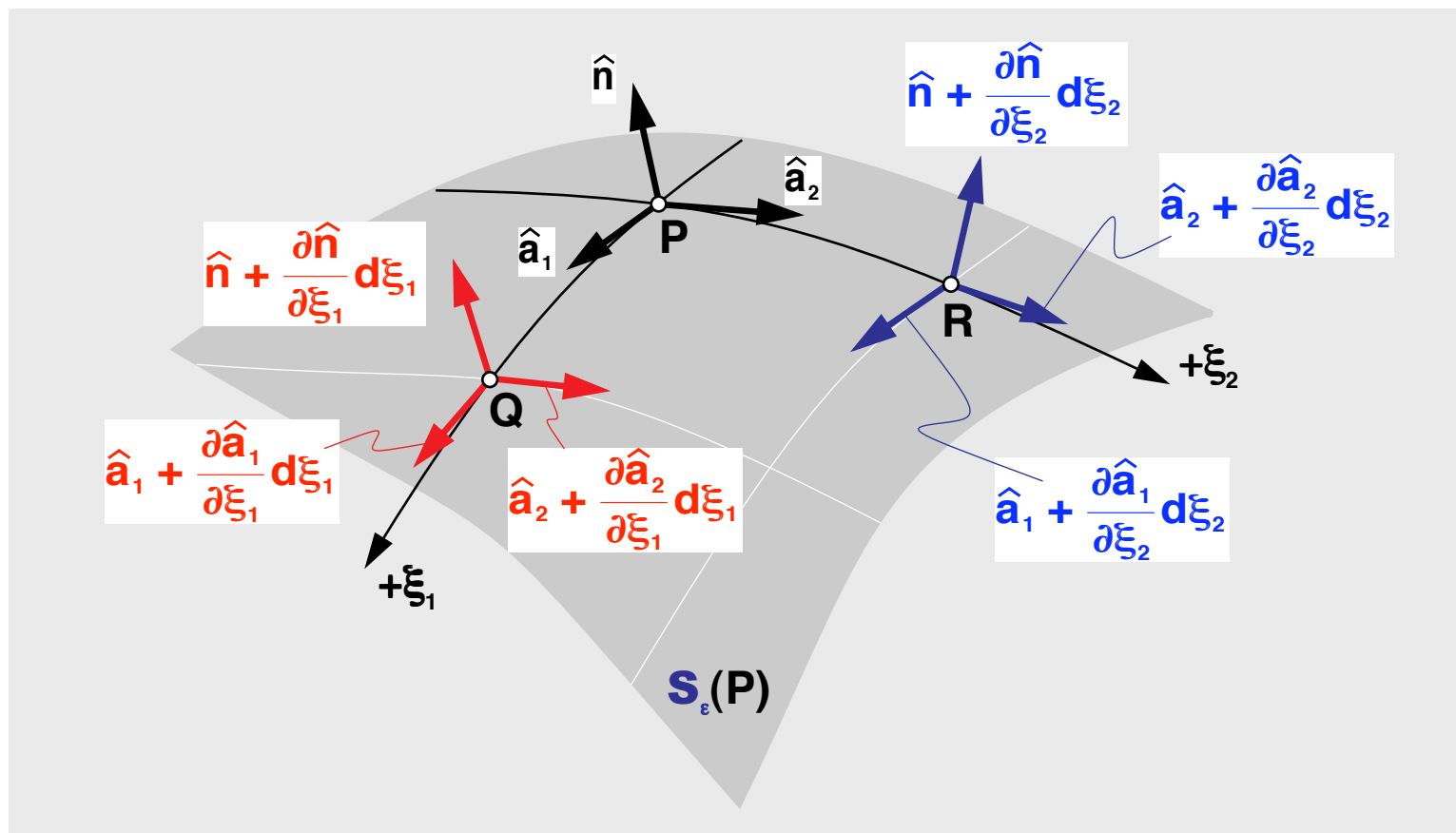
- Inverting these equations gives

$$\hat{\mathbf{a}}_1 = \hat{\mathbf{a}}^1 \csc\theta_{12} + \hat{\mathbf{a}}^2 \cot\theta_{12} \quad \text{and} \quad \hat{\mathbf{a}}_2 = \hat{\mathbf{a}}^1 \cot\theta_{12} + \hat{\mathbf{a}}^2 \csc\theta_{12}$$

- It is important to note that the vectors $\{\hat{\mathbf{a}}^1, \hat{\mathbf{a}}^2, \hat{\mathbf{n}}\}$ are **inherently linearly independent** and, as a result, **also constitute a basis** for vector fields associated with points of the reference surface
- These vector fields are referred to herein as **reciprocal base-vector fields** for convenience although they satisfy $\hat{\mathbf{a}}^\alpha \cdot \hat{\mathbf{a}}_\beta = \delta_\beta^\alpha \sin\theta_{12}$ and not the true reciprocal relation $\hat{\mathbf{a}}^\alpha \cdot \hat{\mathbf{a}}_\beta = \delta_\beta^\alpha$

CURVATURES AND TORSIONS OF $\mathcal{S}_0$

- Consider the infinitesimal changes in the base-vector fields associated with the reference surface, depicted in the figure



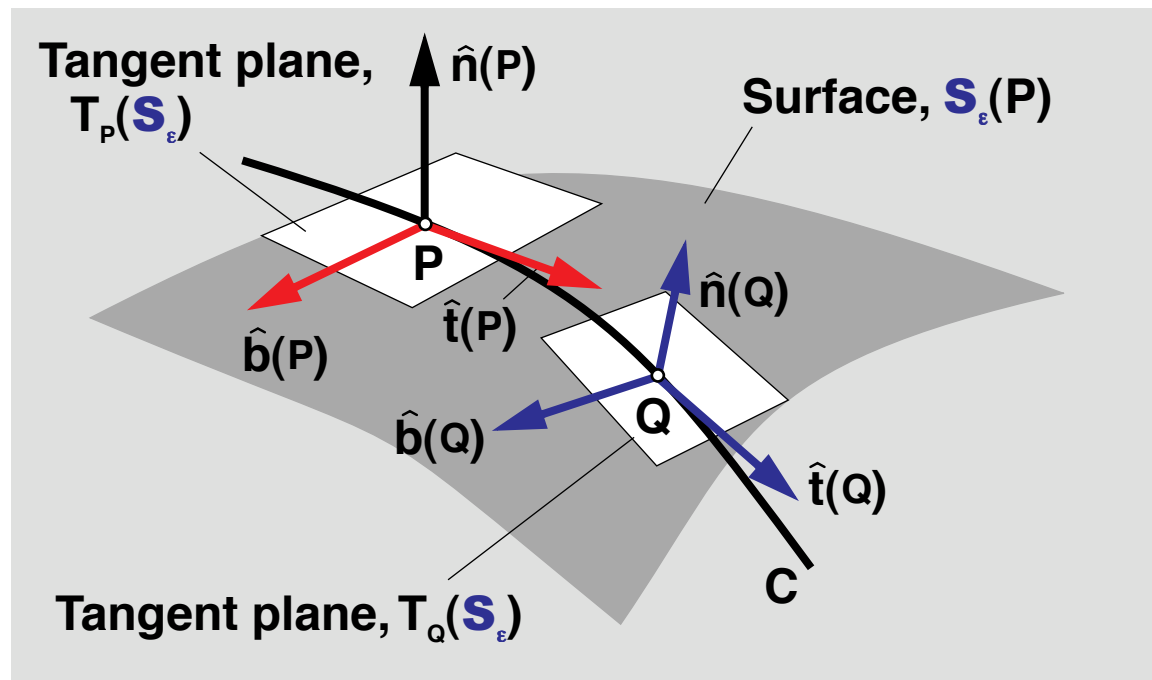
CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- The changes in the orientation of each base vector initially at point P depends generally on how the surface bends and twists, and how the curve being traversed bends within the surface
- The term **“normal curvature”** is used in differential geometry to describe the bending of a surface along a given infinitesimal path of traversal
- Similarly, the term **“surface torsion”** is used to describe the twisting of a surface along a given infinitesimal path of traversal
- The term **“geodesic curvature”** is used to describe how the infinitesimal path of traversal bends within the surface
- These descriptive terms are quantified as follows

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

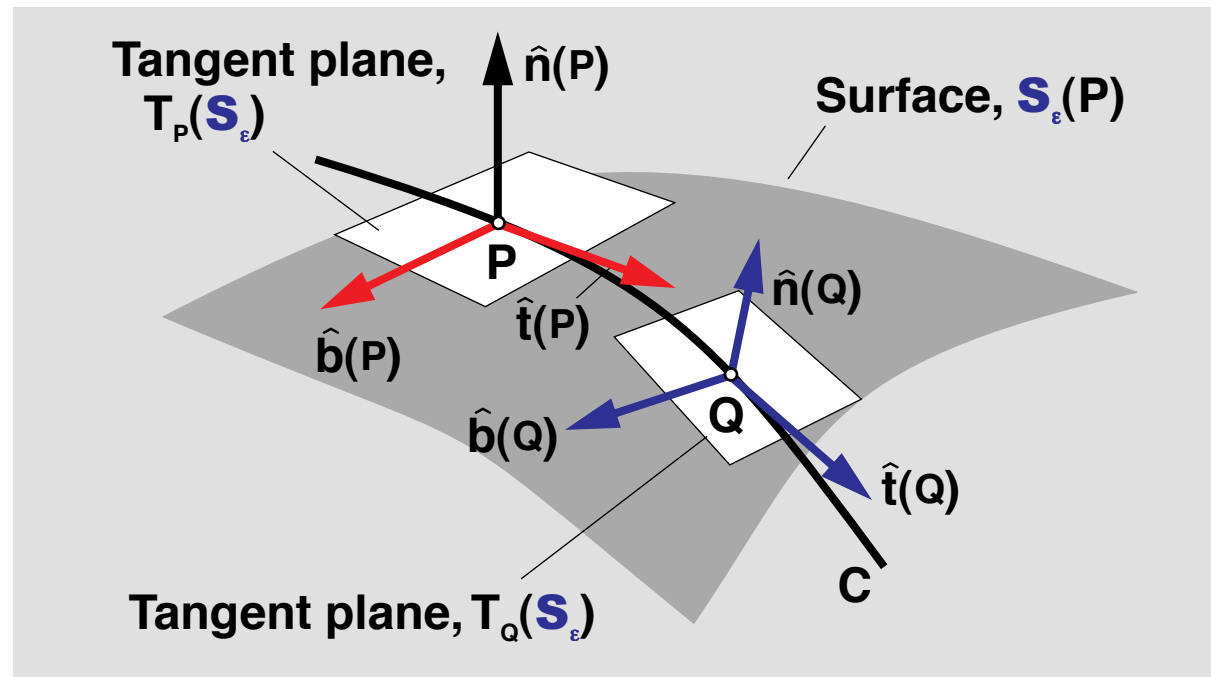
- Let $\hat{\mathbf{t}}(P)$, $\hat{\mathbf{n}}(P)$, and $\hat{\mathbf{b}}(P)$ be the unit-magnitude tangent, normal, and binormal vector fields, respectively, associated with the **surface curve** C and the tangent plane $T_P(\mathbf{S}_\epsilon)$ at point P with coordinates (ξ_1, ξ_2)

- Likewise, let $\hat{\mathbf{t}}(Q)$, $\hat{\mathbf{n}}(Q)$, and $\hat{\mathbf{b}}(Q)$ be the corresponding unit-magnitude tangent, normal, and binormal vector fields, respectively, associated with the tangent plane $T_Q(\mathbf{S}_\epsilon)$ at a point Q in the infinitesimal neighborhood and the **surface curve** C



CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- As the curve C is traversed from point P to point Q , the three vectors at point P come into coincidence with the corresponding ones at point Q , and the tangent plane $T_P(\mathcal{S}_\epsilon)$ becomes coincident with $T_Q(\mathcal{S}_\epsilon)$

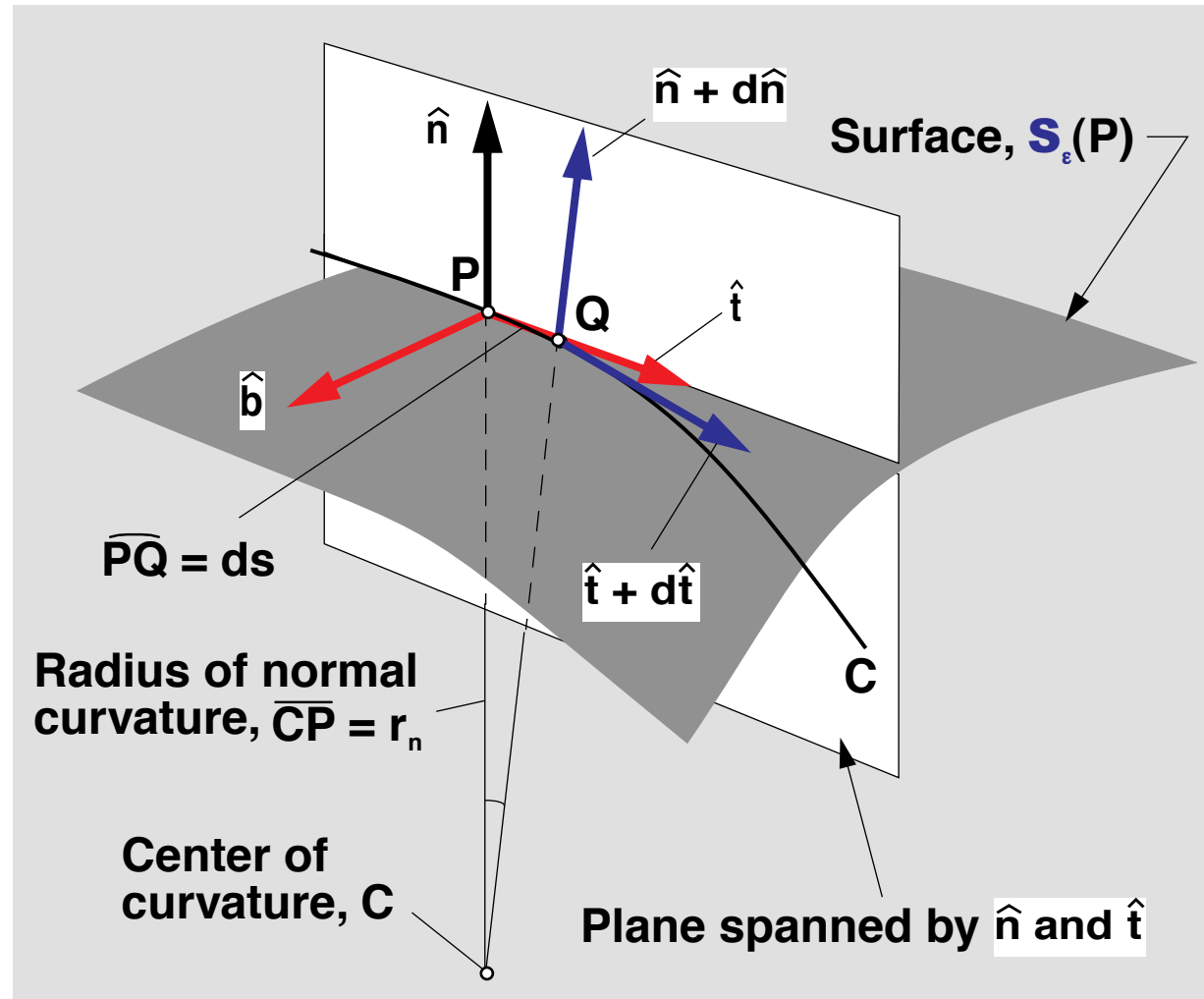


CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- In general, the geometric figure formed by the three orthogonal vectors $\hat{\mathbf{t}}(\mathbf{P})$, $\hat{\mathbf{n}}(\mathbf{P})$, and $\hat{\mathbf{b}}(\mathbf{P})$ associated with the tangent plane $T_p(\mathbf{S}_\varepsilon)$ undergoes **pitch, roll, and yaw** as the surface curve is traversed by an infinitesimal amount
 - **Pitch** is associated with how the surface curves or bends in the direction of traversal from point P to point Q
 - **Roll** is associated with how the surface twists from point P to point Q
 - **Yaw** is associated with how the differential arc $\widehat{PQ}$ bends relative to the tangent plane $T_p(\mathbf{S}_\varepsilon)$ as the curve is traversed from P to Q
- The definitions of the quantities used in **differential geometry** to characterize the geometric properties are described as follows

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- The term “**normal curvature**” is used in differential geometry to describe how $\{\hat{t}, \hat{n}, \hat{b}\}$ **pitch** forward as a surface curve C is traversed by amount ds



CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- Note that in the previous figure, the vector $\hat{n}$ is directed away from the center of curvature
- The normal curvature is defined herein by $\frac{1}{r_n} \equiv \hat{t} \cdot \frac{d\hat{n}}{ds}$, where $r_n(\xi_1, \xi_2)$ is called the radius of normal curvature
- Other definitions of the normal curvature appear in the literature in which $\hat{n}$ is directed toward the center of curvature shown in the previous figure
- The particular definition given above for the normal curvature is used herein so that a sphere with $\hat{n}$ pointing outward, away from the center of the sphere, has a positive value for the normal curvature

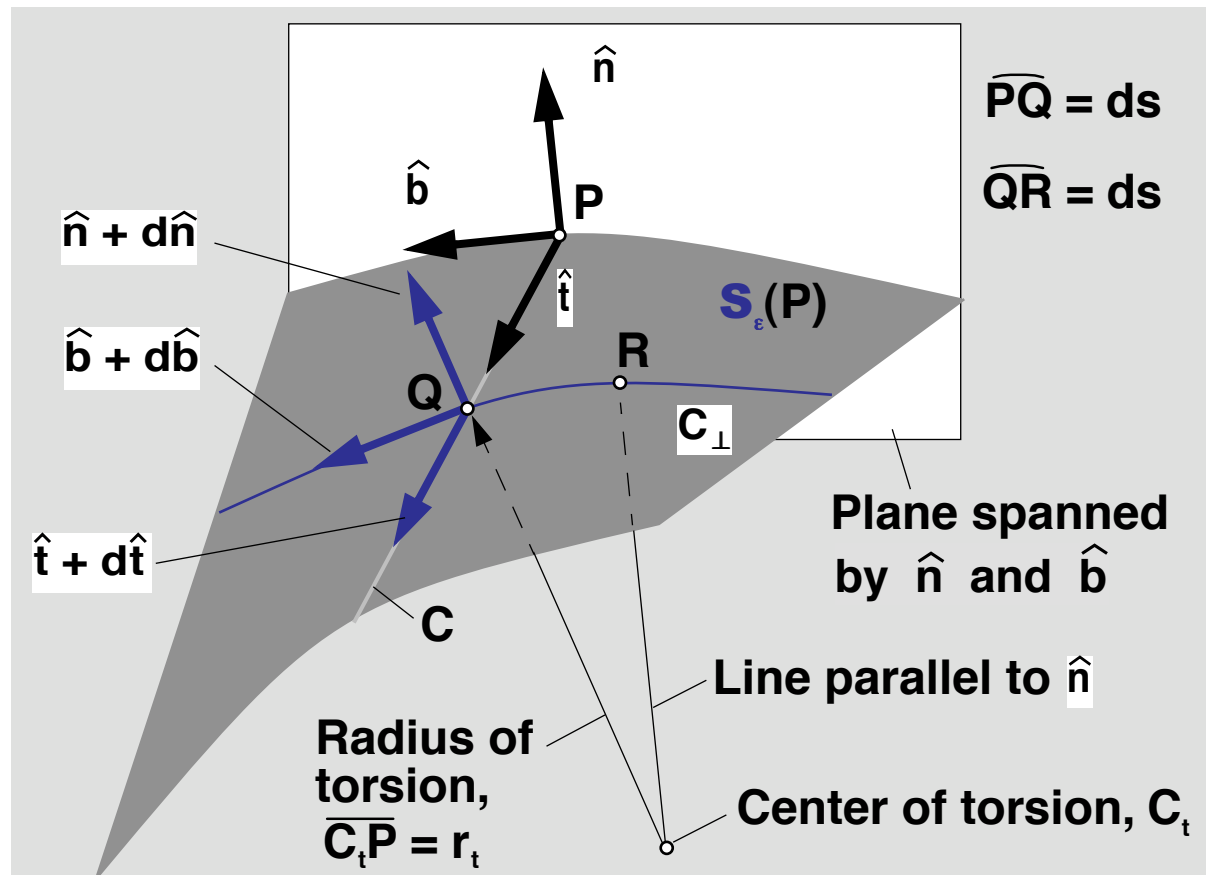
CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- “Surface torsion” is used to characterize how the set $\{\hat{t}, \hat{n}, \hat{b}\}$ rolls as a surface curve C is traversed by amount ds

- In this figure, $C_\perp$ is the surface curve that is tangent to $\hat{b} + d\hat{b}$ at point Q

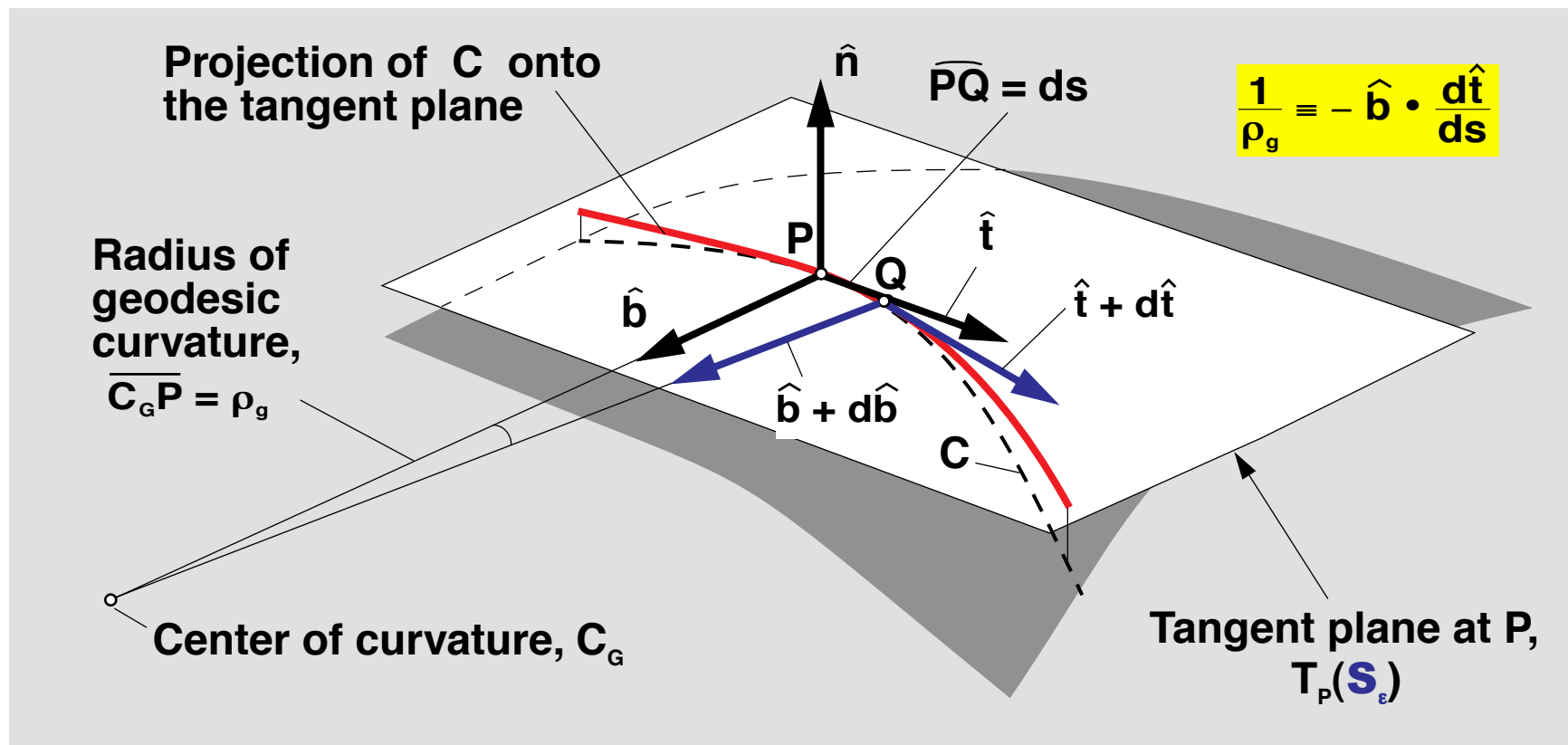
- The radius of torsion $r_t(\xi_1, \xi_2)$ is defined by

$$\frac{1}{r_t} \equiv \hat{b} \cdot \frac{d\hat{n}}{ds}$$



CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- The term “**geodesic curvature**” is used to characterize how the set $\{\hat{t}, \hat{n}, \hat{b}\}$ **yaws** as a surface curve C is traversed by amount ds



CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- The radius of geodesic curvature, $\rho_g(\xi_1, \xi_2)$, is defined by

$$\frac{1}{\rho_g} \equiv -\hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{t}}}{ds}$$

- In general, these definitions apply to any arbitrary, smooth surface curve, and when applied to the ξ_1 - coordinate curve,

$$\{\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}\} \rightarrow \{\hat{\mathbf{a}}_1, \hat{\mathbf{n}}, -\hat{\mathbf{a}}^2\}$$

$$\frac{1}{r_n} \equiv \hat{\mathbf{t}} \cdot \frac{d\hat{\mathbf{n}}}{ds} \rightarrow \frac{1}{r_{11}} = \hat{\mathbf{a}}_1 \cdot \frac{1}{A_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1}$$

$$\frac{1}{r_t} \equiv \hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{n}}}{ds} \rightarrow \frac{1}{r_{12}} = -\hat{\mathbf{a}}^2 \cdot \frac{1}{A_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1}$$

$$\frac{1}{\rho_g} \equiv -\hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{t}}}{ds} \rightarrow \frac{1}{\rho_{11}} = \hat{\mathbf{a}}^2 \cdot \frac{1}{A_1} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1}$$

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- Likewise, for the ξ_2 - coordinate curve,

$$\{\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}\} \quad \rightarrow \quad \{\hat{\mathbf{a}}_2, \hat{\mathbf{n}}, \hat{\mathbf{a}}^1\}$$

$$\frac{1}{r_n} \equiv \hat{\mathbf{t}} \cdot \frac{d\hat{\mathbf{n}}}{ds} \quad \rightarrow \quad \frac{1}{r_{22}} = \hat{\mathbf{a}}_2 \cdot \frac{1}{A_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2}$$

$$\frac{1}{r_t} \equiv \hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{n}}}{ds} \quad \rightarrow \quad \frac{1}{r_{21}} = \hat{\mathbf{a}}^1 \cdot \frac{1}{A_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2}$$

$$\frac{1}{\rho_g} \equiv -\hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{t}}}{ds} \quad \rightarrow \quad \frac{1}{\rho_{22}} = -\hat{\mathbf{a}}^1 \cdot \frac{1}{A_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2}$$

- Thus, the symbols r_{11} and r_{22} denote the *radii of normal curvature* along the ξ_1 - and ξ_2 - coordinate curves, respectively
- The symbols r_{12} and r_{21} denote the *radii of surface torsion*, or twist along the ξ_1 - and ξ_2 - coordinate curves, respectively

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- And, the symbols ρ_{11} and ρ_{22} denote the *radii of geodesic curvature* along the ξ_1 - and ξ_2 - coordinate curves, respectively
- Alternate forms for the surface curvatures are obtained by using

$$\frac{1}{A_1} \frac{\partial}{\partial \xi_1} (\hat{\mathbf{a}}_1 \cdot \hat{\mathbf{n}}) = 0 \quad \text{and} \quad \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\hat{\mathbf{a}}_2 \cdot \hat{\mathbf{n}}) = 0 \quad \text{to get}$$

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{A_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} = - \frac{1}{A_1} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1} \cdot \hat{\mathbf{n}} \quad \text{and} \quad \hat{\mathbf{a}}_2 \cdot \frac{1}{A_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} = - \frac{1}{A_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} \cdot \hat{\mathbf{n}}$$

- These results yield

$$\frac{1}{r_{11}} = \hat{\mathbf{a}}_1 \cdot \frac{1}{A_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \quad \rightarrow \quad \frac{1}{r_{11}} = - \frac{1}{A_1} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1} \cdot \hat{\mathbf{n}}$$

$$\frac{1}{r_{22}} = \hat{\mathbf{a}}_2 \cdot \frac{1}{A_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \quad \rightarrow \quad \frac{1}{r_{22}} = - \frac{1}{A_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} \cdot \hat{\mathbf{n}}$$

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- A very useful relationship between the Gaussian-coordinate curvatures and torsions of the reference surface is obtained by using

$$\hat{a}^1 = \hat{a}_1 \csc\theta_{12} - \hat{a}_2 \cot\theta_{12} \quad \text{and} \quad \hat{a}^2 = \hat{a}_2 \csc\theta_{12} - \hat{a}_1 \cot\theta_{12} \quad \text{with}$$

$$\frac{1}{r_{12}} = -\hat{a}^2 \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \quad \text{and} \quad \frac{1}{r_{21}} = \hat{a}^1 \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} \quad \text{to get}$$

$$\frac{1}{r_{12}} = \left(\hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \right) \cot\theta_{12} - \left(\hat{a}_2 \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \right) \csc\theta_{12} \quad \text{and}$$

$$\frac{1}{r_{21}} = \left(\hat{a}_1 \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} \right) \csc\theta_{12} - \left(\hat{a}_2 \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} \right) \cot\theta_{12}$$

- Next, $\frac{1}{r_{11}} = \hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1}$ and $\frac{1}{r_{22}} = \hat{a}_2 \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2}$ are used to get

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

$$\frac{1}{r_{12}} = \frac{\cot\theta_{12}}{r_{11}} - \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \right) \csc\theta_{12} \quad \text{and} \quad \frac{1}{r_{21}} = \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \right) \csc\theta_{12} - \frac{\cot\theta_{12}}{r_{22}}$$

- From $\hat{\mathbf{a}}_\alpha \cdot \hat{\mathbf{n}} = 0$, it follows that
$$\frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \hat{\mathbf{a}}_\alpha}{\partial \xi_\beta} \cdot \hat{\mathbf{n}} = - \frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_\beta} \cdot \hat{\mathbf{a}}_\alpha$$

- As a result,

$$\frac{1}{r_{12}} = \frac{\cot\theta_{12}}{r_{11}} + \left(\frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} \cdot \hat{\mathbf{n}} \right) \csc\theta_{12} \quad \text{and} \quad \frac{1}{r_{21}} = - \left(\frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} \cdot \hat{\mathbf{n}} \right) \csc\theta_{12} - \frac{\cot\theta_{12}}{r_{22}}$$

- Then, noting that for a smooth surface
$$\frac{\partial}{\partial \xi_2} \left(\frac{\partial \vec{\mathbf{x}}}{\partial \xi_1} \right) = \frac{\partial}{\partial \xi_1} \left(\frac{\partial \vec{\mathbf{x}}}{\partial \xi_2} \right),$$
 it follows

that $\frac{\partial \vec{\mathbf{a}}_1}{\partial \xi_2} = \frac{\partial \vec{\mathbf{a}}_2}{\partial \xi_1}$, or equivalently,
$$\frac{\partial}{\partial \xi_2} (\mathbf{A}_1 \hat{\mathbf{a}}_1) = \frac{\partial}{\partial \xi_1} (\mathbf{A}_2 \hat{\mathbf{a}}_2)$$

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- Taking the dot product of both sides of $\frac{\partial}{\partial \xi_2}(\mathbf{A}_1 \hat{\mathbf{a}}_1) = \frac{\partial}{\partial \xi_1}(\mathbf{A}_2 \hat{\mathbf{a}}_2)$ with $\hat{\mathbf{n}}$

and simplifying gives $\frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} \cdot \hat{\mathbf{n}} = \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} \cdot \hat{\mathbf{n}}$

- Using this result with the previous expression for the radii of surface twist gives the relationship

$$\frac{1}{r_{12}} + \frac{1}{r_{21}} = \cot \theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$$

- Explicit expressions for the radii of geodesic curvature are obtained by first noting that differentiating $\hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}_1 = 1$ and $\hat{\mathbf{a}}_2 \cdot \hat{\mathbf{a}}_2 = 1$ gives the

results $\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1} = 0$ $\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} = 0$ $\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} = 0$ $\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} = 0$

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- In indicial form, these results are expressed as $\hat{\mathbf{a}}_{(\alpha)} \cdot \frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \hat{\mathbf{a}}_\alpha}{\partial \xi_\beta} = 0$
- Substituting $\hat{\mathbf{a}}^1 = \hat{\mathbf{a}}_1 \csc\theta_{12} - \hat{\mathbf{a}}_2 \cot\theta_{12}$ and $\hat{\mathbf{a}}^2 = \hat{\mathbf{a}}_2 \csc\theta_{12} - \hat{\mathbf{a}}_1 \cot\theta_{12}$ into the previous expressions for the radii of geodesic curvature and using

$$\hat{\mathbf{a}}_{(\alpha)} \cdot \frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \hat{\mathbf{a}}_\alpha}{\partial \xi_\beta} = 0 \quad \text{gives} \quad \frac{\sin\theta_{12}}{\rho_{11}} = \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1} \quad \text{and} \quad \frac{\sin\theta_{12}}{\rho_{22}} = -\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2}$$

- Next, taking the dot product of each side of $\frac{\partial}{\partial \xi_2}(\mathbf{A}_1 \hat{\mathbf{a}}_1) = \frac{\partial}{\partial \xi_1}(\mathbf{A}_2 \hat{\mathbf{a}}_2)$ with

$$\hat{\mathbf{a}}_1 \quad \text{and using} \quad \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} \cdot \hat{\mathbf{a}}_1 = 0 \quad \text{gives}$$

$$\frac{1}{\mathbf{A}_1 \mathbf{A}_2} \left(\frac{\partial \mathbf{A}_1}{\partial \xi_2} - \frac{\partial \mathbf{A}_2}{\partial \xi_1} \cos\theta_{12} \right) = \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} \cdot \hat{\mathbf{a}}_1$$

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- Likewise, taking the dot product of each side of $\frac{\partial}{\partial \xi_2}(\mathbf{A}_1 \hat{\mathbf{a}}_1) = \frac{\partial}{\partial \xi_1}(\mathbf{A}_2 \hat{\mathbf{a}}_2)$ with $\hat{\mathbf{a}}_2$ and using $\frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} \cdot \hat{\mathbf{a}}_2 = 0$ gives

$$\frac{1}{\mathbf{A}_1 \mathbf{A}_2} \left(\frac{\partial \mathbf{A}_1}{\partial \xi_2} \cos \theta_{12} - \frac{\partial \mathbf{A}_2}{\partial \xi_1} \right) = - \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} \cdot \hat{\mathbf{a}}_2$$

- From $\frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1}(\hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}_2) = \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1}(\cos \theta_{12})$ and $\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1} = \frac{\sin \theta_{12}}{\rho_{11}}$ it follows that

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} = \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1}(\cos \theta_{12}) - \frac{\sin \theta_{12}}{\rho_{11}}$$

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- Similarly, $\frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}_2) = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\cos \theta_{12})$ and $\hat{\mathbf{a}}_1 \cdot \frac{1}{A_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} = - \frac{\sin \theta_{12}}{\rho_{22}}$
yield

$$\frac{1}{A_2} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} \cdot \hat{\mathbf{a}}_2 = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\cos \theta_{12}) + \frac{\sin \theta_{12}}{\rho_{22}}$$

- Thus, $\frac{1}{A_1 A_2} \left(\frac{\partial A_1}{\partial \xi_2} - \frac{\partial A_2}{\partial \xi_1} \cos \theta_{12} \right) = \frac{1}{A_1} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} \cdot \hat{\mathbf{a}}_1$ becomes

$$\frac{1}{\rho_{11}} = \frac{\csc \theta_{12}}{A_1 A_2} \left(\frac{\partial A_2}{\partial \xi_1} \cos \theta_{12} - \frac{\partial A_1}{\partial \xi_2} \right) + \frac{\csc \theta_{12}}{A_1} \frac{\partial}{\partial \xi_1} (\cos \theta_{12})$$

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- Likewise, $\frac{1}{A_1 A_2} \left(\frac{\partial A_1}{\partial \xi_2} \cos \theta_{12} - \frac{\partial A_2}{\partial \xi_1} \right) = - \frac{1}{A_2} \frac{\partial \hat{a}_1}{\partial \xi_2} \cdot \hat{a}_2$ becomes

$$\frac{1}{\rho_{22}} = \frac{\csc \theta_{12}}{A_1 A_2} \left(\frac{\partial A_2}{\partial \xi_1} - \frac{\partial A_1}{\partial \xi_2} \cos \theta_{12} \right) - \frac{\csc \theta_{12}}{A_2} \frac{\partial}{\partial \xi_2} (\cos \theta_{12})$$

- These two expressions for the geodesic curvatures can be solved to obtain the following useful formulas

$$\frac{\partial A_1}{\partial \xi_2} = A_1 A_2 \left[\frac{\cot \theta_{12}}{\rho_{22}} - \frac{\csc \theta_{12}}{\rho_{11}} - \frac{\csc \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\cot \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right]$$

$$\frac{\partial A_2}{\partial \xi_1} = A_1 A_2 \left[\frac{\csc \theta_{12}}{\rho_{22}} - \frac{\cot \theta_{12}}{\rho_{11}} - \frac{\cot \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\csc \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right]$$

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- Compact expressions for the radii of geodesic curvature are given by

$$\frac{1}{\rho_{11}} = \frac{\csc\theta_{12}}{A_1 A_2} \left(\frac{\partial}{\partial \xi_1} [A_2 \cos\theta_{12}] - \frac{\partial A_1}{\partial \xi_2} \right) \quad \text{and}$$

$$\frac{1}{\rho_{22}} = - \frac{\csc\theta_{12}}{A_1 A_2} \left(\frac{\partial}{\partial \xi_2} [A_1 \cos\theta_{12}] - \frac{\partial A_2}{\partial \xi_1} \right)$$

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- For the special case of **orthogonal Gaussian reference-surface coordinates**,

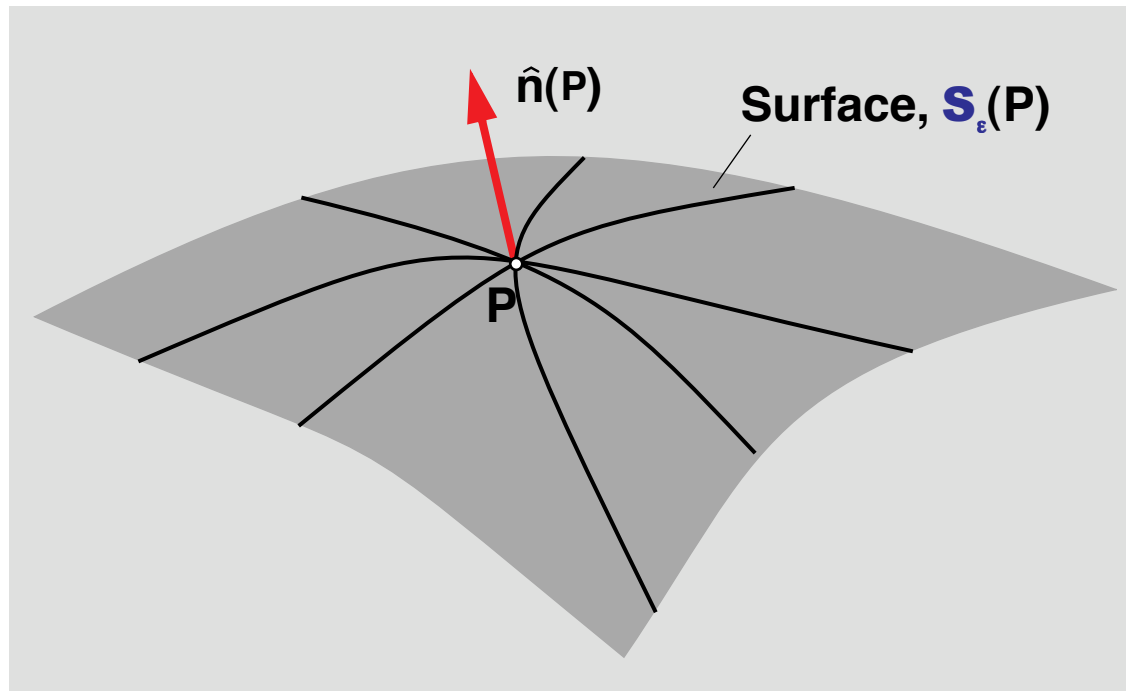
$$\frac{1}{r_{12}} + \frac{1}{r_{21}} = \cot\theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \quad \rightarrow \quad \frac{1}{r_{21}} = - \frac{1}{r_{12}}$$

$$\frac{1}{\rho_{11}} = \frac{\csc\theta_{12}}{A_1 A_2} \left(\frac{\partial}{\partial \xi_1} [A_2 \cos\theta_{12}] - \frac{\partial A_1}{\partial \xi_2} \right) \quad \rightarrow \quad \frac{1}{\rho_{11}} = - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2}$$

$$\frac{1}{\rho_{22}} = - \frac{\csc\theta_{12}}{A_1 A_2} \left(\frac{\partial}{\partial \xi_2} [A_1 \cos\theta_{12}] - \frac{\partial A_2}{\partial \xi_1} \right) \quad \rightarrow \quad \frac{1}{\rho_{22}} = \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \xi_1}$$

CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONTINUED

- Another special case of practical importance consists of orthogonal Gaussian-coordinate curves whose tangent vectors are aligned with the principal directions of surface normal curvature
 - This class of Gaussian coordinates are referred to herein as **principal-curvature coordinates**
- In general, there exists an infinite number of smooth curves that pass through point P of a smooth surface, each of which generally have a different value for the radius of normal curvature at point P



CURVATURES AND TORSIONS OF $\mathcal{S}_0$ - CONCLUDED

- Sufficient conditions for identifying the coordinate curves that correspond to the directions of principal curvature are that they are orthogonal and that

$$\frac{1}{r_{12}} = \frac{1}{r_{21}} = 0$$

- For this very special case, the following notation is used herein

$$r_{11} \rightarrow R_1 \quad \text{and} \quad r_{22} \rightarrow R_2$$

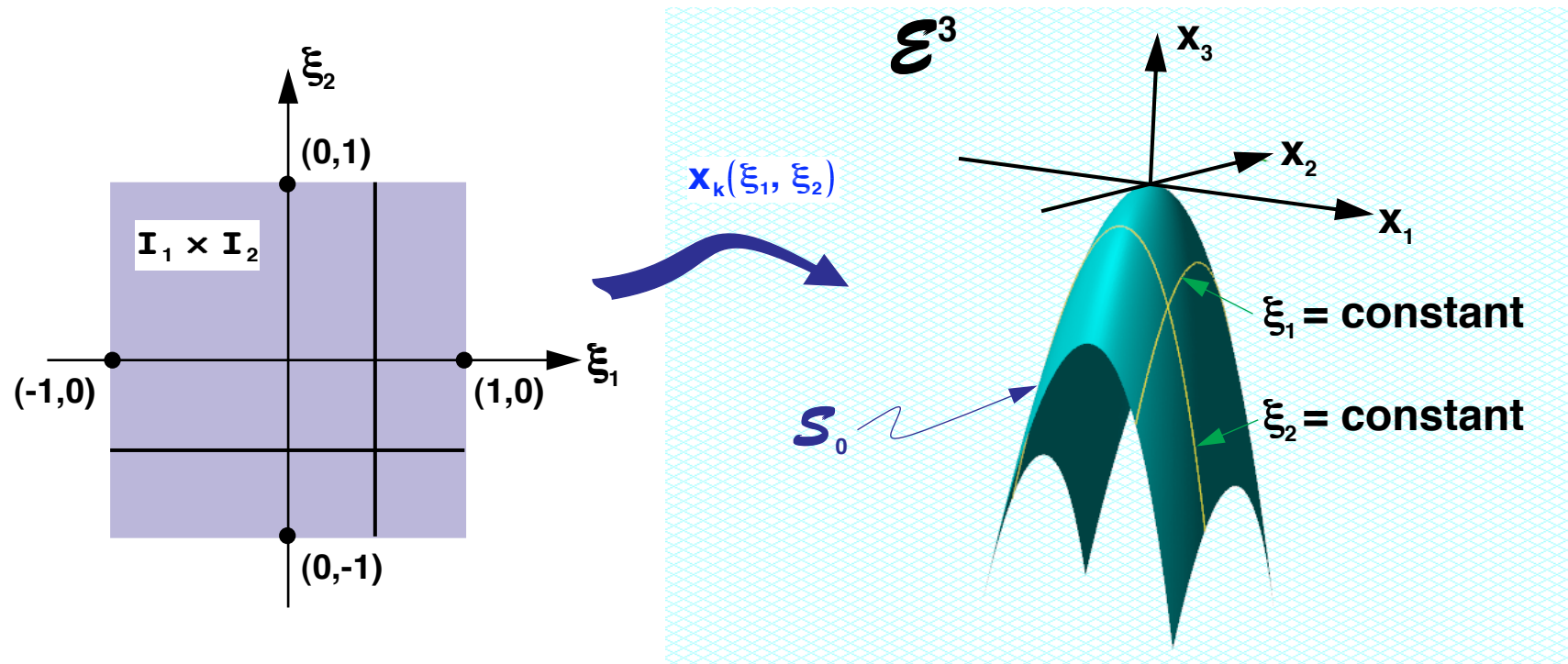
$\mathcal{S}_0$ GEOMETRY EXAMPLE

- A parametric representation of an **elliptic paraboloid** is given by

$$\mathbf{x}_1 = \xi_1, \quad \mathbf{x}_2 = \xi_2, \quad \text{and} \quad \mathbf{x}_3 = -\left[2(\xi_1)^2 + (\xi_2)^2\right]$$

where

$$\xi_1 \in \mathcal{I}_1 = [-1, 1] \quad \text{and} \quad \xi_2 \in \mathcal{I}_2 = [-1, 1]$$



S_0 GEOMETRY EXAMPLE - CONTINUED

- For this parametric representation, $\vec{x} = \xi_1 \hat{i}_1 + \xi_2 \hat{i}_2 - [2(\xi_1)^2 + (\xi_2)^2] \hat{i}_3$

- The natural base-vector fields are obtained as

$$\vec{a}_1 = \frac{\partial \vec{x}}{\partial \xi_1} = \hat{i}_1 - 4\xi_1 \hat{i}_3 \quad \text{and} \quad \vec{a}_2 = \frac{\partial \vec{x}}{\partial \xi_2} = \hat{i}_2 - 2\xi_2 \hat{i}_3$$

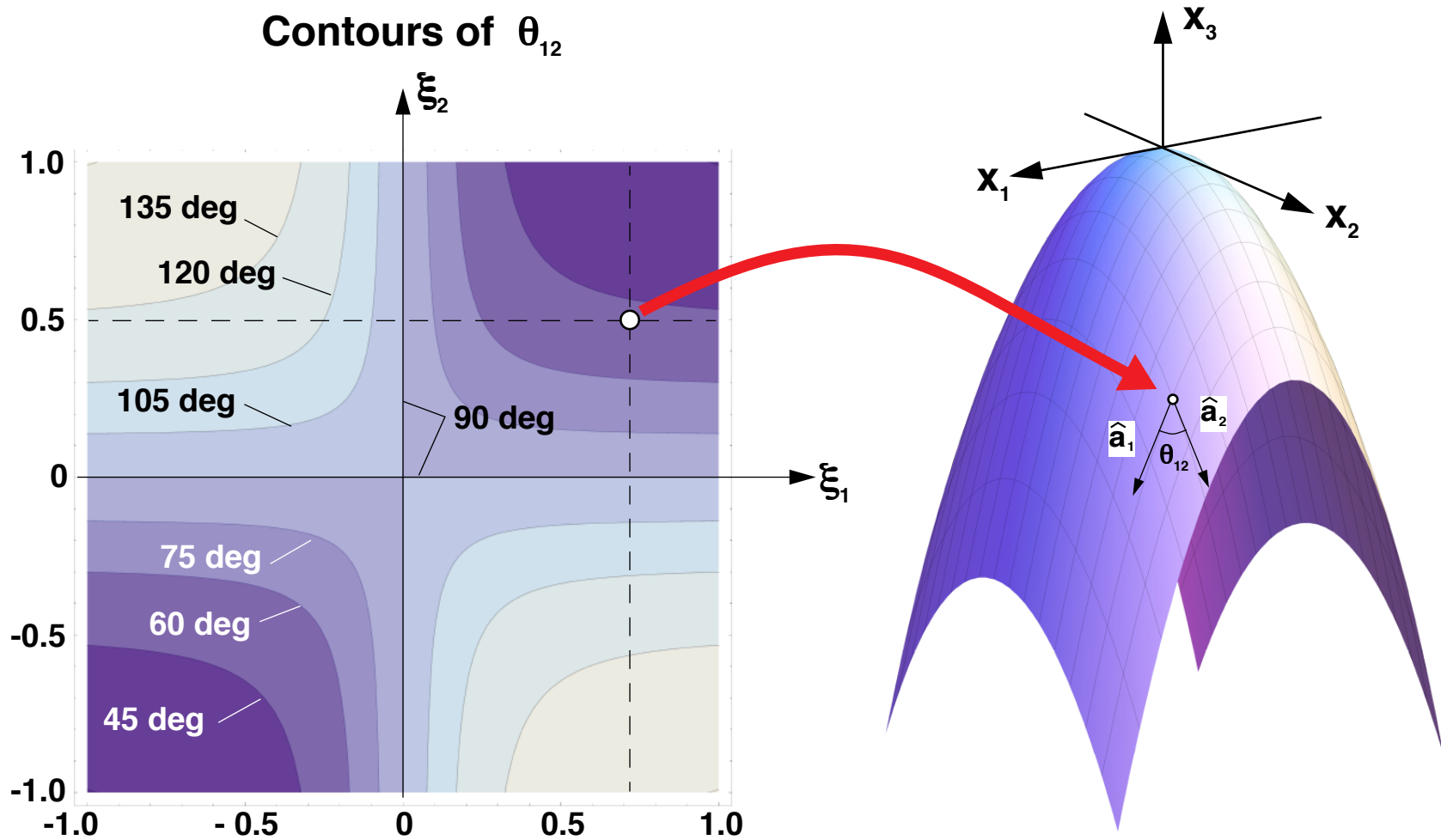
- The corresponding metric coefficients are

$$A_1 = |\vec{a}_1| = \sqrt{1 + (4\xi_1)^2} \quad \text{and} \quad A_2 = |\vec{a}_2| = \sqrt{1 + (2\xi_2)^2}$$

- Thus, $\hat{a}_1 = \frac{\vec{a}_1}{A_1} = \frac{\hat{i}_1 - 4\xi_1 \hat{i}_3}{\sqrt{1 + (4\xi_1)^2}} \quad \text{and} \quad \hat{a}_2 = \frac{\vec{a}_2}{A_2} = \frac{\hat{i}_2 - 2\xi_2 \hat{i}_3}{\sqrt{1 + (2\xi_2)^2}}$

- In addition, $\cos\theta_{12} = \hat{a}_1 \cdot \hat{a}_2 = \frac{8\xi_1\xi_2}{\sqrt{1 + (4\xi_1)^2} \sqrt{1 + (2\xi_2)^2}}$

$\mathcal{S}_0$ GEOMETRY EXAMPLE - CONTINUED



S_0 GEOMETRY EXAMPLE - CONTINUED

- Next, $\vec{a}_1 \times \vec{a}_2 = 4\xi_1 \hat{i}_1 + 2\xi_2 \hat{i}_2 + \hat{i}_3$ and $|\vec{a}_1 \times \vec{a}_2| = \sqrt{1 + (4\xi_1)^2 + (2\xi_2)^2}$

- Thus,
$$\hat{n} = \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|} = \frac{4\xi_1 \hat{i}_1 + 2\xi_2 \hat{i}_2 + \hat{i}_3}{\sqrt{1 + (4\xi_1)^2 + (2\xi_2)^2}}$$

- The reciprocal base-vector fields $\hat{a}^1 \equiv \hat{a}_2 \times \hat{n}$ and $\hat{a}^2 \equiv \hat{n} \times \hat{a}_1$ become

$$\hat{a}^1 = \frac{[1 + (2\xi_2)^2]\hat{i}_1 - 8\xi_1\xi_2 \hat{i}_2 - 4\xi_1 \hat{i}_3}{\sqrt{1 + (2\xi_2)^2} \sqrt{1 + (4\xi_1)^2 + (2\xi_2)^2}} \quad \text{and}$$

$$\hat{a}^2 = \frac{-8\xi_1\xi_2 \hat{i}_1 + [1 + (4\xi_1)^2]\hat{i}_2 - 2\xi_2 \hat{i}_3}{\sqrt{1 + (4\xi_1)^2} \sqrt{1 + (4\xi_1)^2 + (2\xi_2)^2}}$$

- Also,
$$\sin\theta_{12} = \hat{a}^1 \cdot \hat{a}_1 = \frac{\sqrt{1 + (4\xi_1)^2 + (2\xi_2)^2}}{\sqrt{1 + (4\xi_1)^2} \sqrt{1 + (2\xi_2)^2}}$$

$\mathcal{S}_0$ GEOMETRY EXAMPLE - CONTINUED

- Moreover,

$$\cot\theta_{12} = \frac{8\xi_1\xi_2}{\sqrt{1 + (4\xi_1)^2 + (2\xi_2)^2}}$$

- The curvatures and torsions are given by

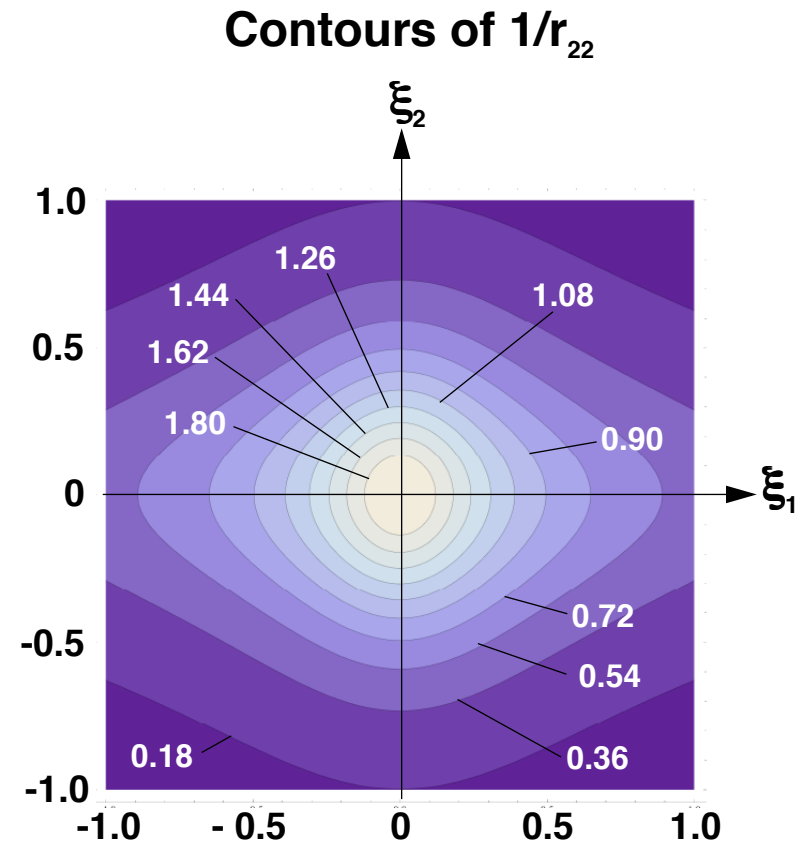
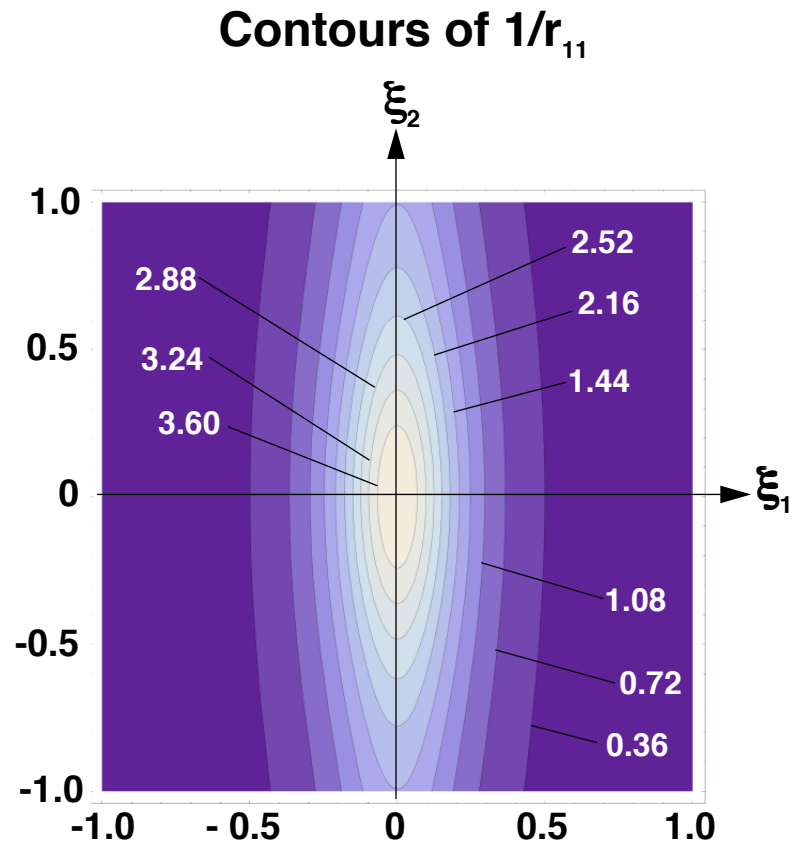
$$\frac{1}{r_{11}} = \hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} = \frac{4}{[1 + (4\xi_1)^2] \sqrt{1 + (4\xi_1)^2 + (2\xi_2)^2}}$$

$$\frac{1}{r_{12}} = -\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} = \frac{32\xi_1\xi_2}{[1 + (4\xi_1)^2][1 + (4\xi_1)^2 + (2\xi_2)^2]}$$

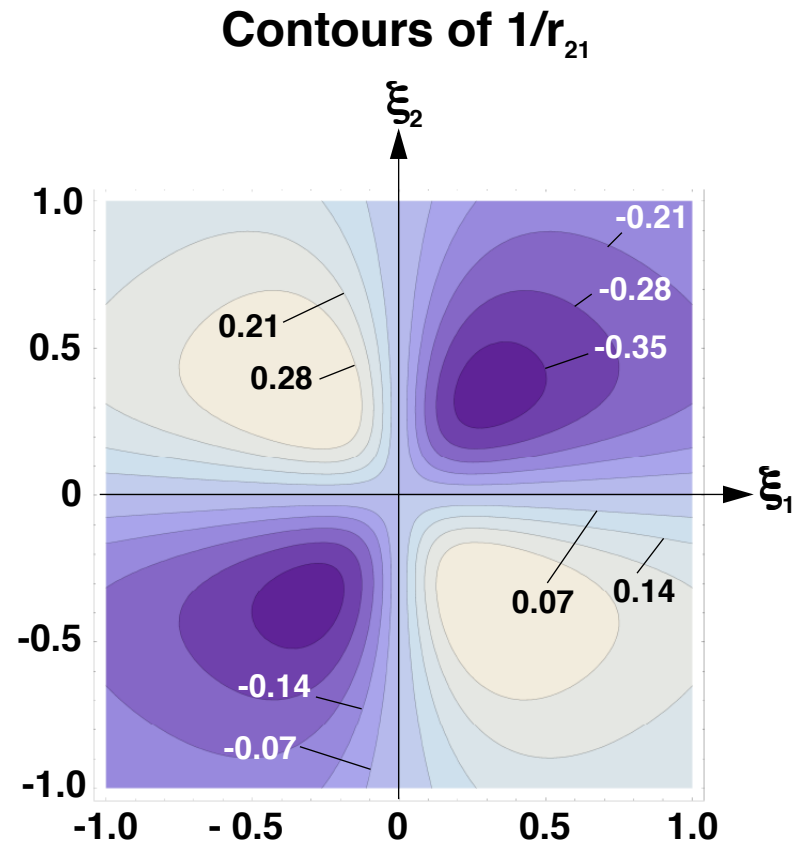
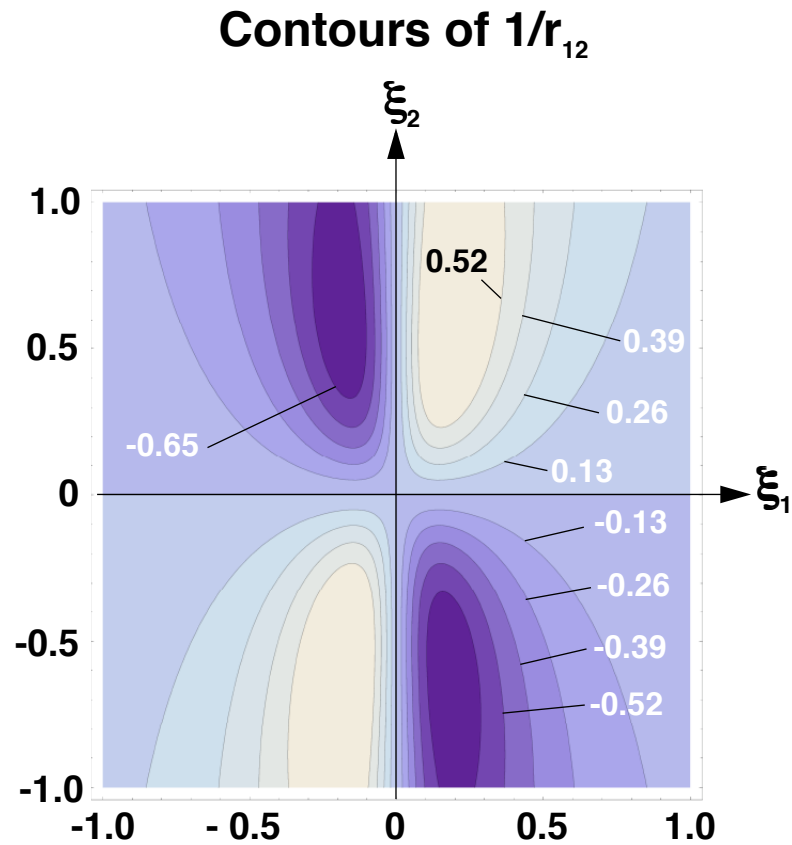
$$\frac{1}{r_{21}} = \hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} = \frac{-16\xi_1\xi_2}{[1 + (2\xi_2)^2][1 + (4\xi_1)^2 + (2\xi_2)^2]}$$

$$\frac{1}{r_{22}} = \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} = \frac{2}{[1 + (2\xi_2)^2] \sqrt{1 + (4\xi_1)^2 + (2\xi_2)^2}}$$

$\mathcal{S}_0$ GEOMETRY EXAMPLE - CONTINUED



$\mathcal{S}_0$ GEOMETRY EXAMPLE - CONTINUED



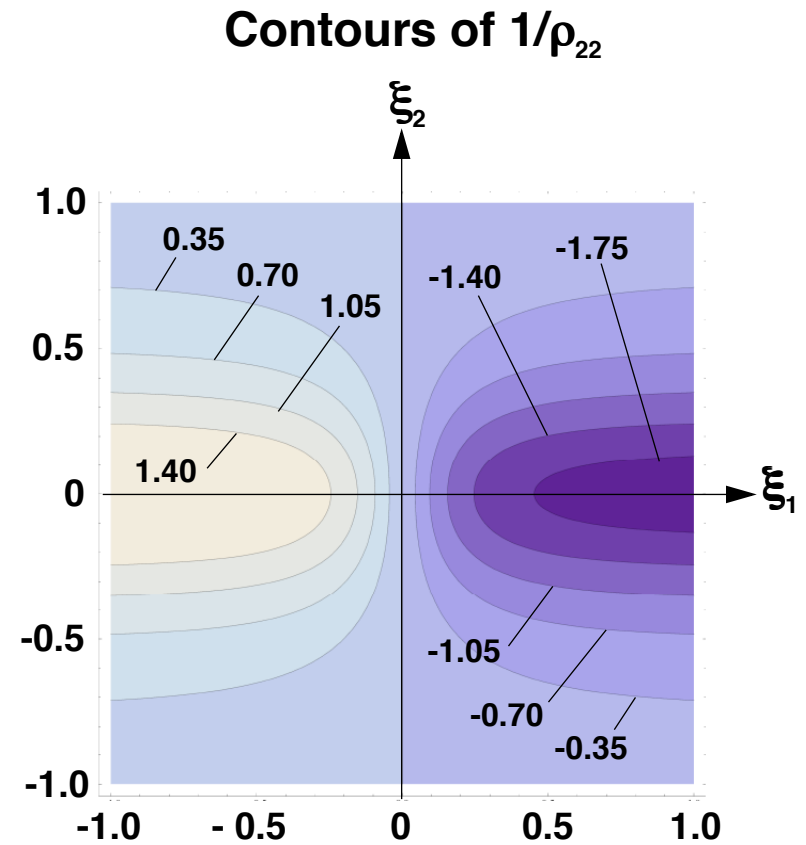
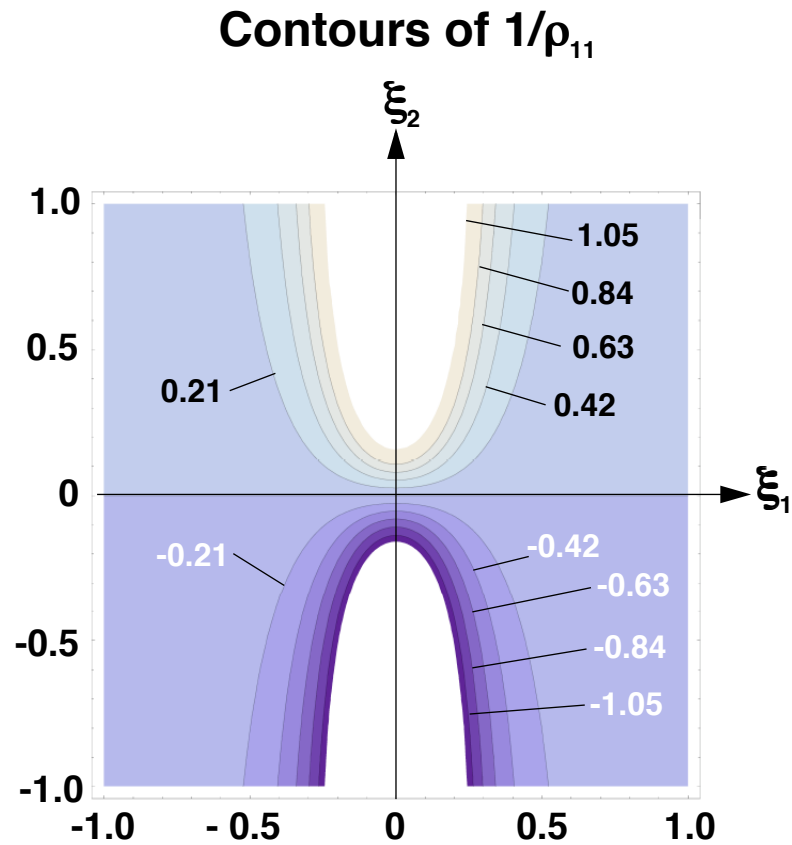
$\mathcal{S}_0$ GEOMETRY EXAMPLE - CONTINUED

- Likewise, the geodesic curvatures are given by

$$\frac{1}{\rho_{11}} = \hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1} = \frac{8\xi_2}{\left[1 + (4\xi_1)^2\right]^{\frac{3}{2}} \sqrt{1 + (4\xi_1)^2 + (2\xi_2)^2}}$$

$$\frac{1}{\rho_{22}} = -\hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} = \frac{-8\xi_1}{\left[1 + (2\xi_2)^2\right]^{\frac{3}{2}} \sqrt{1 + (4\xi_1)^2 + (2\xi_2)^2}}$$

$\mathcal{S}_0$ GEOMETRY EXAMPLE - CONCLUDED



DERIVATIVES OF BASE-VECTOR FIELDS ON $\mathcal{S}_0$

- The set of vector fields $\{\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{n}}\}$ are functions of the Gaussian reference-surface coordinates that form a **pointwise basis** for representing vector fields associated with the points of the shell reference surface
- Thus, to determine the derivatives of a reference-surface vector field, the derivatives of $\{\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{n}}\}$ are needed
- Because $\{\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{n}}\}$ form a basis at a given point of the reference surface, the partial derivatives of $\{\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{n}}\}$ evaluated at that point can be expressed as a linear combination of $\{\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{n}}\}$; that is,

$$\frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \hat{\mathbf{a}}_\alpha}{\partial \xi_\beta} = \mathbf{C}_\beta^{\alpha\gamma} \hat{\mathbf{a}}_\gamma + \mathbf{C}_\beta^{\alpha 3} \hat{\mathbf{n}} \quad \text{and} \quad \frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_\beta} = \mathbf{C}_\beta^{3\gamma} \hat{\mathbf{a}}_\gamma + \mathbf{C}_\beta^{33} \hat{\mathbf{n}}$$

DERIVATIVES OF BASE-VECTOR FIELDS ON $\mathcal{S}_0$

CONTINUED

- In expanded form, the linear combinations are given by

$$\frac{1}{A_1} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1} = C_1^{11} \hat{\mathbf{a}}_1 + C_1^{12} \hat{\mathbf{a}}_2 + C_1^{13} \hat{\mathbf{n}}$$

$$\frac{1}{A_2} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} = C_2^{11} \hat{\mathbf{a}}_1 + C_2^{12} \hat{\mathbf{a}}_2 + C_2^{13} \hat{\mathbf{n}}$$

$$\frac{1}{A_1} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} = C_1^{21} \hat{\mathbf{a}}_1 + C_1^{22} \hat{\mathbf{a}}_2 + C_1^{23} \hat{\mathbf{n}}$$

$$\frac{1}{A_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} = C_2^{21} \hat{\mathbf{a}}_1 + C_2^{22} \hat{\mathbf{a}}_2 + C_2^{23} \hat{\mathbf{n}}$$

$$\frac{1}{A_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} = C_1^{31} \hat{\mathbf{a}}_1 + C_1^{32} \hat{\mathbf{a}}_2 + C_1^{33} \hat{\mathbf{n}}$$

$$\frac{1}{A_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} = C_2^{31} \hat{\mathbf{a}}_1 + C_2^{32} \hat{\mathbf{a}}_2 + C_2^{33} \hat{\mathbf{n}}$$

- The functions C_i^{jk} are found by using $\hat{\mathbf{a}}^\alpha \cdot \hat{\mathbf{a}}_\beta = \delta_\beta^\alpha \sin \theta_{12}$ and $\hat{\mathbf{a}}^\alpha \cdot \hat{\mathbf{n}} = 0$

- In particular,

$$C_1^{11} = \left(\frac{1}{A_1} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1} \cdot \hat{\mathbf{a}}^1 \right) \csc \theta_{12}$$

$$C_1^{12} = \left(\frac{1}{A_1} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1} \cdot \hat{\mathbf{a}}^2 \right) \csc \theta_{12}$$

$$C_1^{13} = \frac{1}{A_1} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1} \cdot \hat{\mathbf{n}}$$

DERIVATIVES OF BASE-VECTOR FIELDS ON $\mathcal{S}_0$

CONTINUED

- Likewise,

$$\mathbf{C}_2^{11} = \left(\frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} \cdot \hat{\mathbf{a}}^1 \right) \csc \theta_{12}$$

$$\mathbf{C}_2^{12} = \left(\frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} \cdot \hat{\mathbf{a}}^2 \right) \csc \theta_{12}$$

$$\mathbf{C}_2^{13} = \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} \cdot \hat{\mathbf{n}}$$

$$\mathbf{C}_1^{21} = \left(\frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} \cdot \hat{\mathbf{a}}^1 \right) \csc \theta_{12}$$

$$\mathbf{C}_1^{22} = \left(\frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} \cdot \hat{\mathbf{a}}^2 \right) \csc \theta_{12}$$

$$\mathbf{C}_1^{23} = \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} \cdot \hat{\mathbf{n}}$$

$$\mathbf{C}_2^{21} = \left(\frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} \cdot \hat{\mathbf{a}}^1 \right) \csc \theta_{12}$$

$$\mathbf{C}_2^{22} = \left(\frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} \cdot \hat{\mathbf{a}}^2 \right) \csc \theta_{12}$$

$$\mathbf{C}_2^{23} = \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} \cdot \hat{\mathbf{n}}$$

$$\mathbf{C}_1^{31} = \left(\frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \cdot \hat{\mathbf{a}}^1 \right) \csc \theta_{12}$$

$$\mathbf{C}_1^{32} = \left(\frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \cdot \hat{\mathbf{a}}^2 \right) \csc \theta_{12}$$

$$\mathbf{C}_1^{33} = \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \cdot \hat{\mathbf{n}}$$

$$\mathbf{C}_2^{31} = \left(\frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \cdot \hat{\mathbf{a}}^1 \right) \csc \theta_{12}$$

$$\mathbf{C}_2^{32} = \left(\frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \cdot \hat{\mathbf{a}}^2 \right) \csc \theta_{12}$$

$$\mathbf{C}_2^{33} = \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \cdot \hat{\mathbf{n}}$$

DERIVATIVES OF BASE-VECTOR FIELDS ON $\mathcal{S}_0$

CONTINUED

- Explicit expressions for the functions C_i^{jk} are found by using the following previously derived results

$$\hat{a}^1 = \hat{a}_1 \csc\theta_{12} - \hat{a}_2 \cot\theta_{12}$$

$$\hat{a}^2 = \hat{a}_2 \csc\theta_{12} - \hat{a}_1 \cot\theta_{12}$$

$$\frac{1}{A_1} \frac{\partial \hat{a}_1}{\partial \xi_1} \cdot \hat{a}_1 = 0$$

$$\frac{1}{A_1} \frac{\partial \hat{a}_1}{\partial \xi_1} \cdot \hat{a}_2 = \frac{\sin\theta_{12}}{\rho_{11}}$$

$$\frac{1}{A_1} \frac{\partial \hat{a}_1}{\partial \xi_1} \cdot \hat{n} = -\frac{1}{r_{11}}$$

$$\frac{1}{A_2} \frac{\partial \hat{a}_1}{\partial \xi_2} \cdot \hat{a}_1 = 0$$

$$\frac{1}{A_2} \frac{\partial \hat{a}_1}{\partial \xi_2} \cdot \hat{a}_2 = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\cos\theta_{12}) + \frac{\sin\theta_{12}}{\rho_{22}}$$

$$\frac{1}{A_2} \frac{\partial \hat{a}_1}{\partial \xi_2} \cdot \hat{n} = -\frac{\sin\theta_{12}}{r_{21}} - \frac{\cos\theta_{12}}{r_{22}}$$

$$\frac{1}{A_1} \frac{\partial \hat{a}_2}{\partial \xi_1} \cdot \hat{a}_1 = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (\cos\theta_{12}) - \frac{\sin\theta_{12}}{\rho_{11}}$$

$$\frac{1}{A_1} \frac{\partial \hat{a}_2}{\partial \xi_1} \cdot \hat{a}_2 = 0$$

$$\frac{1}{A_1} \frac{\partial \hat{a}_2}{\partial \xi_1} \cdot \hat{n} = \frac{\sin\theta_{12}}{r_{12}} - \frac{\cos\theta_{12}}{r_{11}}$$

DERIVATIVES OF BASE-VECTOR FIELDS ON $\mathcal{S}_0$

CONTINUED

$$\frac{1}{A_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} \cdot \hat{\mathbf{a}}_1 = -\frac{\sin \theta_{12}}{\rho_{22}}$$

$$\frac{1}{A_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} \cdot \hat{\mathbf{a}}_2 = 0$$

$$\frac{1}{A_2} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} \cdot \hat{\mathbf{n}} = -\frac{1}{r_{22}}$$

$$\frac{1}{A_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \cdot \hat{\mathbf{a}}_1 = \frac{1}{r_{11}}$$

$$\frac{1}{A_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \cdot \hat{\mathbf{a}}_2 = \frac{\cos \theta_{12}}{r_{11}} - \frac{\sin \theta_{12}}{r_{12}}$$

$$\frac{1}{A_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \cdot \hat{\mathbf{a}}_1 = \frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}}$$

$$\frac{1}{A_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \cdot \hat{\mathbf{a}}_2 = \frac{1}{r_{22}}$$

- In addition, differentiating $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$ gives the additional results

$$\frac{1}{A_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \cdot \hat{\mathbf{n}} = 0 \quad \text{and} \quad \frac{1}{A_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \cdot \hat{\mathbf{n}} = 0$$

- Applying these dot-product relations yields the following results for the dot products involving the reciprocal basis

DERIVATIVES OF BASE-VECTOR FIELDS ON $\mathcal{S}_0$

CONTINUED

$$\frac{1}{A_1} \frac{\partial \hat{a}_1}{\partial \xi_1} \cdot \hat{a}^1 = - \frac{\cos \theta_{12}}{\rho_{11}}$$

$$\frac{1}{A_2} \frac{\partial \hat{a}_1}{\partial \xi_2} \cdot \hat{a}^1 = - \frac{\cot \theta_{12}}{A_2} \frac{\partial}{\partial \xi_2} (\cos \theta_{12}) - \frac{\cos \theta_{12}}{\rho_{22}}$$

$$\frac{1}{A_1} \frac{\partial \hat{a}_1}{\partial \xi_1} \cdot \hat{a}^2 = \frac{1}{\rho_{11}}$$

$$\frac{1}{A_2} \frac{\partial \hat{a}_1}{\partial \xi_2} \cdot \hat{a}^2 = \frac{\csc \theta_{12}}{A_2} \frac{\partial}{\partial \xi_2} (\cos \theta_{12}) + \frac{1}{\rho_{22}}$$

$$\frac{1}{A_1} \frac{\partial \hat{a}_2}{\partial \xi_1} \cdot \hat{a}^1 = \frac{\csc \theta_{12}}{A_1} \frac{\partial}{\partial \xi_1} (\cos \theta_{12}) - \frac{1}{\rho_{11}}$$

$$\frac{1}{A_2} \frac{\partial \hat{a}_2}{\partial \xi_2} \cdot \hat{a}^1 = - \frac{1}{\rho_{22}}$$

$$\frac{1}{A_1} \frac{\partial \hat{a}_2}{\partial \xi_1} \cdot \hat{a}^2 = \frac{\cos \theta_{12}}{\rho_{11}} - \frac{\cot \theta_{12}}{A_1} \frac{\partial}{\partial \xi_1} (\cos \theta_{12})$$

$$\frac{1}{A_2} \frac{\partial \hat{a}_2}{\partial \xi_2} \cdot \hat{a}^2 = \frac{\cos \theta_{12}}{\rho_{22}}$$

$$\frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \cdot \hat{a}^1 = \frac{\sin \theta_{12}}{r_{11}} + \frac{\cos \theta_{12}}{r_{12}}$$

$$\frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} \cdot \hat{a}^1 = \frac{1}{r_{21}}$$

$$\frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \cdot \hat{a}^2 = - \frac{1}{r_{12}}$$

$$\frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} \cdot \hat{a}^2 = \frac{\sin \theta_{12}}{r_{22}} - \frac{\cos \theta_{12}}{r_{21}}$$

DERIVATIVES OF BASE-VECTOR FIELDS ON $\mathcal{S}_0$

CONTINUED

- In addition,

$$C_1^{11} = -\frac{\cot\theta_{12}}{\rho_{11}} \quad C_1^{12} = \frac{\csc\theta_{12}}{\rho_{11}} \quad C_1^{13} = -\frac{1}{r_{11}} \quad C_2^{12} = \frac{\csc^2\theta_{12}}{A_2} \frac{\partial}{\partial \xi_2}(\cos\theta_{12}) + \frac{\csc\theta_{12}}{\rho_{22}}$$

$$C_2^{11} = -\left(\frac{\csc\theta_{12} \cot\theta_{12}}{A_2} \frac{\partial}{\partial \xi_2}(\cos\theta_{12}) + \frac{\cot\theta_{12}}{\rho_{22}} \right) \quad C_2^{13} = -\frac{\sin\theta_{12}}{r_{21}} - \frac{\cos\theta_{12}}{r_{22}}$$

$$C_1^{21} = \frac{\csc^2\theta_{12}}{A_1} \frac{\partial}{\partial \xi_1}(\cos\theta_{12}) - \frac{\csc\theta_{12}}{\rho_{11}} \quad C_1^{22} = \frac{\cot\theta_{12}}{\rho_{11}} - \frac{\csc\theta_{12} \cot\theta_{12}}{A_1} \frac{\partial}{\partial \xi_1}(\cos\theta_{12})$$

$$C_1^{23} = \frac{\sin\theta_{12}}{r_{12}} - \frac{\cos\theta_{12}}{r_{11}} \quad C_2^{21} = -\frac{\csc\theta_{12}}{\rho_{22}} \quad C_2^{22} = \frac{\cot\theta_{12}}{\rho_{22}} \quad C_2^{23} = -\frac{1}{r_{22}}$$

$$C_1^{31} = \frac{1}{r_{11}} + \frac{\cot\theta_{12}}{r_{12}} \quad C_1^{32} = -\frac{\csc\theta_{12}}{r_{12}} \quad C_1^{33} = 0$$

$$C_2^{31} = \frac{\csc\theta_{12}}{r_{21}} \quad C_2^{32} = \frac{1}{r_{22}} - \frac{\cot\theta_{12}}{r_{21}} \quad C_2^{33} = 0$$

DERIVATIVES OF BASE-VECTOR FIELDS ON $\mathcal{S}_0$

CONTINUED

- Thus, the derivatives of the unit-magnitude base-vector fields become

$$\frac{1}{A_1} \frac{\partial \hat{a}_1}{\partial \xi_1} = -\frac{\cot \theta_{12}}{\rho_{11}} \hat{a}_1 + \frac{\csc \theta_{12}}{\rho_{11}} \hat{a}_2 - \frac{1}{r_{11}} \hat{n}$$

$$\begin{aligned} \frac{1}{A_2} \frac{\partial \hat{a}_1}{\partial \xi_2} = & - \left(\frac{\csc \theta_{12} \cot \theta_{12}}{A_2} \frac{\partial \cos \theta_{12}}{\partial \xi_2} + \frac{\cot \theta_{12}}{\rho_{22}} \right) \hat{a}_1 + \left(\frac{\csc^2 \theta_{12}}{A_2} \frac{\partial \cos \theta_{12}}{\partial \xi_2} + \frac{\csc \theta_{12}}{\rho_{22}} \right) \hat{a}_2 \\ & - \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) \hat{n} \end{aligned}$$

$$\begin{aligned} \frac{1}{A_1} \frac{\partial \hat{a}_2}{\partial \xi_1} = & \left(\frac{\csc^2 \theta_{12}}{A_1} \frac{\partial \cos \theta_{12}}{\partial \xi_1} - \frac{\csc \theta_{12}}{\rho_{11}} \right) \hat{a}_1 + \left(\frac{\cot \theta_{12}}{\rho_{11}} - \frac{\csc \theta_{12} \cot \theta_{12}}{A_1} \frac{\partial \cos \theta_{12}}{\partial \xi_1} \right) \hat{a}_2 \\ & + \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right) \hat{n} \end{aligned}$$

$$\frac{1}{A_2} \frac{\partial \hat{a}_2}{\partial \xi_2} = -\frac{\csc \theta_{12}}{\rho_{22}} \hat{a}_1 + \frac{\cot \theta_{12}}{\rho_{22}} \hat{a}_2 - \frac{1}{r_{22}} \hat{n}$$

DERIVATIVES OF BASE-VECTOR FIELDS ON $\mathcal{S}_0$

CONCLUDED

$$\frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} = \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) \hat{a}_1 - \frac{\csc \theta_{12}}{r_{12}} \hat{a}_2$$

$$\frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} = \frac{\csc \theta_{12}}{r_{21}} \hat{a}_1 + \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right) \hat{a}_2$$

- For the special case of **orthogonal Gaussian reference-surface coordinates**, these expressions reduce to

$$\frac{1}{A_1} \frac{\partial \hat{a}_1}{\partial \xi_1} = \frac{1}{\rho_{11}} \hat{a}_2 - \frac{1}{r_{11}} \hat{n}$$

$$\frac{1}{A_2} \frac{\partial \hat{a}_1}{\partial \xi_2} = \frac{1}{\rho_{22}} \hat{a}_2 + \frac{1}{r_{12}} \hat{n}$$

$$\frac{1}{A_1} \frac{\partial \hat{a}_2}{\partial \xi_1} = -\frac{1}{\rho_{11}} \hat{a}_1 + \frac{1}{r_{12}} \hat{n}$$

$$\frac{1}{A_2} \frac{\partial \hat{a}_2}{\partial \xi_2} = -\frac{1}{\rho_{22}} \hat{a}_1 - \frac{1}{r_{22}} \hat{n}$$

$$\frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} = \frac{1}{r_{11}} \hat{a}_1 - \frac{1}{r_{12}} \hat{a}_2$$

$$\frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} = -\frac{1}{r_{12}} \hat{a}_1 + \frac{1}{r_{22}} \hat{a}_2$$

where $r_{21} = -r_{12}$, $\frac{1}{\rho_{11}} = -\frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2}$, and $\frac{1}{\rho_{22}} = \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \xi_1}$

DERIVATIVES OF VECTOR FIELDS ON $\mathcal{S}_0$

- Let $\vec{V}(\xi_1, \xi_2)$ denote any vector field that is associated with the points of the reference surface, $P \in \mathcal{S}_0$, and that occupies three-dimensional **Euclidean space** $\mathcal{E}^3$
- In general, $\vec{V}(\xi_1, \xi_2)$ has several component forms that are convenient for various purposes
- As indicated previously, $\vec{V}(\xi_1, \xi_2)$ is expressed herein in terms of the **unit-magnitude base vector fields** as

$$\vec{V}(\xi_1, \xi_2) = V_\alpha(\xi_1, \xi_2)\hat{a}_\alpha(\xi_1, \xi_2) + V_3(\xi_1, \xi_2)\hat{n}(\xi_1, \xi_2)$$

- The derivatives of the vector field are then given by

$$\frac{1}{A_{(\beta)}} \frac{\partial \vec{V}}{\partial \xi_\beta} = \frac{1}{A_{(\beta)}} \frac{\partial}{\partial \xi_\beta} (V_\alpha \hat{a}_\alpha + V_3 \hat{n})$$

DERIVATIVES OF VECTOR FIELDS ON $\mathcal{S}_0$ - CONTINUED

- The product rule of differentiation gives

$$\frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \vec{V}}{\partial \xi_\beta} = \frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial V_\alpha}{\partial \xi_\beta} \hat{\mathbf{a}}_\alpha + V_\alpha \frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \hat{\mathbf{a}}_\alpha}{\partial \xi_\beta} + \frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial V_3}{\partial \xi_\beta} \hat{\mathbf{n}} + V_3 \frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_\beta}$$

- The previously derived expressions for the derivatives of the reference-surface base vector fields are then used to obtain

$$\frac{1}{\mathbf{A}_1} \frac{\partial \vec{V}}{\partial \xi_1} \equiv \mathbf{v}^{(1)} \Big|_{\langle 1 \rangle} \hat{\mathbf{a}}_1 + \mathbf{v}^{(2)} \Big|_{\langle 1 \rangle} \hat{\mathbf{a}}_2 + \mathbf{v}^{(3)} \Big|_{\langle 1 \rangle} \hat{\mathbf{n}} \quad \text{and}$$

$$\frac{1}{\mathbf{A}_2} \frac{\partial \vec{V}}{\partial \xi_2} \equiv \mathbf{v}^{(1)} \Big|_{\langle 2 \rangle} \hat{\mathbf{a}}_1 + \mathbf{v}^{(2)} \Big|_{\langle 2 \rangle} \hat{\mathbf{a}}_2 + \mathbf{v}^{(3)} \Big|_{\langle 2 \rangle} \hat{\mathbf{n}} \quad \text{or}$$

$$\frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \vec{V}}{\partial \xi_\beta} \equiv \mathbf{v}^{(\alpha)} \Big|_{\langle \beta \rangle} \hat{\mathbf{a}}_\alpha + \mathbf{v}^{(3)} \Big|_{\langle \beta \rangle} \hat{\mathbf{n}} \quad \text{where}$$

DERIVATIVES OF VECTOR FIELDS ON $\mathcal{S}_0$ - CONTINUED

$$\mathbf{v}^{(1)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{V}_1}{\partial \xi_1} - \frac{\mathbf{V}_2 \mathbf{csc} \theta_{12}}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\mathbf{csc} \theta_{12}}{\rho_{11}} (\mathbf{V}_1 \cos \theta_{12} + \mathbf{V}_2) + \mathbf{V}_3 \left(\frac{1}{\mathbf{r}_{11}} + \frac{\cot \theta_{12}}{\mathbf{r}_{12}} \right)$$

$$\mathbf{v}^{(2)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{V}_2}{\partial \xi_1} + \frac{\mathbf{csc} \theta_{12}}{\rho_{11}} (\mathbf{V}_1 + \mathbf{V}_2 \cos \theta_{12}) + \frac{\mathbf{V}_2 \cot \theta_{12}}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \mathbf{V}_3 \frac{\mathbf{csc} \theta_{12}}{\mathbf{r}_{12}}$$

$$\mathbf{v}^{(3)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{V}_3}{\partial \xi_1} - \frac{\mathbf{V}_1}{\mathbf{r}_{11}} + \mathbf{V}_2 \left(\frac{\sin \theta_{12}}{\mathbf{r}_{12}} - \frac{\cos \theta_{12}}{\mathbf{r}_{11}} \right)$$

$$\mathbf{v}^{(1)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{V}_1}{\partial \xi_2} + \frac{\mathbf{V}_1 \cot \theta_{12}}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} - \frac{\mathbf{csc} \theta_{12}}{\rho_{22}} (\mathbf{V}_1 \cos \theta_{12} + \mathbf{V}_2) + \mathbf{V}_3 \frac{\mathbf{csc} \theta_{12}}{\mathbf{r}_{21}}$$

$$\mathbf{v}^{(2)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{V}_2}{\partial \xi_2} - \frac{\mathbf{V}_1 \mathbf{csc} \theta_{12}}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} + \frac{\mathbf{csc} \theta_{12}}{\rho_{22}} (\mathbf{V}_1 + \mathbf{V}_2 \cos \theta_{12}) + \mathbf{V}_3 \left(\frac{1}{\mathbf{r}_{22}} - \frac{\cot \theta_{12}}{\mathbf{r}_{21}} \right)$$

$$\mathbf{v}^{(3)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{V}_3}{\partial \xi_2} - \mathbf{V}_1 \left(\frac{\sin \theta_{12}}{\mathbf{r}_{21}} + \frac{\cos \theta_{12}}{\mathbf{r}_{22}} \right) - \frac{\mathbf{V}_2}{\mathbf{r}_{22}}$$

DERIVATIVES OF VECTOR FIELDS ON $\mathcal{S}_0$ - CONTINUED

- For **orthogonal Gaussian coordinates**, these expressions reduce to

$$\mathbf{V}^{(1)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{V}_1}{\partial \xi_1} - \frac{\mathbf{V}_2}{\rho_{11}} + \frac{\mathbf{V}_3}{\mathbf{r}_{11}}$$

$$\mathbf{V}^{(2)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{V}_2}{\partial \xi_1} + \frac{\mathbf{V}_1}{\rho_{11}} - \frac{\mathbf{V}_3}{\mathbf{r}_{12}}$$

$$\mathbf{V}^{(3)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{V}_3}{\partial \xi_1} - \frac{\mathbf{V}_1}{\mathbf{r}_{11}} + \frac{\mathbf{V}_2}{\mathbf{r}_{12}}$$

$$\mathbf{V}^{(1)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{V}_1}{\partial \xi_2} - \frac{\mathbf{V}_2}{\rho_{22}} - \frac{\mathbf{V}_3}{\mathbf{r}_{12}}$$

$$\mathbf{V}^{(2)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{V}_2}{\partial \xi_2} + \frac{\mathbf{V}_1}{\rho_{22}} + \frac{\mathbf{V}_3}{\mathbf{r}_{22}}$$

$$\mathbf{V}^{(3)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{V}_3}{\partial \xi_2} + \frac{\mathbf{V}_1}{\mathbf{r}_{12}} - \frac{\mathbf{V}_2}{\mathbf{r}_{22}}$$

- Likewise, the expressions for the derivatives reduce to

$$\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{V}}}{\partial \xi_1} = \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{V}_1}{\partial \xi_1} - \frac{\mathbf{V}_2}{\rho_{11}} + \frac{\mathbf{V}_3}{\mathbf{r}_{11}} \right) \hat{\mathbf{a}}_1 + \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{V}_2}{\partial \xi_1} + \frac{\mathbf{V}_1}{\rho_{11}} - \frac{\mathbf{V}_3}{\mathbf{r}_{12}} \right) \hat{\mathbf{a}}_2 + \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{V}_3}{\partial \xi_1} - \frac{\mathbf{V}_1}{\mathbf{r}_{11}} + \frac{\mathbf{V}_2}{\mathbf{r}_{12}} \right) \hat{\mathbf{n}}$$

$$\frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{V}}}{\partial \xi_2} = \left(\frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{V}_1}{\partial \xi_2} - \frac{\mathbf{V}_2}{\rho_{22}} - \frac{\mathbf{V}_3}{\mathbf{r}_{12}} \right) \hat{\mathbf{a}}_1 + \left(\frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{V}_2}{\partial \xi_2} + \frac{\mathbf{V}_1}{\rho_{22}} + \frac{\mathbf{V}_3}{\mathbf{r}_{22}} \right) \hat{\mathbf{a}}_2 + \left(\frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{V}_3}{\partial \xi_2} + \frac{\mathbf{V}_1}{\mathbf{r}_{12}} - \frac{\mathbf{V}_2}{\mathbf{r}_{22}} \right) \hat{\mathbf{n}}$$

DERIVATIVES OF VECTOR FIELDS ON $\mathcal{S}_0$ - CONCLUDED

- From these derivative expressions, it also follows that

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{A_1} \frac{\partial \vec{V}}{\partial \xi_1} = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (V_1 + V_2 \cos \theta_{12}) - \frac{V_2 \sin \theta_{12}}{\rho_{11}} + \frac{V_3}{r_{11}}$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{A_1} \frac{\partial \vec{V}}{\partial \xi_1} = \left(\frac{1}{A_1} \frac{\partial V_1}{\partial \xi_1} + \frac{V_3}{r_{11}} \right) \cos \theta_{12} + \frac{1}{A_1} \frac{\partial V_2}{\partial \xi_1} + \left(\frac{V_1}{\rho_{11}} - \frac{V_3}{r_{12}} \right) \sin \theta_{12}$$

$$\hat{\mathbf{n}} \cdot \frac{1}{A_1} \frac{\partial \vec{V}}{\partial \xi_1} = \frac{1}{A_1} \frac{\partial V_3}{\partial \xi_1} - \frac{V_1}{r_{11}} + V_2 \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right)$$

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{A_2} \frac{\partial \vec{V}}{\partial \xi_2} = \frac{1}{A_2} \frac{\partial V_1}{\partial \xi_2} + \left(\frac{1}{A_2} \frac{\partial V_2}{\partial \xi_2} + \frac{V_3}{r_{22}} \right) \cos \theta_{12} + \left(\frac{V_3}{r_{21}} - \frac{V_2}{\rho_{22}} \right) \sin \theta_{12}$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{A_2} \frac{\partial \vec{V}}{\partial \xi_2} = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (V_2 + V_1 \cos \theta_{12}) + \frac{V_1 \sin \theta_{12}}{\rho_{22}} + \frac{V_3}{r_{22}}$$

$$\hat{\mathbf{n}} \cdot \frac{1}{A_2} \frac{\partial \vec{V}}{\partial \xi_2} = \frac{1}{A_2} \frac{\partial V_3}{\partial \xi_2} - V_1 \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) - \frac{V_2}{r_{22}}$$

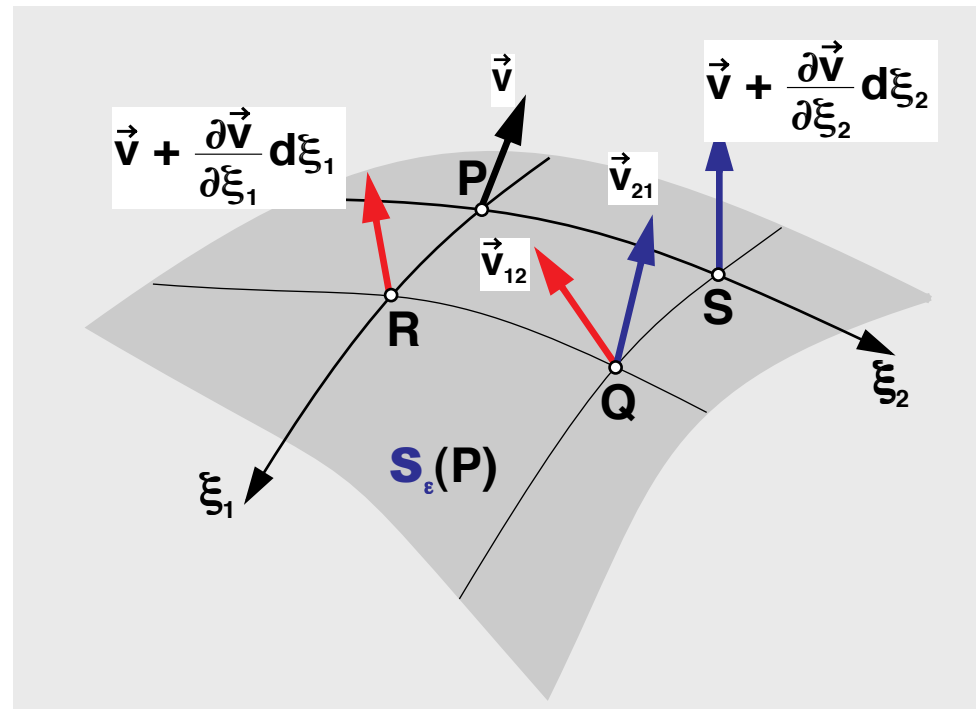
COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$

- So far in the present study, it has been shown that when a simple surface is given by $\vec{x} = x_k(\xi_1, \xi_2) \hat{i}_k$, in order for the vector fields $\vec{a}_1 = \frac{\partial \vec{x}}{\partial \xi_1}$ and $\vec{a}_2 = \frac{\partial \vec{x}}{\partial \xi_2}$ to exist and be continuous, $\vec{x}(\xi_1, \xi_2)$ must have continuous first partial derivatives; denoted by $\vec{x}(\xi_1, \xi_2) \in \mathbf{C}^1$
- It follows that $\vec{a}_1 \times \vec{a}_2$ is a continuous vector field and $|\vec{a}_1 \times \vec{a}_2|$ is a continuous function of (ξ_1, ξ_2)
- Thus, $\hat{n}(\xi_1, \xi_2) = \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|}$ is also a continuous vector field when $\vec{x}(\xi_1, \xi_2) \in \mathbf{C}^1$ is satisfied

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

- For $\vec{a}_1$ and $\vec{a}_2$ to be unique at every point of the reference surface, they must also be single-valued vector fields
- Let $\vec{v}(\xi_1, \xi_2)$ be any arbitrary differentiable vector field associated with the reference surface
- From the calculus of vector functions, it follows that the value of the vector field at point R is $\vec{v} + \frac{\partial \vec{v}}{\partial \xi_1} d\xi_1$ and the value at point S is

$$\vec{v} + \frac{\partial \vec{v}}{\partial \xi_2} d\xi_2$$



COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

- As the surface is traversed from point R to point Q, the value of the vector field becomes $\vec{v}_{12} = \vec{v} + \frac{\partial \vec{v}}{\partial \xi_1} d\xi_1 + \frac{\partial}{\partial \xi_2} \left(\vec{v} + \frac{\partial \vec{v}}{\partial \xi_1} d\xi_1 \right) d\xi_2$
- Similarly, as the surface is traversed from point S to point Q, the value of the vector field becomes $\vec{v}_{21} = \vec{v} + \frac{\partial \vec{v}}{\partial \xi_2} d\xi_2 + \frac{\partial}{\partial \xi_1} \left(\vec{v} + \frac{\partial \vec{v}}{\partial \xi_2} d\xi_2 \right) d\xi_1$
- For the vector field to be single valued, it follows that $\vec{v}_{12} = \vec{v}_{21}$ is a necessary condition
- Therefore, for $\vec{v}(\xi_1, \xi_2)$ to be a single-valued vector field, defined for points of the reference surface, the following **necessary condition** must be met

$$\frac{\partial}{\partial \xi_1} \left(\frac{\partial \vec{v}}{\partial \xi_2} \right) = \frac{\partial}{\partial \xi_2} \left(\frac{\partial \vec{v}}{\partial \xi_1} \right)$$

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

- Thus, single-valuedness of $\vec{a}_1$ and $\vec{a}_2$ requires

$$\frac{\partial}{\partial \xi_1} \left(\frac{\partial \vec{a}_1}{\partial \xi_2} \right) = \frac{\partial}{\partial \xi_2} \left(\frac{\partial \vec{a}_1}{\partial \xi_1} \right) \quad \text{and} \quad \frac{\partial}{\partial \xi_1} \left(\frac{\partial \vec{a}_2}{\partial \xi_2} \right) = \frac{\partial}{\partial \xi_2} \left(\frac{\partial \vec{a}_2}{\partial \xi_1} \right)$$

and ensures single-valuedness of $\hat{n}$

- In terms of $\vec{x}(\xi_1, \xi_2)$, single-valuedness of $\vec{a}_1$ and $\vec{a}_2$ requires

$$\frac{\partial}{\partial \xi_1} \left(\frac{\partial}{\partial \xi_2} \left(\frac{\partial \vec{x}}{\partial \xi_1} \right) \right) = \frac{\partial}{\partial \xi_2} \left(\frac{\partial}{\partial \xi_1} \left(\frac{\partial \vec{x}}{\partial \xi_1} \right) \right) \quad \text{and} \quad \frac{\partial}{\partial \xi_1} \left(\frac{\partial}{\partial \xi_2} \left(\frac{\partial \vec{x}}{\partial \xi_2} \right) \right) = \frac{\partial}{\partial \xi_2} \left(\frac{\partial}{\partial \xi_1} \left(\frac{\partial \vec{x}}{\partial \xi_2} \right) \right);$$

that is, $\vec{x}(\xi_1, \xi_2)$ is required to have continuous third partial derivatives, which is denoted herein by $\vec{x}(\xi_1, \xi_2) \in \mathbf{C}^3$

- Note that differentiability of $\vec{a}_1$ and $\vec{a}_2$ ensures differentiability of $\hat{n}$

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

- Consider the following formulas for the geometric quantities associated with the reference surface, expressed in terms of $\vec{\mathbf{x}}(\xi_1, \xi_2)$

$$\mathbf{A}_1(\xi_1, \xi_2) = \sqrt{\vec{\mathbf{a}}_1 \cdot \vec{\mathbf{a}}_1} \quad \rightarrow \quad \mathbf{A}_1(\xi_1, \xi_2) = \sqrt{\frac{\partial \vec{\mathbf{x}}}{\partial \xi_1} \cdot \frac{\partial \vec{\mathbf{x}}}{\partial \xi_1}}$$

$$\mathbf{A}_2(\xi_1, \xi_2) = \sqrt{\vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_2} \quad \rightarrow \quad \mathbf{A}_2(\xi_1, \xi_2) = \sqrt{\frac{\partial \vec{\mathbf{x}}}{\partial \xi_2} \cdot \frac{\partial \vec{\mathbf{x}}}{\partial \xi_2}}$$

$$\cos\theta_{12} = \frac{\vec{\mathbf{a}}_1 \cdot \vec{\mathbf{a}}_2}{\mathbf{A}_1 \mathbf{A}_2} \quad \rightarrow \quad \cos\theta_{12} = \frac{1}{\mathbf{A}_1 \mathbf{A}_2} \left(\frac{\partial \vec{\mathbf{x}}}{\partial \xi_1} \cdot \frac{\partial \vec{\mathbf{x}}}{\partial \xi_2} \right)$$

$$\hat{\mathbf{n}}(\xi_1, \xi_2) = \frac{\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2}{|\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2|} \quad \rightarrow \quad \hat{\mathbf{n}}(\xi_1, \xi_2) = \frac{1}{\mathbf{A}_1 \mathbf{A}_2 \sin\theta_{12}} \left(\frac{\partial \vec{\mathbf{x}}}{\partial \xi_1} \times \frac{\partial \vec{\mathbf{x}}}{\partial \xi_2} \right)$$

$$\frac{1}{r_{11}} = -\frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1} \cdot \hat{\mathbf{n}} \quad \rightarrow \quad \frac{1}{r_{11}} = -\frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\mathbf{A}_1 \frac{\partial \vec{\mathbf{x}}}{\partial \xi_1} \right) \cdot \hat{\mathbf{n}}$$

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

$$\frac{1}{r_{22}} = - \frac{1}{A_2} \frac{\partial \hat{a}_2}{\partial \xi_2} \cdot \hat{n} \quad \rightarrow \quad \frac{1}{r_{22}} = - \frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(A_2 \frac{\partial \vec{x}}{\partial \xi_2} \right) \cdot \hat{n}$$

$$\frac{1}{r_{12}} = \frac{\cot \theta_{12}}{r_{11}} + \left(\frac{1}{A_1} \frac{\partial \hat{a}_2}{\partial \xi_1} \cdot \hat{n} \right) \csc \theta_{12}$$



$$\frac{1}{r_{12}} = \frac{\cot \theta_{12}}{r_{11}} + \left[\frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(A_2 \frac{\partial \vec{x}}{\partial \xi_2} \right) \cdot \hat{n} \right] \csc \theta_{12}$$

$$\frac{1}{r_{21}} = - \left(\frac{1}{A_2} \frac{\partial \hat{a}_1}{\partial \xi_2} \cdot \hat{n} \right) \csc \theta_{12} - \frac{\cot \theta_{12}}{r_{22}}$$



$$\frac{1}{r_{21}} = - \left[\frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(A_1 \frac{\partial \vec{x}}{\partial \xi_1} \right) \cdot \hat{n} \right] \csc \theta_{12} - \frac{\cot \theta_{12}}{r_{22}}$$

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

- In addition,
$$\frac{1}{\rho_{11}} = \frac{\csc\theta_{12}}{\mathbf{A}_1\mathbf{A}_2} \left(\frac{\partial}{\partial\xi_1} [\mathbf{A}_2 \cos\theta_{12}] - \frac{\partial\mathbf{A}_1}{\partial\xi_2} \right) \text{ and}$$

$$\frac{1}{\rho_{22}} = - \frac{\csc\theta_{12}}{\mathbf{A}_1\mathbf{A}_2} \left(\frac{\partial}{\partial\xi_2} [\mathbf{A}_1 \cos\theta_{12}] - \frac{\partial\mathbf{A}_2}{\partial\xi_1} \right)$$

- Examination of these equations indicates that if $\vec{\mathbf{x}}(\xi_1, \xi_2) \in \mathbf{C}^3$, then every element the set of geometric quantities

$$\left\{ \mathbf{A}_1, \mathbf{A}_2, \theta_{12}, \mathbf{r}_{11}, \mathbf{r}_{12}, \mathbf{r}_{21}, \mathbf{r}_{22}, \rho_{11}, \rho_{22} \right\} \text{ is differentiable}$$

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

- Now suppose that $\{ \mathbf{A}_{11}, \mathbf{A}_{22}, \theta_{12}, \mathbf{r}_{11}, \mathbf{r}_{12}, \mathbf{r}_{21}, \mathbf{r}_{22}, \rho_{11}, \rho_{22} \}$ are given, without any reference to a surface defined by $\vec{\mathbf{x}}(\xi_1, \xi_2)$, but are required to be single-valued functions
- The issue of **surface compatibility** is concerned with determining inter-relation requirements for these geometric quantities that must be enforced in order for them to correspond to a well-defined simple smooth surface $\vec{\mathbf{x}}(\xi_1, \xi_2)$
- First note that ρ_{11} and ρ_{22} are not independent variables, as indicated by their definitions
- Next, recall that $\vec{\mathbf{x}}(\xi_1, \xi_2) \in \mathbf{C}^3$ must be fulfilled for $\vec{\mathbf{x}}(\xi_1, \xi_2)$ to be a valid surface, which implies that the necessary conditions

$$\frac{\partial}{\partial \xi_1} \left(\frac{\partial \vec{\mathbf{a}}_1}{\partial \xi_2} \right) = \frac{\partial}{\partial \xi_2} \left(\frac{\partial \vec{\mathbf{a}}_1}{\partial \xi_1} \right) \quad \text{and} \quad \frac{\partial}{\partial \xi_1} \left(\frac{\partial \vec{\mathbf{a}}_2}{\partial \xi_2} \right) = \frac{\partial}{\partial \xi_2} \left(\frac{\partial \vec{\mathbf{a}}_2}{\partial \xi_1} \right) \quad \text{must be satisfied}$$

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

- Therefore, surface compatibility requirements can be obtained by expressing the partial derivatives of $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$ in terms of the geometric quantities $\{ \mathbf{A}_1, \mathbf{A}_2, \theta_{12}, \mathbf{r}_{11}, \mathbf{r}_{12}, \mathbf{r}_{21}, \mathbf{r}_{22} \}$ and then enforcing

$$\frac{\partial}{\partial \xi_1} \left(\frac{\partial \vec{\mathbf{a}}_1}{\partial \xi_2} \right) = \frac{\partial}{\partial \xi_2} \left(\frac{\partial \vec{\mathbf{a}}_1}{\partial \xi_1} \right) \quad \text{and} \quad \frac{\partial}{\partial \xi_1} \left(\frac{\partial \vec{\mathbf{a}}_2}{\partial \xi_2} \right) = \frac{\partial}{\partial \xi_2} \left(\frac{\partial \vec{\mathbf{a}}_2}{\partial \xi_1} \right)$$

- In terms of the unit-magnitude vector fields, these expressions are

$$\mathbf{A}_1 \frac{\partial}{\partial \xi_1} \left(\frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} \right) = \mathbf{A}_1 \frac{\partial}{\partial \xi_2} \left(\frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1} \right) + \left[\frac{\partial}{\partial \xi_2} \left(\frac{\partial \mathbf{A}_1}{\partial \xi_1} \right) - \frac{\partial}{\partial \xi_1} \left(\frac{\partial \mathbf{A}_1}{\partial \xi_2} \right) \right] \hat{\mathbf{a}}_1 \quad \text{and}$$

$$\mathbf{A}_2 \frac{\partial}{\partial \xi_1} \left(\frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} \right) = \mathbf{A}_2 \frac{\partial}{\partial \xi_2} \left(\frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} \right) + \left[\frac{\partial}{\partial \xi_2} \left(\frac{\partial \mathbf{A}_2}{\partial \xi_1} \right) - \frac{\partial}{\partial \xi_1} \left(\frac{\partial \mathbf{A}_2}{\partial \xi_2} \right) \right] \hat{\mathbf{a}}_2$$

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

- Because A_1 and A_2 are required to be single-valued functions, it

follows that $\frac{\partial}{\partial \xi_2} \left(\frac{\partial A_1}{\partial \xi_1} \right) = \frac{\partial}{\partial \xi_1} \left(\frac{\partial A_1}{\partial \xi_2} \right)$ and $\frac{\partial}{\partial \xi_2} \left(\frac{\partial A_2}{\partial \xi_1} \right) = \frac{\partial}{\partial \xi_1} \left(\frac{\partial A_2}{\partial \xi_2} \right)$

- Therefore, compatibility of the surface geometric quantities is also ensured by enforcing

$$\frac{\partial}{\partial \xi_1} \left(\frac{\partial \hat{a}_1}{\partial \xi_2} \right) = \frac{\partial}{\partial \xi_2} \left(\frac{\partial \hat{a}_1}{\partial \xi_1} \right) \quad \text{and} \quad \frac{\partial}{\partial \xi_1} \left(\frac{\partial \hat{a}_2}{\partial \xi_2} \right) = \frac{\partial}{\partial \xi_2} \left(\frac{\partial \hat{a}_2}{\partial \xi_1} \right)$$

- Recall, that the general expressions for the derivatives of $\hat{a}_1$ and $\hat{a}_2$ are given by

$$\frac{1}{A_1} \frac{\partial \hat{a}_1}{\partial \xi_1} = - \frac{\cot \theta_{12}}{\rho_{11}} \hat{a}_1 + \frac{\csc \theta_{12}}{\rho_{11}} \hat{a}_2 - \frac{1}{r_{11}} \hat{n}$$

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

$$\frac{1}{A_2} \frac{\partial \hat{a}_1}{\partial \xi_2} = - \left(\frac{\csc \theta_{12} \cot \theta_{12}}{A_2} \frac{\partial \cos \theta_{12}}{\partial \xi_2} + \frac{\cot \theta_{12}}{\rho_{22}} \right) \hat{a}_1 + \left(\frac{\csc^2 \theta_{12}}{A_2} \frac{\partial \cos \theta_{12}}{\partial \xi_2} + \frac{\csc \theta_{12}}{\rho_{22}} \right) \hat{a}_2 - \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) \hat{n}$$

$$\frac{1}{A_1} \frac{\partial \hat{a}_2}{\partial \xi_1} = \left(\frac{\csc^2 \theta_{12}}{A_1} \frac{\partial \cos \theta_{12}}{\partial \xi_1} - \frac{\csc \theta_{12}}{\rho_{11}} \right) \hat{a}_1 + \left(\frac{\cot \theta_{12}}{\rho_{11}} - \frac{\csc \theta_{12} \cot \theta_{12}}{A_1} \frac{\partial \cos \theta_{12}}{\partial \xi_1} \right) \hat{a}_2 + \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right) \hat{n}$$

$$\frac{1}{A_2} \frac{\partial \hat{a}_2}{\partial \xi_2} = - \frac{\csc \theta_{12}}{\rho_{22}} \hat{a}_1 + \frac{\cot \theta_{12}}{\rho_{22}} \hat{a}_2 - \frac{1}{r_{22}} \hat{n}$$

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

- Enforcing the requirement that $\frac{\partial}{\partial \xi_2} \left(\frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_1} \right) = \frac{\partial}{\partial \xi_1} \left(\frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} \right)$ yields three nontrivial scalar equations

- The coefficients of $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ are identical and reduce to

$$\begin{aligned} \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\rho_{11}} \right) - \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\rho_{22}} \right) - \frac{\mathbf{A}_1 \mathbf{A}_2}{\mathbf{r}_{11}} \left(\frac{\sin \theta_{12}}{\mathbf{r}_{22}} - \frac{\cos \theta_{12}}{\mathbf{r}_{21}} \right) \\ - \frac{\mathbf{A}_1 \mathbf{A}_2}{\mathbf{r}_{12}} \left(\frac{\sin \theta_{12}}{\mathbf{r}_{21}} + \frac{\cos \theta_{12}}{\mathbf{r}_{22}} \right) + \frac{\partial}{\partial \xi_1} \left(\frac{\partial \theta_{12}}{\partial \xi_2} \right) = 0 \end{aligned}$$

- The simplified coefficient equation of $\hat{\mathbf{n}}$ is given by

$$\begin{aligned} \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\mathbf{r}_{22}} \right) \cos \theta_{12} + \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\mathbf{r}_{21}} \right) \sin \theta_{12} - \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\mathbf{r}_{11}} \right) + \frac{\mathbf{A}_1 \mathbf{A}_2}{\rho_{11}} \left(\frac{\cos \theta_{12}}{\mathbf{r}_{21}} - \frac{\sin \theta_{12}}{\mathbf{r}_{22}} \right) \\ - \frac{\mathbf{A}_1 \mathbf{A}_2}{\rho_{22} \mathbf{r}_{12}} + \left[\frac{\mathbf{A}_2}{\mathbf{r}_{21}} \cos \theta_{12} - \frac{\mathbf{A}_2}{\mathbf{r}_{22}} \sin \theta_{12} \right] \frac{\partial \theta_{12}}{\partial \xi_1} + \left(\frac{\mathbf{A}_1}{\mathbf{r}_{12}} \right) \frac{\partial \theta_{12}}{\partial \xi_2} = 0 \end{aligned}$$

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

- Likewise, enforcing the requirement that $\frac{\partial}{\partial \xi_2} \left(\frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} \right) = \frac{\partial}{\partial \xi_1} \left(\frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_2} \right)$ also yields three nontrivial scalar equations

- The coefficients of $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ are identical and reduce to

$$\begin{aligned} \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\rho_{11}} \right) - \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\rho_{22}} \right) - \frac{\mathbf{A}_1 \mathbf{A}_2}{\mathbf{r}_{11}} \left(\frac{\sin \theta_{12}}{\mathbf{r}_{22}} - \frac{\cos \theta_{12}}{\mathbf{r}_{21}} \right) \\ - \frac{\mathbf{A}_1 \mathbf{A}_2}{\mathbf{r}_{12}} \left(\frac{\sin \theta_{12}}{\mathbf{r}_{21}} + \frac{\cos \theta_{12}}{\mathbf{r}_{22}} \right) + \frac{\partial}{\partial \xi_1} \left(\frac{\partial \theta_{12}}{\partial \xi_2} \right) = 0 \end{aligned}$$

- The simplified coefficient equation of $\hat{\mathbf{n}}$ is given by

$$\begin{aligned} \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\mathbf{r}_{22}} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\mathbf{r}_{12}} \right) \sin \theta_{12} - \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\mathbf{r}_{11}} \right) \cos \theta_{12} - \frac{\mathbf{A}_1 \mathbf{A}_2}{\rho_{22}} \left(\frac{\cos \theta_{12}}{\mathbf{r}_{12}} + \frac{\sin \theta_{12}}{\mathbf{r}_{11}} \right) \\ + \frac{\mathbf{A}_1 \mathbf{A}_2}{\rho_{11} \mathbf{r}_{21}} + \left(\frac{\mathbf{A}_2}{\mathbf{r}_{21}} \right) \frac{\partial \theta_{12}}{\partial \xi_1} + \left[\frac{\mathbf{A}_1}{\mathbf{r}_{12}} \cos \theta_{12} + \frac{\mathbf{A}_1}{\mathbf{r}_{11}} \sin \theta_{12} \right] \frac{\partial \theta_{12}}{\partial \xi_2} = 0 \end{aligned}$$

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

- Thus, three independent compatibility equations are obtained:

$$\begin{aligned} \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\rho_{11}} \right) - \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\rho_{22}} \right) - \frac{\mathbf{A}_1 \mathbf{A}_2}{r_{11}} \left(\frac{\sin \theta_{12}}{r_{22}} - \frac{\cos \theta_{12}}{r_{21}} \right) \\ - \frac{\mathbf{A}_1 \mathbf{A}_2}{r_{12}} \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) + \frac{\partial}{\partial \xi_1} \left(\frac{\partial \theta_{12}}{\partial \xi_2} \right) = 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{r_{22}} \right) \cos \theta_{12} + \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{r_{21}} \right) \sin \theta_{12} - \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{r_{11}} \right) + \frac{\mathbf{A}_1 \mathbf{A}_2}{\rho_{11}} \left(\frac{\cos \theta_{12}}{r_{21}} - \frac{\sin \theta_{12}}{r_{22}} \right) \\ - \frac{\mathbf{A}_1 \mathbf{A}_2}{\rho_{22} r_{12}} + \left[\frac{\mathbf{A}_2}{r_{21}} \cos \theta_{12} - \frac{\mathbf{A}_2}{r_{22}} \sin \theta_{12} \right] \frac{\partial \theta_{12}}{\partial \xi_1} + \left(\frac{\mathbf{A}_1}{r_{12}} \right) \frac{\partial \theta_{12}}{\partial \xi_2} = 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{r_{22}} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{r_{12}} \right) \sin \theta_{12} - \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{r_{11}} \right) \cos \theta_{12} - \frac{\mathbf{A}_1 \mathbf{A}_2}{\rho_{22}} \left(\frac{\cos \theta_{12}}{r_{12}} + \frac{\sin \theta_{12}}{r_{11}} \right) \\ + \frac{\mathbf{A}_1 \mathbf{A}_2}{\rho_{11} r_{21}} + \left(\frac{\mathbf{A}_2}{r_{21}} \right) \frac{\partial \theta_{12}}{\partial \xi_1} + \left[\frac{\mathbf{A}_1}{r_{12}} \cos \theta_{12} + \frac{\mathbf{A}_1}{r_{11}} \sin \theta_{12} \right] \frac{\partial \theta_{12}}{\partial \xi_2} = 0 \end{aligned}$$

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONTINUED

- In addition, recall that previously, $\frac{\partial \vec{a}_1}{\partial \xi_2} = \frac{\partial \vec{a}_2}{\partial \xi_1}$ was enforced, which

yields
$$\frac{1}{r_{12}} + \frac{1}{r_{21}} = \cot \theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$$

- Thus, there are three independent compatibility equations and six independent surface geometry quantities
- The first compatibility equation, known as **Gauss' equation**, is sometimes expressed as

$$\frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\rho_{11}} \right) - \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\rho_{22}} \right) + \frac{\partial}{\partial \xi_1} \left(\frac{\partial \theta_{12}}{\partial \xi_2} \right) = \mathbf{A}_1 \mathbf{A}_2 \sin \theta_{12} K_G \quad \text{where}$$

$$K_G \equiv \frac{1}{r_{11} r_{22}} + \frac{1}{r_{12} r_{21}} + \cot \theta_{12} \left(\frac{1}{r_{12} r_{22}} - \frac{1}{r_{11} r_{21}} \right) \quad \text{is called the Gaussian curvature}$$

COMPATIBILITY CONDITIONS FOR $\mathcal{S}_0$ - CONCLUDED

- For **orthogonal** Gaussian reference-surface coordinates, $\frac{1}{r_{21}} = -\frac{1}{r_{12}}$, and the compatibility equations reduce to

$$\frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{r_{12}} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{r_{11}} \right) - \frac{1}{r_{22}} \frac{\partial \mathbf{A}_1}{\partial \xi_2} + \frac{1}{r_{12}} \frac{\partial \mathbf{A}_2}{\partial \xi_1} = 0$$

$$\frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{r_{22}} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{r_{12}} \right) + \frac{1}{r_{12}} \frac{\partial \mathbf{A}_1}{\partial \xi_2} - \frac{1}{r_{11}} \frac{\partial \mathbf{A}_2}{\partial \xi_1} = 0$$

$$\frac{\partial}{\partial \xi_1} \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{A}_2}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} \right) + \frac{\mathbf{A}_1 \mathbf{A}_2}{r_{11} r_{22} - (r_{12})^2} = 0$$

for the five surface geometric quantities $\left\{ \mathbf{A}_1, \mathbf{A}_2, r_{11}, r_{12}, r_{22} \right\}$

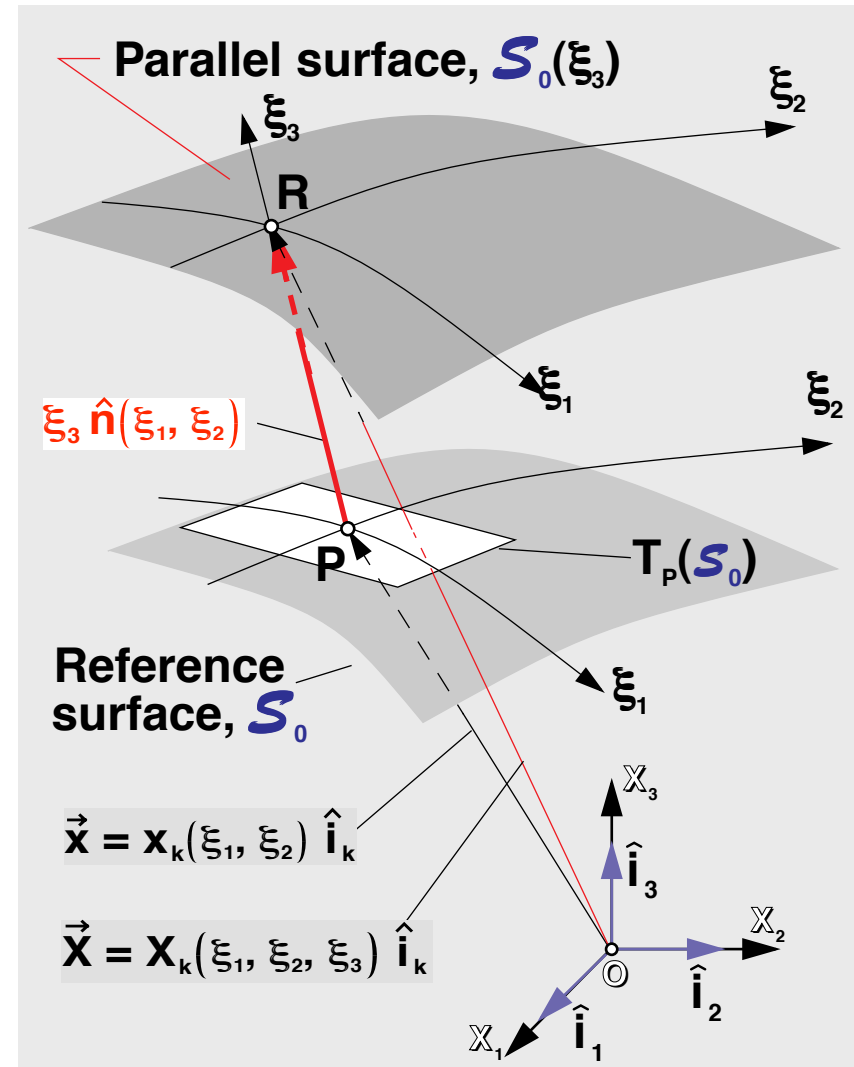
**MATHEMATICAL DESCRIPTION OF THE
SHELL SPACE, $\mathcal{R}_0$, IN TERMS OF
REFERENCE-SURFACE PROPERTIES**

MATHEMATICAL DESCRIPTION OF $\mathcal{R}_0$

- Previously, a shell was defined herein as a family of **contiguous parallel surfaces** that fill the three-dimensional region of space occupied by the shell; that is, the region $\mathcal{R}_0 \subset \mathcal{E}^3$
- As such, points of $\mathcal{R}_0$ are located herein by the position vector $\vec{X} = X_k(\xi_1, \xi_2, \xi_3)\hat{i}_k$, where a specified ordered pair (ξ_1, ξ_2) corresponds to a given point of the reference surface, $P \in \mathcal{S}_0$, and a specified value of the coordinate ξ_3 defines the corresponding parallel surface in $\mathcal{R}_0$
 - Coordinate values of ξ_3 are measured perpendicular to the tangent plane $T_p(\mathcal{S}_0)$, as shown in the following figure
- Thus, a generic point of the shell corresponds to a given point of a parallel surface, $R \in \mathcal{S}_0(\xi_3) \subset \mathcal{R}_0$

MATHEMATICAL DESCRIPTION OF $\mathcal{R}_0$ - CONTINUED

- The reason for using this approach to locate points of the shell is that the metrical properties, derived from a generic differential arc length emanating from point R in the figure, are determined from base-vector fields associated with the parallel surface, $\mathcal{S}_0(\xi_3)$, and the corresponding normal vector field
- These vector fields can then be related to corresponding vector fields for the reference surface, $\mathcal{S}_0$, which gives the metrical properties of the shell in terms of those of the reference surface



MATHEMATICAL DESCRIPTION OF $\mathcal{R}_0$ - CONTINUED

- Thus, points of the parallel surfaces are located by the curvilinear coordinates (ξ_1, ξ_2, ξ_3) , where (ξ_1, ξ_2) are specified to be the corresponding **Gaussian coordinates** of the reference surface
- A family of parallel surfaces, $\mathcal{S}_0(\xi_3)$, are defined by constant values of the coordinate ξ_3
- The **position vector** to a point R of a given parallel surface $\mathcal{S}_0(\xi_3)$ is specified as

$$\vec{\mathbf{X}}(\xi_1, \xi_2, \xi_3) = \vec{\mathbf{x}}(\xi_1, \xi_2) + \xi_3 \hat{\mathbf{n}}(\xi_1, \xi_2) \quad \text{with} \quad \mathbf{c}_3^-(\xi_1, \xi_2) \leq \xi_3 \leq \mathbf{c}_3^+(\xi_1, \xi_2)$$
- The shell thickness is then given by $h(\xi_1, \xi_2) = \mathbf{c}_3^+ - \mathbf{c}_3^- > 0$
- With the coordinate ξ_3 held constant, this representation induces curvilinear surface coordinates on the parallel surface $\mathcal{S}_0(\xi_3)$

MATHEMATICAL DESCRIPTION OF $\mathcal{R}_0$ - CONCLUDED

- The corresponding Cartesian coordinates (X_1, X_2, X_3) , with respect to the frame $O - X_1 - X_2 - X_3$, are given by

$$\mathbf{X}_k(\xi_1, \xi_2, \xi_3) = \mathbf{x}_k(\xi_1, \xi_2) + \xi_3 \mathbf{n}_k(\xi_1, \xi_2)$$

where $\mathbf{n}_k \equiv \hat{\mathbf{n}} \cdot \hat{\mathbf{i}}_k$

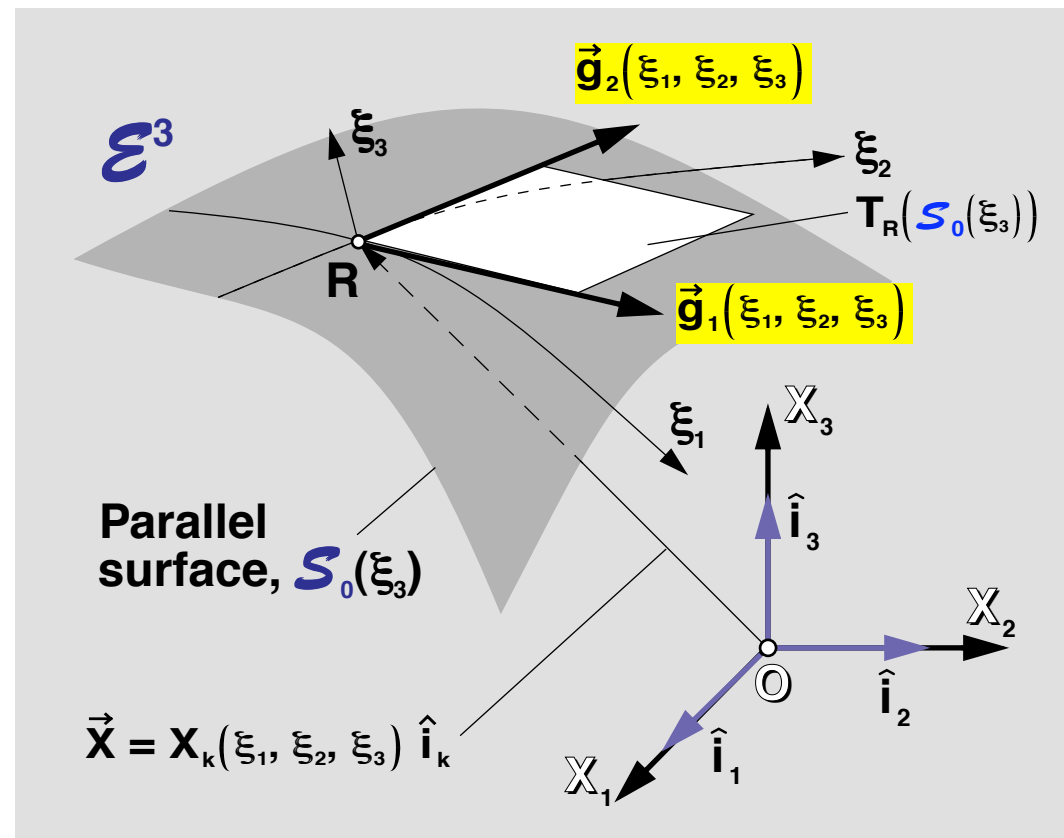
VECTOR FIELDS DEFINED ON $\mathcal{R}_0$

- Henceforth, let $\mathcal{S}_0(\xi_3) \subset \mathcal{R}_0 \subset \mathcal{E}^3$ be a **smooth parallel surface** that is defined by the *position vector* $\vec{X}(\xi_1, \xi_2, \xi_3)$

- The corresponding **natural base-vector fields** of $\mathcal{S}_0(\xi_3)$, that span the tangent plane, $T_R(\mathcal{S}_0(\xi_3))$, at every point $R \in \mathcal{S}_0(\xi_3)$, are then given by

$$\vec{g}_1(\xi_1, \xi_2, \xi_3) \equiv \frac{\partial \vec{X}}{\partial \xi_1} \quad \text{and}$$

$$\vec{g}_2(\xi_1, \xi_2, \xi_3) \equiv \frac{\partial \vec{X}}{\partial \xi_2}$$



VECTOR FIELDS DEFINED ON $\mathcal{R}_0$ - CONTINUED

- At a given point of $\mathcal{S}_0(\xi_3)$, the base vectors $\vec{g}_1$ and $\vec{g}_2$ are tangent to the Gaussian-coordinate curves, as shown in the previous figure

- Using

$$\vec{X}(\xi_1, \xi_2, \xi_3) = \vec{x}(\xi_1, \xi_2) + \xi_3 \hat{n}(\xi_1, \xi_2) \quad \text{and} \quad \vec{a}_\alpha(\xi_1, \xi_2) = \frac{\partial \vec{x}}{\partial \xi_\alpha}, \quad \text{it is found that}$$

$$\vec{g}_\alpha(\xi_1, \xi_2, \xi_3) = \vec{a}_\alpha + \xi_3 \frac{\partial \hat{n}}{\partial \xi_\alpha}$$

- Writing the last equation as $\vec{g}_\alpha = A_{(\alpha)} \left(\hat{a}_\alpha + \xi_3 \frac{1}{A_{(\alpha)}} \frac{\partial \hat{n}}{\partial \xi_\alpha} \right)$ and using the derivative expressions

$$\frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} = \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) \hat{a}_1 - \frac{\csc \theta_{12}}{r_{12}} \hat{a}_2 \quad \text{and}$$

$$\frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} = \frac{\csc \theta_{12}}{r_{21}} \hat{a}_1 + \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right) \hat{a}_2 \quad \text{gives}$$

VECTOR FIELDS DEFINED ON $\mathcal{R}_0$ - CONTINUED

$$\vec{g}_1 = A_1 \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \hat{a}_1 - \frac{\xi_3 \csc \theta_{12}}{r_{12}} \hat{a}_2 \right] \text{ and}$$

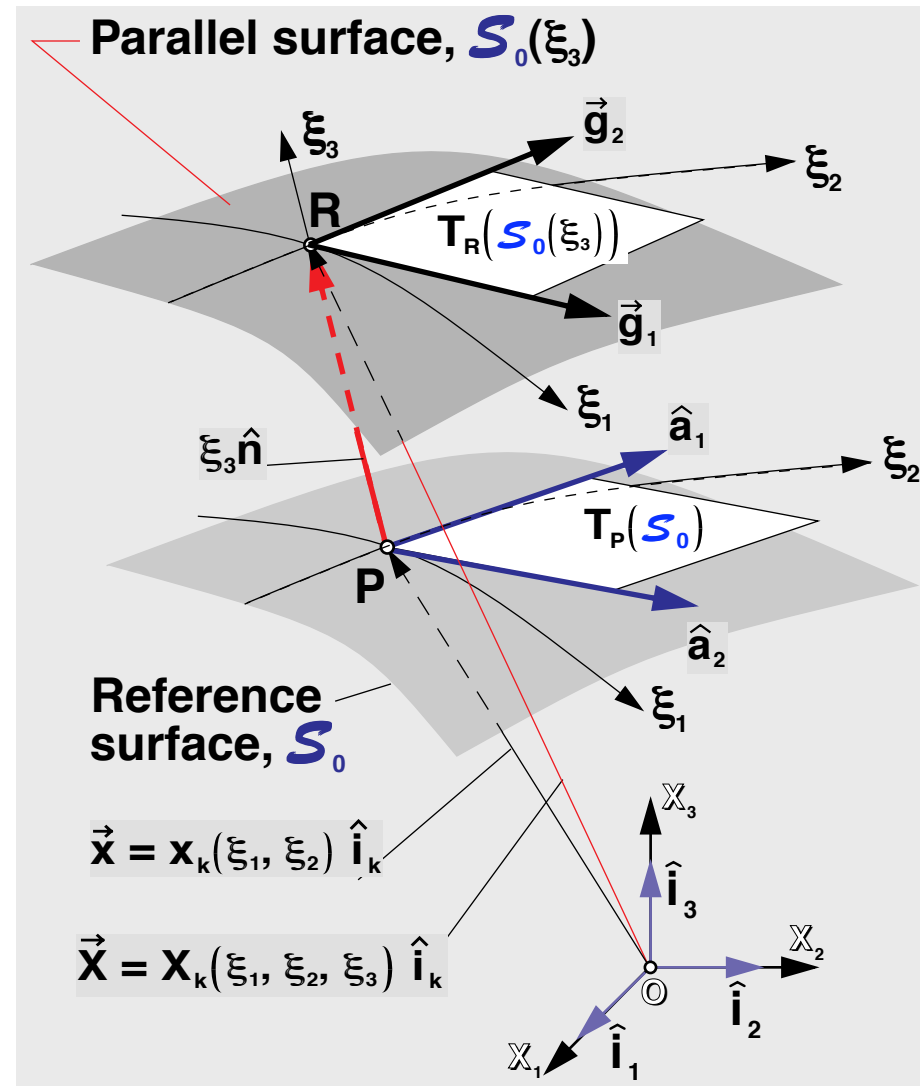
$$\vec{g}_2 = A_2 \left[\frac{\xi_3 \csc \theta_{12}}{r_{21}} \hat{a}_1 + \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \hat{a}_2 \right]$$

- Likewise, $\vec{g}_3(\xi_1, \xi_2, \xi_3) = \frac{\partial \vec{X}}{\partial \xi_3} = \hat{n}(\xi_1, \xi_2)$ for all values of ξ_3 and the cross product is given by

$$\frac{\vec{g}_1 \times \vec{g}_2}{A_1 A_2} = \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) + \left(\frac{\xi_3}{r_{12}} \right) \left(\frac{\xi_3}{r_{21}} \right) \csc^2 \theta_{12} \right] \hat{n}$$

VECTOR FIELDS DEFINED ON $\mathcal{R}_0$ - CONTINUED

- The cross-product equation verifies that the tangent plane of the parallel surface, spanned by $\vec{g}_1$ and $\vec{g}_2$, is indeed parallel to the tangent plane spanned by $\hat{a}_1$ and $\hat{a}_2$
- That is, $T_R(\mathcal{S}_0(\xi_3))$ is parallel to $T_P(\mathcal{S}_0)$

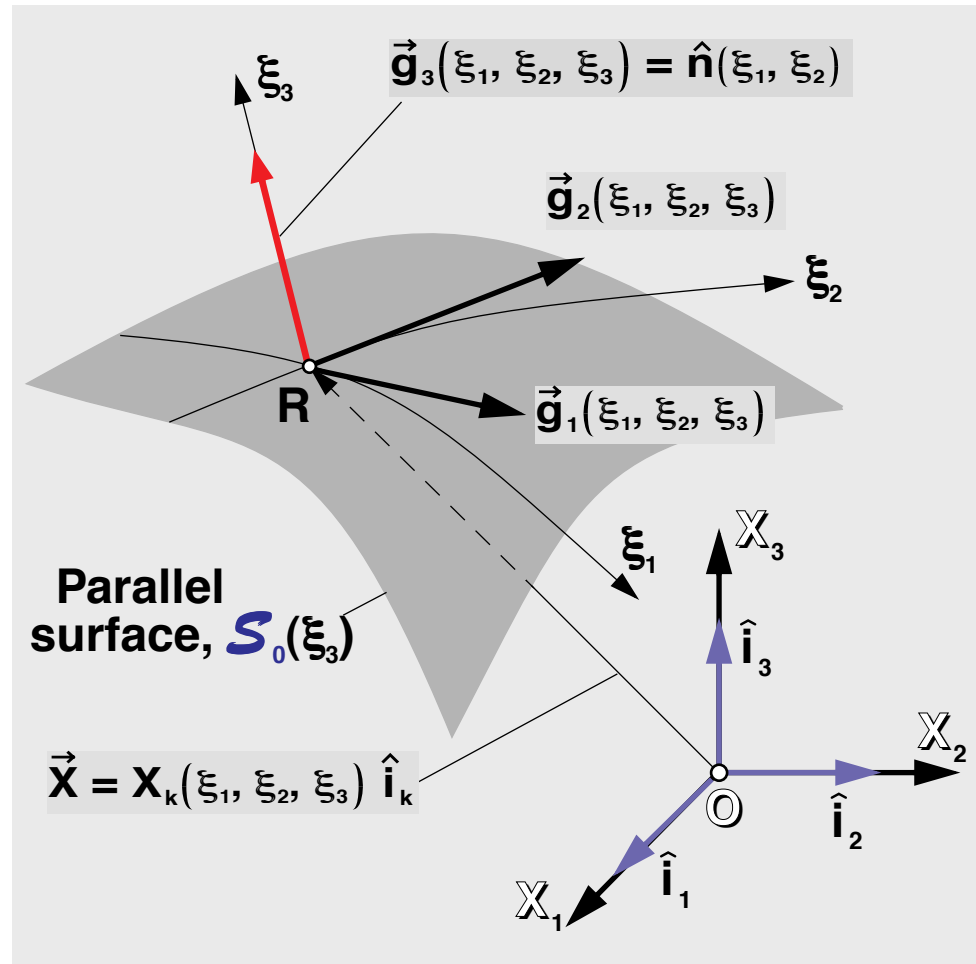


VECTOR FIELDS DEFINED ON $\mathcal{R}_0$ - CONTINUED

- Because $\vec{g}_1$ and $\vec{g}_2$ span the tangent plane $T_R(\mathcal{S}_0(\xi_3))$ at every point $R \in \mathcal{S}_0(\xi_3)$, and the vector $\vec{g}_3 = \hat{n}(\xi_1, \xi_2)$ is perpendicular to the tangent plane, it follows that

$$\{\vec{g}_1, \vec{g}_2, \hat{n}\}$$

form a **basis** for vector fields associated with the points of the shell space $\mathcal{R}_0$



VECTOR FIELDS DEFINED ON $\mathcal{R}_0$ - CONTINUED

- Previously, **unit-magnitude vector fields** $\{\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \hat{\mathbf{g}}_3\}$ were defined by

$$\hat{\mathbf{g}}_1(\xi_1, \xi_2, \xi_3) \equiv \frac{\vec{\mathbf{g}}_1}{H_1}$$

$$\hat{\mathbf{g}}_2(\xi_1, \xi_2, \xi_3) \equiv \frac{\vec{\mathbf{g}}_2}{H_2}$$

$$\hat{\mathbf{g}}_3(\xi_1, \xi_2, \xi_3) \equiv \frac{\vec{\mathbf{g}}_3}{H_3}$$

where

$$H_\alpha(\xi_1, \xi_2, \xi_3) \equiv |\vec{\mathbf{g}}_\alpha| = \sqrt{\vec{\mathbf{g}}_\alpha \cdot \vec{\mathbf{g}}_{(\alpha)}}$$

- Using

$$\vec{\mathbf{g}}_1 = A_1 \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \hat{\mathbf{a}}_1 - \frac{\xi_3 \csc \theta_{12}}{r_{12}} \hat{\mathbf{a}}_2 \right],$$

$$\vec{\mathbf{g}}_2 = A_2 \left[\frac{\xi_3 \csc \theta_{12}}{r_{21}} \hat{\mathbf{a}}_1 + \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \hat{\mathbf{a}}_2 \right], \text{ and}$$

$$\vec{\mathbf{g}}_3 = \hat{\mathbf{n}}(\xi_1, \xi_2) \text{ gives}$$

VECTOR FIELDS DEFINED ON $\mathcal{R}_0$ - CONTINUED

$$H_1 = A_1 \sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}}\right)^2}$$

$$H_2 = A_2 \sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}}\right)^2}$$

$$H_3 = 1$$

- The **unit-magnitude vector fields** $\{\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \hat{\mathbf{g}}_3\}$ are now given by

$$\hat{\mathbf{g}}_1 = \frac{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right) \hat{\mathbf{a}}_1 - \frac{\xi_3 \csc \theta_{12}}{r_{12}} \hat{\mathbf{a}}_2}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}}\right)^2}}$$

VECTOR FIELDS DEFINED ON $\mathcal{R}_0$ - CONTINUED

$$\hat{\mathbf{g}}_2 = \frac{\frac{\xi_3 \csc\theta_{12}}{r_{21}} \hat{\mathbf{a}}_1 + \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right) \hat{\mathbf{a}}_2}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}} \quad \text{and} \quad \hat{\mathbf{g}}_3 = \hat{\mathbf{n}}$$

- It is convenient to introduce the following quantities, that are often referred to in the technical literature as **“shifters”** or **“translators,”** such that the first two unit-magnitude base vectors are given by

$$\hat{\mathbf{g}}_1 = \mu_{11} \hat{\mathbf{a}}_1 + \mu_{12} \hat{\mathbf{a}}_2 \quad \text{and} \quad \hat{\mathbf{g}}_2 = \mu_{21} \hat{\mathbf{a}}_1 + \mu_{22} \hat{\mathbf{a}}_2$$

where $\mu_{\alpha\beta} = \mu_{\alpha\beta}(\xi_1, \xi_2, \xi_3)$

VECTOR FIELDS DEFINED ON $\mathcal{R}_0$ - CONTINUED

- The expressions for the shifters are given by

$$\mu_{11} = \frac{1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}}\right)^2}}$$

$$\mu_{12} = \frac{-\frac{\xi_3 \csc \theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}}\right)^2}}$$

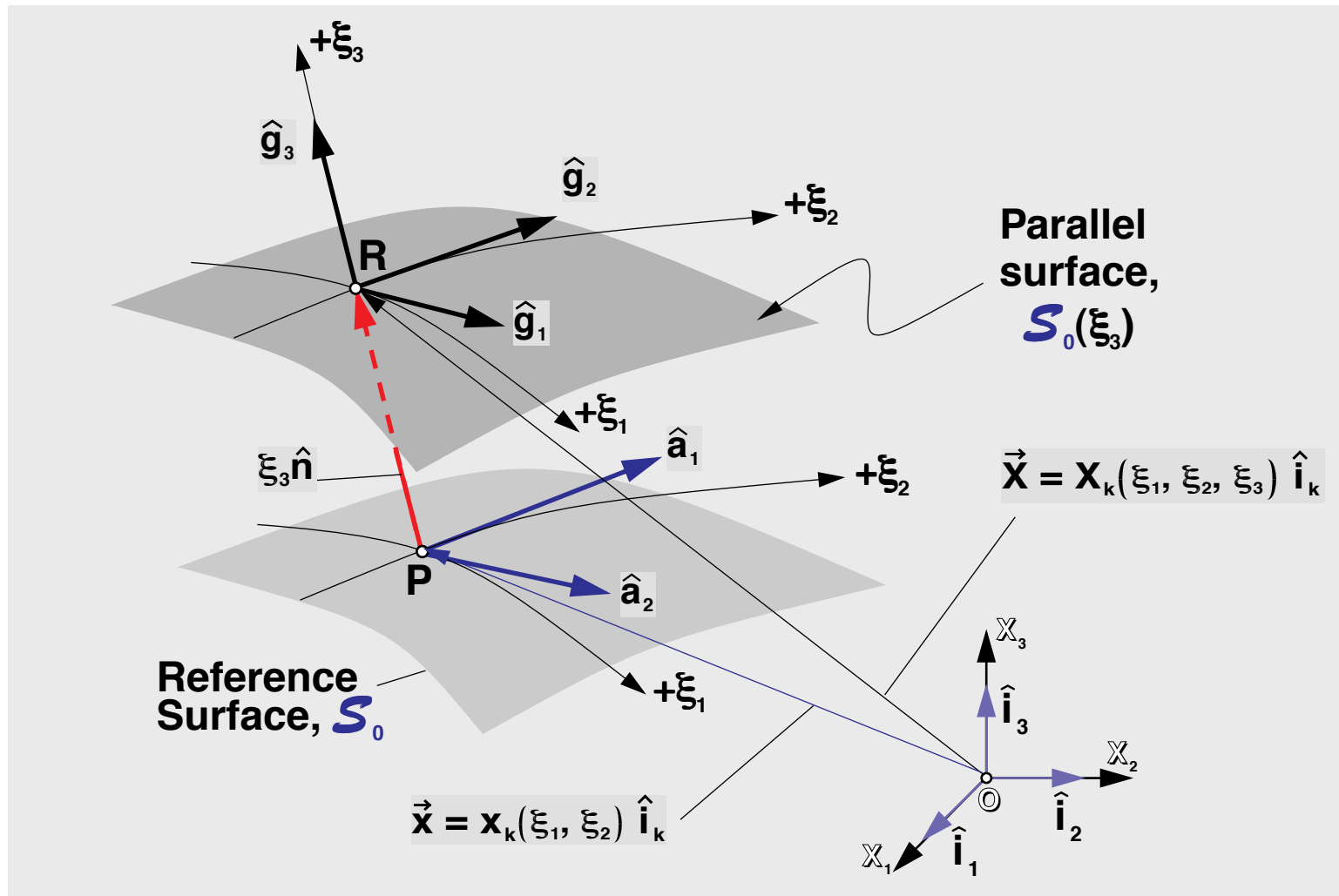
$$\mu_{21} = \frac{\frac{\xi_3 \csc \theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}}\right)^2}}$$

VECTOR FIELDS DEFINED ON $\mathcal{R}_0$ - CONTINUED

$$\mu_{22} = \frac{1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}}$$

- Thus, the shifters relate the vectors $\{\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \hat{\mathbf{g}}_3\}$, at point $\mathbf{R}$ of the parallel surface $\mathcal{S}_0(\xi_3)$, to the corresponding vectors $\{\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{n}}\}$ of the reference surface, at the corresponding point $\mathbf{P}$

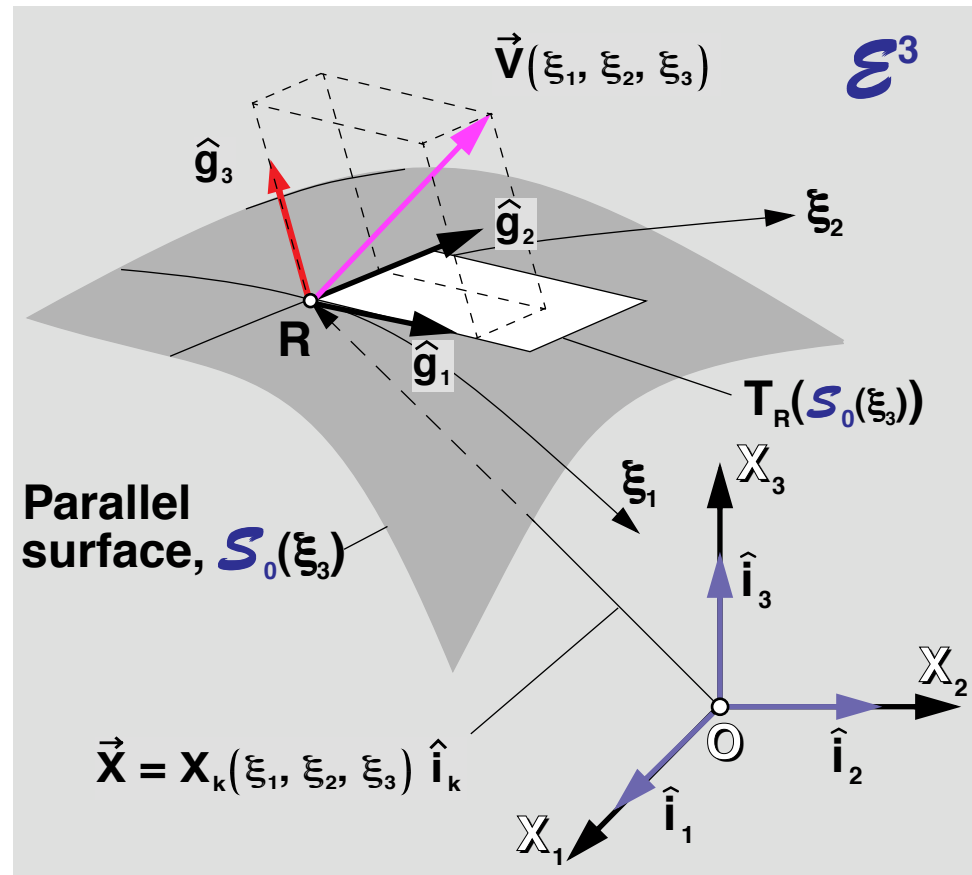
VECTOR FIELDS DEFINED ON $\mathcal{R}_0$ - CONTINUED



VECTOR FIELDS DEFINED ON $\mathcal{R}_0$ - CONCLUDED

- The set $\{\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \hat{\mathbf{g}}_3\}$ also forms a **basis** for vector fields in $\mathcal{E}^3$ that are associated with the points of the shell space $\mathcal{R}_0$
- Thus, any vector $\vec{\mathbf{V}}(\xi_1, \xi_2, \xi_3)$ in $\mathcal{E}^3$ that is associated with a point $R \in \mathcal{R}_0$ can be expressed as

$$\vec{\mathbf{V}} = V_k(\xi_1, \xi_2, \xi_3) \hat{\mathbf{g}}_k(\xi_1, \xi_2, \xi_3)$$



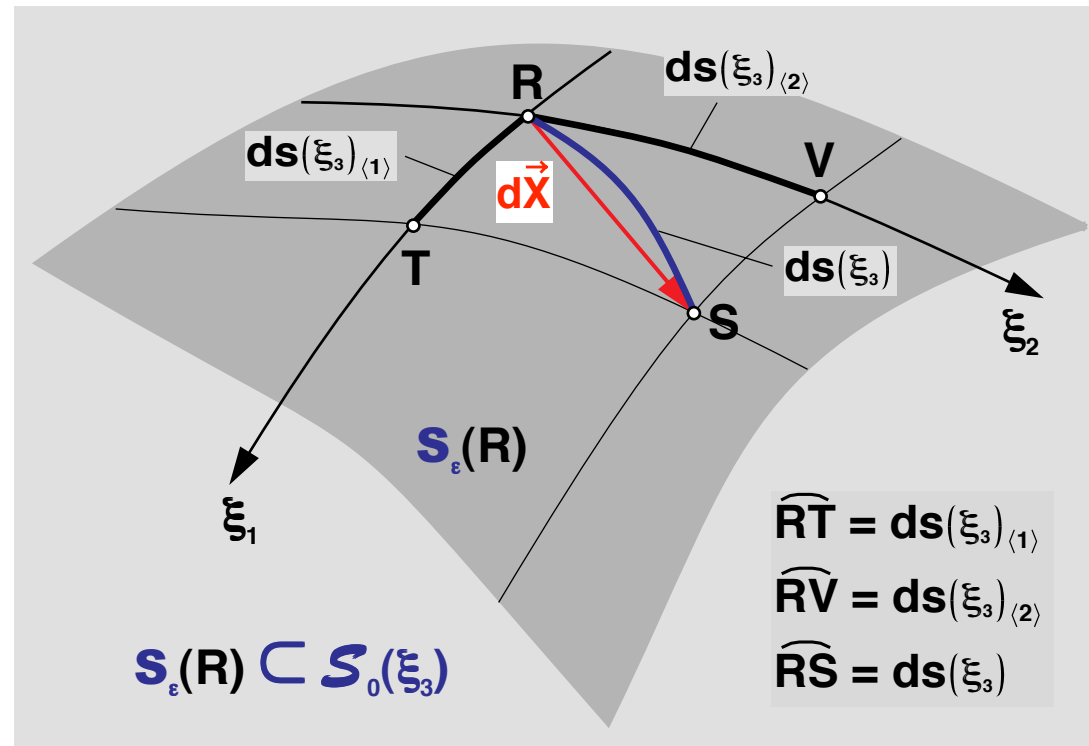
METRIC COEFFICIENTS OF A PARALLEL SURFACE

- Let $\mathbf{S}_\epsilon(R)$ denote a small, *infinitesimal neighborhood* of an arbitrary point R of a parallel surface, $\mathbf{S}_0(\xi_3)$
- Likewise, let point S be in $\mathbf{S}_\epsilon(R)$ and C denote a smooth *surface curve* that connects points R and S

- The *position vector* to point S is

$$\vec{X}(\xi_1 + d\xi_1, \xi_2 + d\xi_2, \xi_3) = \vec{X}(\xi_1, \xi_2, \xi_3) + d\vec{X}$$

- The vector from point R to point S is $d\vec{X}$ and the corresponding length of surface arc is $ds(\xi_3) = \widehat{RS}$



METRIC COEFFICIENTS OF A PARALLEL SURFACE

CONTINUED

- The arc length $ds(\xi_3)$ of the two infinitesimally close points, R and S, is given by the inner product $[ds(\xi_3)]^2 = d\vec{X} \cdot d\vec{X}$

- Noting that

$$d\vec{X} = \frac{\partial \vec{X}}{\partial \xi_\alpha} d\xi_\alpha \quad \text{it follows that} \quad [ds(\xi_3)]^2 = (\vec{g}_\alpha \cdot \vec{g}_\beta) d\xi_\alpha d\xi_\beta$$

- Using the previous definitions $H_\alpha(\xi_1, \xi_2, \xi_3) \equiv |\vec{g}_\alpha| = \sqrt{\vec{g}_\alpha \cdot \vec{g}_{(\alpha)}}$, yields

$$[ds(\xi_3)]^2 = (H_1 d\xi_1)^2 + 2H_1 H_2 \cos\Theta_{12} d\xi_1 d\xi_2 + (H_2 d\xi_2)^2$$

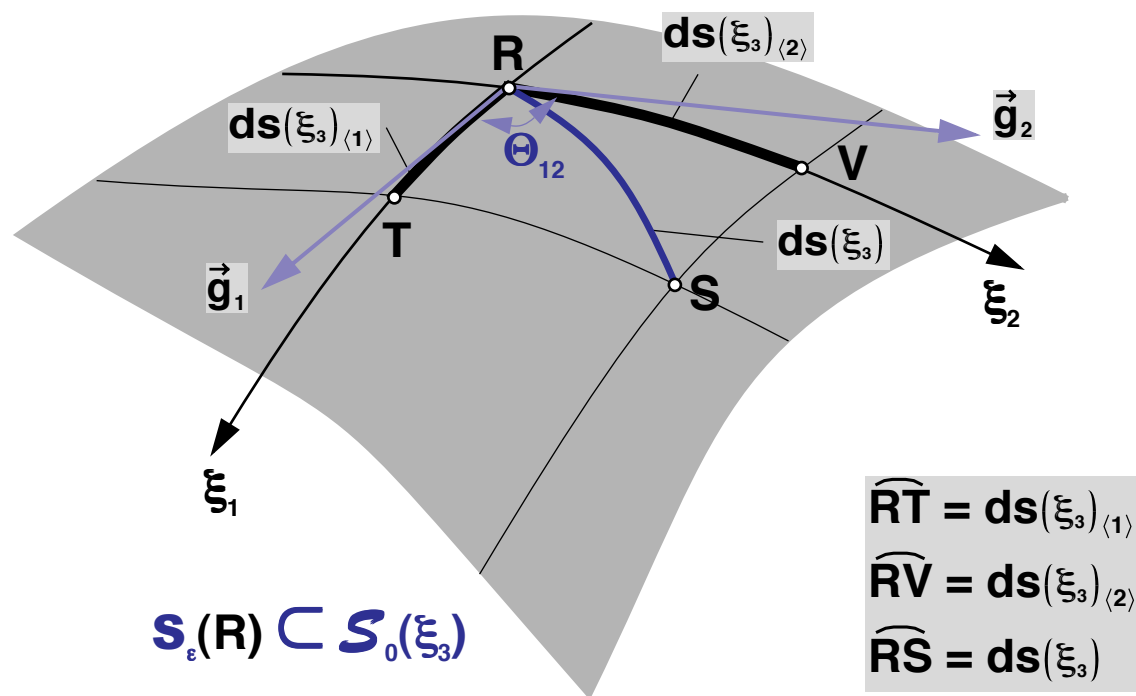
where it is noted that $H_1 H_2 \cos\Theta_{12} = \vec{g}_1 \cdot \vec{g}_2 = |\vec{g}_1| |\vec{g}_2| \cos\Theta_{12}$ and

$$\Theta_{12} = \Theta_{12}(\xi_1, \xi_2, \xi_3)$$

- The functions $H_\alpha(\xi_1, \xi_2, \xi_3)$ are called the **Lame' parameters** of the parallel surface, but are often referred to a metric coefficients

METRIC COEFFICIENTS OF A PARALLEL SURFACE CONTINUED

- $\Theta_{12}(\xi_1, \xi_2, \xi_3)$ is the angle between $\vec{g}_1(\xi_1, \xi_2, \xi_3)$ and $\vec{g}_2(\xi_1, \xi_2, \xi_3)$, as shown in the following figure



METRIC COEFFICIENTS OF A PARALLEL SURFACE CONTINUED

- Noting that $\cos\Theta_{12} = \hat{\mathbf{g}}_1 \cdot \hat{\mathbf{g}}_2$ and using $\hat{\mathbf{g}}_1 = \mu_{11} \hat{\mathbf{a}}_1 + \mu_{12} \hat{\mathbf{a}}_2$ and $\hat{\mathbf{g}}_2 = \mu_{21} \hat{\mathbf{a}}_1 + \mu_{22} \hat{\mathbf{a}}_2$ gives

$$\cos\Theta_{12} = \mu_{11}\mu_{21}(\hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}_1) + [\mu_{11}\mu_{22} + \mu_{12}\mu_{21}](\hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}_2) + \mu_{12}\mu_{22}(\hat{\mathbf{a}}_2 \cdot \hat{\mathbf{a}}_2)$$

- Then, using $\hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}_1 = 1$, $\hat{\mathbf{a}}_2 \cdot \hat{\mathbf{a}}_2 = 1$, and $\hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}_2 = \cos\theta_{12}$ gives

$$\cos\Theta_{12} = \mu_{11}\mu_{21} + \mu_{12}\mu_{22} + [\mu_{11}\mu_{22} + \mu_{12}\mu_{21}]\cos\theta_{12}$$

- Next, using the expressions for the shifters yields

$$\cos\Theta_{12} = \frac{\left(1 + \frac{\xi_3}{r_{11}}\right)\left(1 + \frac{\xi_3}{r_{22}}\right)\cos\theta_{12} + \left[\left(1 + \frac{\xi_3}{r_{11}}\right)\frac{\xi_3}{r_{21}} - \left(1 + \frac{\xi_3}{r_{22}}\right)\frac{\xi_3}{r_{12}}\right]\sin\theta_{12} + \frac{\xi_3}{r_{12}}\frac{\xi_3}{r_{21}}\cos\theta_{12}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{12}}\right)^2} \sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}}$$

METRIC COEFFICIENTS OF A PARALLEL SURFACE CONTINUED

- In addition, using

$$H_1 = A_1 \sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}}\right)^2}$$

$$H_2 = A_2 \sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}}\right)^2} \quad \text{gives}$$

$$\begin{aligned} \frac{H_1 H_2}{A_1 A_2} \cos \Theta_{12} = & \left[\left(1 + \frac{\xi_3}{r_{11}}\right) \left(1 + \frac{\xi_3}{r_{22}}\right) + \frac{\xi_3}{r_{12}} \frac{\xi_3}{r_{21}} \right] \cos \theta_{12} \\ & + \left[\left(1 + \frac{\xi_3}{r_{11}}\right) \frac{\xi_3}{r_{21}} - \left(1 + \frac{\xi_3}{r_{22}}\right) \frac{\xi_3}{r_{12}} \right] \sin \theta_{12} \end{aligned}$$

METRIC COEFFICIENTS OF A PARALLEL SURFACE

CONTINUED

- The arc length $\widehat{RS}$ is expressible as

$$\begin{aligned}
 [ds(\xi_3)]^2 = & \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}} \right)^2 \right] (A_1 d\xi_1)^2 \\
 & + \left[\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}} \right)^2 \right] (A_2 d\xi_2)^2 \\
 & + 2 \left[\left[\left(1 + \frac{\xi_3}{r_{11}} \right) \left(1 + \frac{\xi_3}{r_{22}} \right) + \frac{\xi_3}{r_{12}} \frac{\xi_3}{r_{21}} \right] \cos \theta_{12} + \left[\left(1 + \frac{\xi_3}{r_{11}} \right) \frac{\xi_3}{r_{21}} - \left(1 + \frac{\xi_3}{r_{22}} \right) \frac{\xi_3}{r_{12}} \right] \sin \theta_{12} \right] A_1 d\xi_1 A_2 d\xi_2
 \end{aligned}$$

- Along the ξ_1 -coordinate curve, ξ_2 is a constant value and, as a result $d\xi_2 = 0$; thus, $ds(\xi_3)_{\langle 1 \rangle} = H_1 d\xi_1$ or

$$ds(\xi_3)_{\langle 1 \rangle} = \sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}} \right)^2} A_1 d\xi_1$$

METRIC COEFFICIENTS OF A PARALLEL SURFACE CONTINUED

- Likewise,

$$ds(\xi_3)_{\langle 2 \rangle} = H_2 d\xi_2 = \sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}}\right)^2} A_2 d\xi_2$$

- For orthogonal Gaussian reference-surface coordinates,

$$H_1 = A_1 \sqrt{\left(1 + \frac{\xi_3}{r_{11}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}, \quad H_2 = A_2 \sqrt{\left(1 + \frac{\xi_3}{r_{22}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}, \quad \text{and}$$

$$\cos \Theta_{12} = \frac{-\frac{\xi_3}{r_{12}} \left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}}\right)}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2} \sqrt{\left(1 + \frac{\xi_3}{r_{22}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

METRIC COEFFICIENTS OF A PARALLEL SURFACE CONTINUED

- In addition,

$$[ds(\xi_3)]^2 = \left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right] (A_1 d\xi_1)^2 + \left[\left(1 + \frac{\xi_3}{r_{22}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right] (A_2 d\xi_2)^2 - 2A_1 A_2 \frac{\xi_3}{r_{12}} \left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) d\xi_1 d\xi_2,$$

$$ds(\xi_3)_{\langle 1 \rangle} = \sqrt{\left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2} A_1 d\xi_1 \quad \text{along the } \xi_1\text{-coordinate curve,}$$

and

$$ds(\xi_3)_{\langle 2 \rangle} = \sqrt{\left(1 + \frac{\xi_3}{r_{22}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2} A_2 d\xi_2 \quad \text{along the } \xi_2\text{-coordinate curve}$$

METRIC COEFFICIENTS OF A PARALLEL SURFACE CONTINUED

- The shifters reduce to

$$\mu_{11} = \frac{1 + \frac{\xi_3}{r_{11}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

$$\mu_{12} = \frac{-\frac{\xi_3}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

$$\mu_{21} = \frac{-\frac{\xi_3}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

$$\mu_{22} = \frac{1 + \frac{\xi_3}{r_{22}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

- Furthermore, the **unit-magnitude natural base vector fields** reduce to

$$\hat{\mathbf{g}}_1 = \frac{\left(1 + \frac{\xi_3}{r_{11}}\right)\hat{\mathbf{a}}_1 - \frac{\xi_3}{r_{12}}\hat{\mathbf{a}}_2}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

and

$$\hat{\mathbf{g}}_2 = \frac{\left(1 + \frac{\xi_3}{r_{22}}\right)\hat{\mathbf{a}}_2 - \frac{\xi_3}{r_{12}}\hat{\mathbf{a}}_1}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

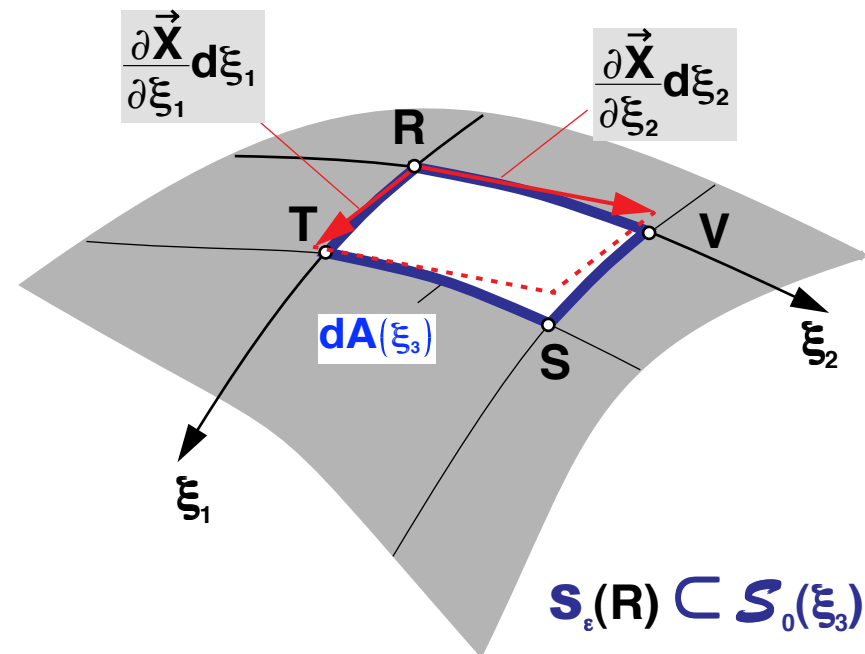
METRIC COEFFICIENTS OF A PARALLEL SURFACE CONTINUED

- Another useful quantity is the differential surface area $dA(\xi_3)$ enclosed by the arcs $\widehat{RT}$, $\widehat{RV}$, $\widehat{TS}$, and $\widehat{VS}$ shown in the figure
- To a first approximation in differentials, $dA(\xi_3)$ is given by

$$dA(\xi_3) = \left| \frac{\partial \vec{X}}{\partial \xi_1} d\xi_1 \times \frac{\partial \vec{X}}{\partial \xi_2} d\xi_2 \right|$$

- Using $\vec{g}_\alpha = \frac{\partial \vec{X}}{\partial \xi_\alpha}$ gives

$$dA(\xi_3) = \left| \vec{g}_1 \times \vec{g}_2 \right| d\xi_1 d\xi_2$$



METRIC COEFFICIENTS OF A PARALLEL SURFACE CONTINUED

- Then, using

$$\frac{\vec{g}_1 \times \vec{g}_2}{A_1 A_2} = \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) + \left(\frac{\xi_3}{r_{12}} \right) \left(\frac{\xi_3}{r_{21}} \right) \csc^2 \theta_{12} \right] \hat{n} \quad \text{gives}$$

$$dA(\xi_3) = \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) + \left(\frac{\xi_3}{r_{12}} \right) \left(\frac{\xi_3}{r_{21}} \right) \csc^2 \theta_{12} \right] A_1 A_2 d\xi_1 d\xi_2$$

- For orthogonal Gaussian reference-surface coordinates,

$$\frac{\vec{g}_1 \times \vec{g}_2}{A_1 A_2} = \left[\left(1 + \frac{\xi_3}{r_{11}} \right) \left(1 + \frac{\xi_3}{r_{22}} \right) - \left(\frac{\xi_3}{r_{12}} \right)^2 \right] \hat{n} \quad \text{and}$$

$$dA(\xi_3) = \left[\left(1 + \frac{\xi_3}{r_{11}} \right) \left(1 + \frac{\xi_3}{r_{22}} \right) - \left(\frac{\xi_3}{r_{12}} \right)^2 \right] A_1 A_2 d\xi_1 d\xi_2$$

METRIC COEFFICIENTS OF A PARALLEL SURFACE

CONTINUED

- For **orthogonal reference-surface Gaussian coordinates with negligible torsion**, $\cos\Theta_{12} = 0$; thus, the coordinate mesh on every parallel surface is also orthogonal
- This simplification is why **Principal-Curvature Coordinates**, which are orthogonal and have no torsion, are often used
- For these coordinates $\frac{1}{r_{12}} = 0$, $r_{11} \rightarrow R_1$, and $r_{22} \rightarrow R_2$
- Thus, for **Principal-Curvature Coordinates**, $\hat{g}_1 = \hat{a}_1$ and $\hat{g}_2 = \hat{a}_2$

METRIC COEFFICIENTS OF A PARALLEL SURFACE

CONCLUDED

- Likewise, for **Principal-Curvature Coordinates**,

$$[ds(\xi_3)]^2 = \left(1 + \frac{\xi_3}{R_1}\right)^2 (A_1 d\xi_1)^2 + \left(1 + \frac{\xi_3}{R_2}\right)^2 (A_2 d\xi_2)^2$$

$$ds(\xi_3)_{\langle 1 \rangle} = \left(1 + \frac{\xi_3}{R_1}\right) A_1 d\xi_1 \quad \text{and}$$

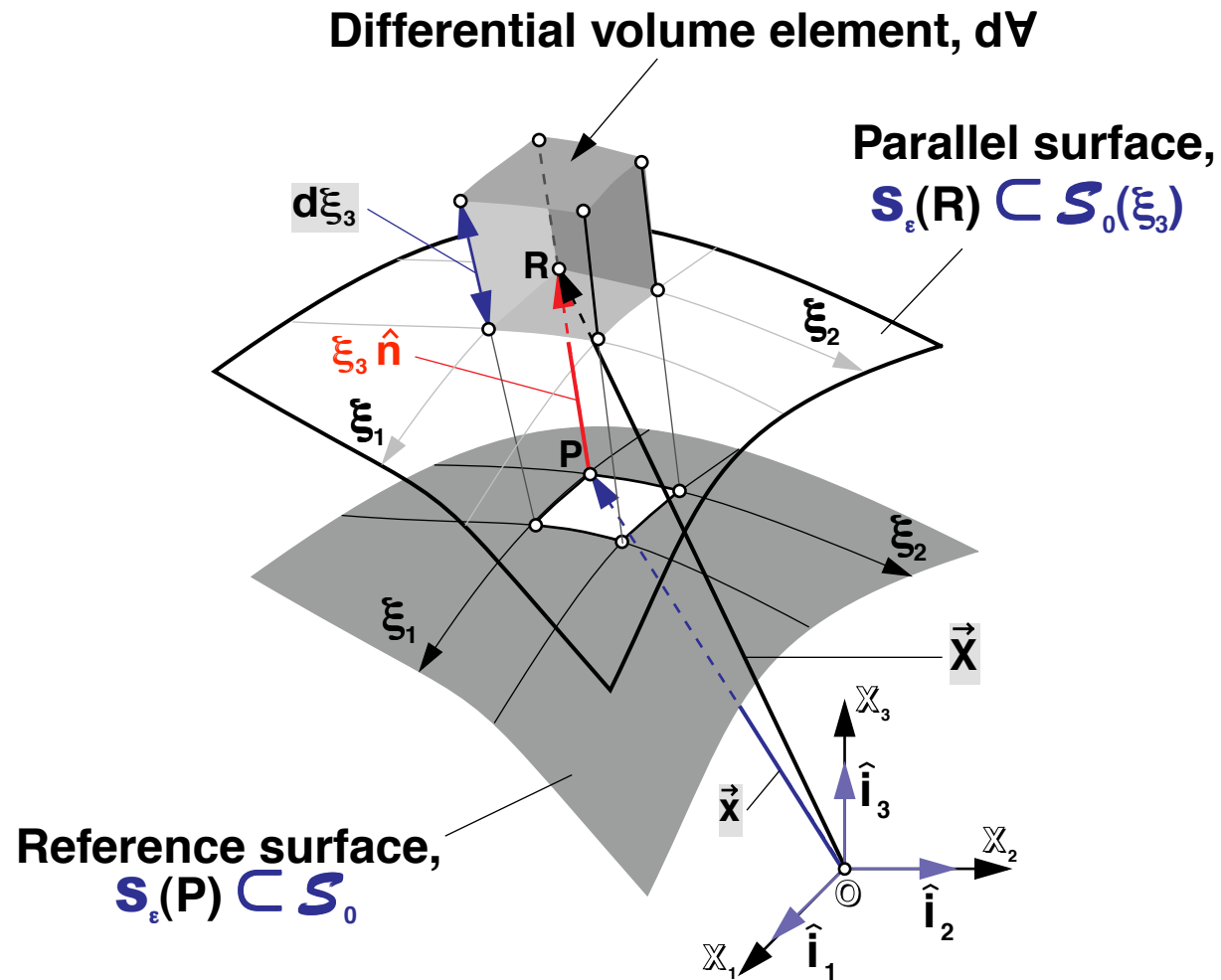
$$ds(\xi_3)_{\langle 2 \rangle} = \left(1 + \frac{\xi_3}{R_2}\right) A_2 d\xi_2 \quad \text{and}$$

$$dA(\xi_3) = \left(1 + \frac{\xi_3}{R_1}\right) \left(1 + \frac{\xi_3}{R_2}\right) A_1 A_2 d\xi_1 d\xi_2$$

METRIC COEFFICIENTS OF $\mathcal{R}_0$

- **Metric coefficients** for the undeformed-shell configuration, $\mathcal{R}_0$, are the quantities that are needed to measure the arc length of material particles forming curves, the area of material-particle surfaces, and the volumes of material-particle regions within the undeformed shell
- In particular, consider a differential element at point $R \in \mathcal{S}_0(\xi_3)$, with coordinates (ξ_1, ξ_2, ξ_3) , that is bounded by the parallel surfaces $\mathcal{S}_0(\xi_3)$ and $\mathcal{S}_0(\xi_3 + d\xi_3)$, as shown in the next figure
- $\mathcal{S}_\varepsilon(R)$ denotes a small, *infinitesimal neighborhood* of R
- The point P shown in the figure is the corresponding point of the reference surface, $\mathcal{S}_0$, and $\mathcal{S}_\varepsilon(P)$ denotes a corresponding small, *infinitesimal neighborhood* of the point P

METRIC COEFFICIENTS OF $\mathcal{R}_0$ - CONTINUED



METRIC COEFFICIENTS OF $\mathcal{R}_0$ - CONTINUED

- The position vectors to points P and R are $\vec{x} = x_k(\xi_1, \xi_2) \hat{i}_k$ and $\vec{X}(\xi_1, \xi_2, \xi_3) = \vec{x}(\xi_1, \xi_2) + \xi_3 \hat{n}(\xi_1, \xi_2)$, respectively

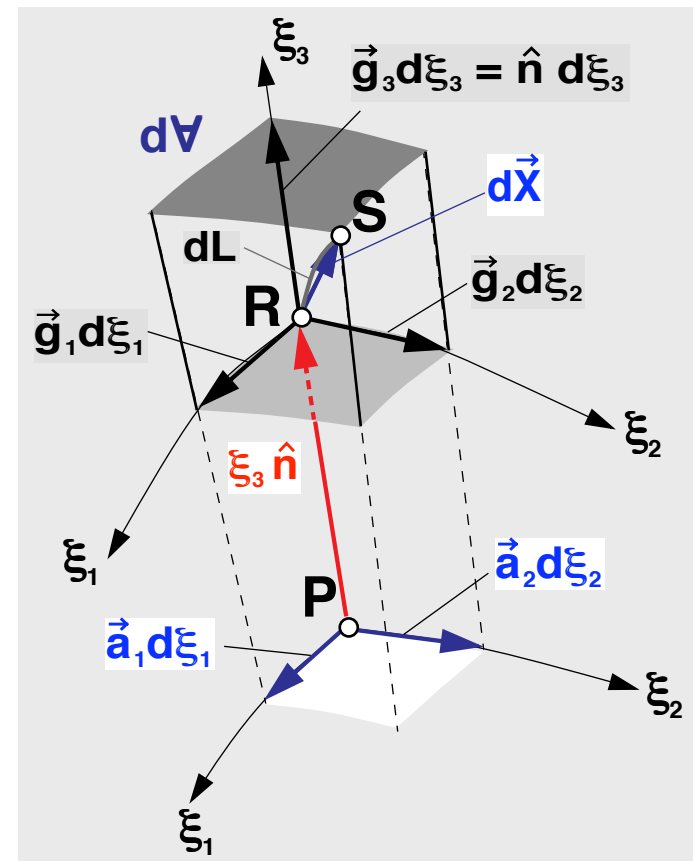
- The position vector to point S is

$$\vec{X}(\xi_1 + d\xi_1, \xi_2 + d\xi_2, \xi_3 + d\xi_3) = \vec{X}(\xi_1, \xi_2, \xi_3) + d\vec{X}$$

- The vector from point R to point S is $d\vec{X}$
- The arc length of the material curve between points R and S is

$$dL = \widehat{RS} = |d\vec{X}|,$$

to a first approximation in differentials



METRIC COEFFICIENTS OF $\mathcal{R}_0$ - CONTINUED

- The **chain rule of differentiation** gives

$$d\vec{X}(\xi_1, \xi_2, \xi_3) = \frac{\partial \vec{X}}{\partial \xi_k} d\xi_k = \vec{g}_k(\xi_1, \xi_2, \xi_3) d\xi_k$$

- From $dL = |d\vec{X}|$, it follows that $dL^2 = d\vec{X} \cdot d\vec{X} = (\vec{g}_j \cdot \vec{g}_k) d\xi_j d\xi_k$
- Using $H_k(\xi_1, \xi_2, \xi_3) = \sqrt{\vec{g}_k \cdot \vec{g}_{(k)}}$ and the presumption of nonorthogonal coordinates on $\mathcal{R}_0$ gives the general form

$$dL^2 = (H_1 d\xi_1)^2 + 2H_1 H_2 \cos\Theta_{12} d\xi_1 d\xi_2 + (H_2 d\xi_2)^2 + 2H_1 H_3 \cos\Theta_{13} d\xi_1 d\xi_3 + 2H_2 H_3 \cos\Theta_{23} d\xi_2 d\xi_3 + (H_3 d\xi_3)^2$$

- In this equation, $\Theta_{jk}(\xi_1, \xi_2, \xi_3)$ are the angles between the generally nonorthogonal intersecting coordinate curves of the undeformed shell that are defined by $H_j H_k \cos\Theta_{(jk)} = \vec{g}_j \cdot \vec{g}_k$

METRIC COEFFICIENTS OF $\mathcal{R}_0$ - CONCLUDED

- Next, recall that $\vec{g}_3 = \hat{n}(\xi_1, \xi_2)$ and $H_3 = \sqrt{\vec{g}_3 \cdot \vec{g}_3} = 1$ for the shell space defined by $\vec{X}(\xi_1, \xi_2, \xi_3) = \vec{x}(\xi_1, \xi_2) + \xi_3 \hat{n}(\xi_1, \xi_2)$

- In addition, the vectors $\vec{g}_1$ and $\vec{g}_2$ are perpendicular to $\vec{g}_3$; thus,

$$H_1 H_3 \cos \Theta_{13} = \vec{g}_1 \cdot \vec{g}_3 = 0 \quad \text{and} \quad H_2 H_3 \cos \Theta_{23} = \vec{g}_2 \cdot \vec{g}_3 = 0$$

- Therefore, $\Theta_{13} = \Theta_{23} = \frac{\pi}{2}$ and the general expression for the arc length, $dL \subset \mathcal{R}_0$, of the material curve between points R and S reduces to

$$dL^2 = (H_1 d\xi_1)^2 + 2H_1 H_2 \cos \Theta_{12} d\xi_1 d\xi_2 + (H_2 d\xi_2)^2 + (d\xi_3)^2$$

for this very special coordinate system

DIFFERENTIAL AREAS AND VOLUMES IN $\mathcal{R}_0$

- The differential area of the face $\xi_2 = 0$ of the differential volume element shown in the figure is given by

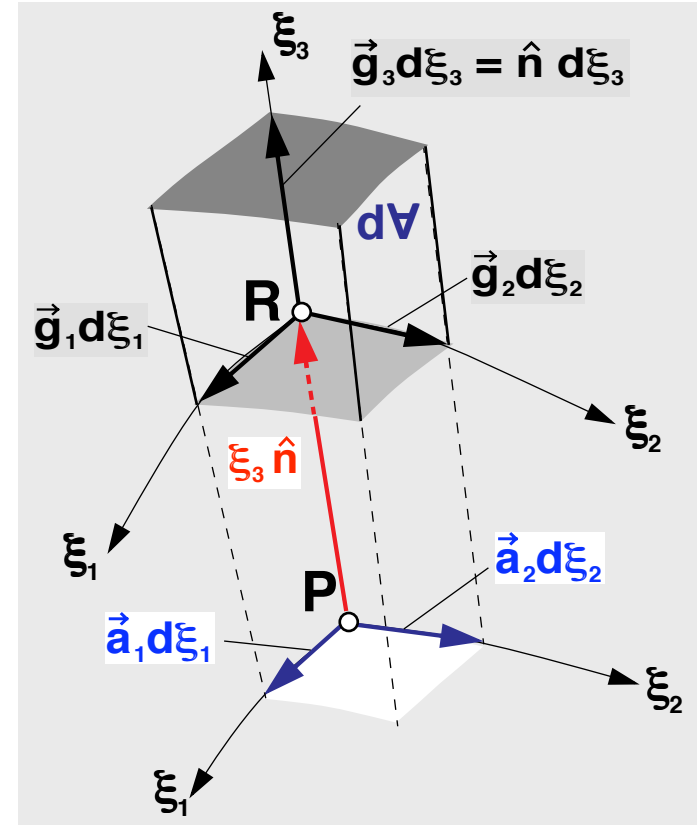
$$dA_{\langle 2 \rangle} = \left| \frac{\partial \vec{X}}{\partial \xi_1} d\xi_1 \times \frac{\partial \vec{X}}{\partial \xi_3} d\xi_3 \right|,$$

to a first approximation in differentials

- Substituting $\vec{g}_k = \frac{\partial \vec{X}}{\partial \xi_k}$ into the previous expression gives

$$dA_{\langle 2 \rangle} = \left| \vec{g}_1 \times \vec{g}_3 \right| d\xi_1 d\xi_3$$

- Next, $\vec{g}_1 = A_1 \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \hat{a}_1 - \frac{\xi_3 \csc \theta_{12}}{r_{12}} \hat{a}_2 \right]$ and $\vec{g}_3 = \hat{n}$ are used



DIFFERENTIAL AREAS AND VOLUMES IN $\mathcal{R}_0$

CONTINUED

to get
$$dA_{\langle 2 \rangle} = A_1 \sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}}\right)^2} d\xi_1 d\xi_3$$

- Similarly, the differential area of the face of the differential element given by $\xi_1 = 0$ is given by

$$dA_{\langle 1 \rangle} = \left| \frac{\partial \vec{X}}{\partial \xi_2} d\xi_2 \times \frac{\partial \vec{X}}{\partial \xi_3} d\xi_3 \right| = \left| \vec{g}_2 \times \vec{g}_3 \right| d\xi_2 d\xi_3$$

- Then,
$$\vec{g}_2 = A_2 \left[\frac{\xi_3 \csc \theta_{12}}{r_{21}} \hat{a}_1 + \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right) \hat{a}_2 \right]$$
 and $\vec{g}_3 = \hat{n}$ are used to get

$$dA_{\langle 1 \rangle} = A_2 \sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}}\right)^2} d\xi_2 d\xi_3$$

DIFFERENTIAL AREAS AND VOLUMES IN $\mathcal{R}_0$

CONTINUED

- The volume of the differential element is given by

$$dV = (\vec{g}_1 \times \vec{g}_2) \cdot \vec{g}_3 d\xi_1 d\xi_2 d\xi_3$$

- Using $\frac{\vec{g}_1 \times \vec{g}_2}{A_1 A_2} = \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}} \right) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}} \right) + \left(\frac{\xi_3}{r_{12}} \right) \left(\frac{\xi_3}{r_{21}} \right) \csc^2\theta_{12} \right] \hat{n}$

and $\vec{g}_3 = \hat{n}$ gives

$$dV = \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}} \right) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}} \right) + \left(\frac{\xi_3}{r_{12}} \right) \left(\frac{\xi_3}{r_{21}} \right) \csc^2\theta_{12} \right] A_1 A_2 d\xi_1 d\xi_2 d\xi_3$$

DIFFERENTIAL AREAS AND VOLUMES IN $\mathcal{R}_0$

CONTINUED

- For orthogonal reference-surface Gaussian coordinates,

$$dA_{\langle 1 \rangle} = A_2 \sqrt{\left(1 + \frac{\xi_3}{r_{22}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2} d\xi_2 d\xi_3$$

$$dA_{\langle 2 \rangle} = A_1 \sqrt{\left(1 + \frac{\xi_3}{r_{11}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2} d\xi_1 d\xi_3$$

$$dV = \left[\left(1 + \frac{\xi_3}{r_{11}}\right) \left(1 + \frac{\xi_3}{r_{22}}\right) - \left(\frac{\xi_3}{r_{12}}\right)^2 \right] A_1 A_2 d\xi_1 d\xi_2 d\xi_3$$

DIFFERENTIAL AREAS AND VOLUMES IN $\mathcal{R}_0$

CONCLUDED

- For principal-curvature Gaussian coordinates,

$$dA_{\langle 1 \rangle} = A_2 \left(1 + \frac{\xi_3}{R_2} \right) d\xi_2 d\xi_3$$

$$dA_{\langle 2 \rangle} = A_1 \left(1 + \frac{\xi_3}{R_1} \right) d\xi_1 d\xi_3$$

$$dV = \left(1 + \frac{\xi_3}{R_1} \right) \left(1 + \frac{\xi_3}{R_2} \right) A_1 A_2 d\xi_1 d\xi_2 d\xi_3$$

RECIPROCAL BASIS FOR $\mathcal{R}_0$

- A reciprocal basis is an algebraic construct that is particularly useful when **the curvilinear coordinates for $\mathcal{R}_0$ are non-orthogonal**
- They are defined to provide orthogonality relations that simplify algebraic computations

- Let the vector fields $\{\vec{g}^1, \vec{g}^2, \vec{g}^3\}$ be defined such that

$$\vec{g}_1 \times \vec{g}_2 = \Lambda_3 \vec{g}^3$$

$$\vec{g}_3 \times \vec{g}_1 = \Lambda_2 \vec{g}^2$$

$$\vec{g}_2 \times \vec{g}_3 = \Lambda_1 \vec{g}^1$$

and $\vec{g}^m \cdot \vec{g}_n = \delta_n^m$, where δ_n^m is the **Kronecker delta symbol**

- Since $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ form a basis at each point of $\mathcal{R}_0$, it follows that $\{\vec{g}^1, \vec{g}^2, \vec{g}^3\}$ can be expressed by the **linear combinations** $\vec{g}^k = g^{kp} \vec{g}_p$

RECIPROCAL BASIS FOR $\mathcal{R}_0$ - CONTINUED

- The **orthogonality relations** $\vec{g}^m \cdot \vec{g}_n = \delta_n^m$ and $\vec{g}^k = g^{kp} \vec{g}_p$ give

$$\vec{g}^k \cdot \vec{g}^n = g^{kp} \vec{g}_p \cdot \vec{g}^n = g^{kp} \delta_p^n = g^{kn}$$
- In addition, applying the **commutative property** of dot-product multiplication to $\vec{g}^k \cdot \vec{g}^n = g^{kn}$ yields the **symmetry property** $g^{kn} = g^{nk}$
- Next, the orthogonality relations $\vec{g}^m \cdot \vec{g}_n = \delta_n^m$ are applied to the previous cross-product definitions to get

$$(\vec{g}_2 \times \vec{g}_3) \cdot \vec{g}_1 = \Lambda_1 \vec{g}^1 \cdot \vec{g}_1 = \Lambda_1$$

$$(\vec{g}_3 \times \vec{g}_1) \cdot \vec{g}_2 = \Lambda_2 \vec{g}^2 \cdot \vec{g}_2 = \Lambda_2$$

$$(\vec{g}_1 \times \vec{g}_2) \cdot \vec{g}_3 = \Lambda_3 \vec{g}^3 \cdot \vec{g}_3 = \Lambda_3$$

RECIPROCAL BASIS FOR $\mathcal{R}_0$ - CONTINUED

- Noting that $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$ for any three distinct vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ it follows that

$$(\vec{g}_1 \times \vec{g}_2) \cdot \vec{g}_3 = (\vec{g}_2 \times \vec{g}_3) \cdot \vec{g}_1 = (\vec{g}_3 \times \vec{g}_1) \cdot \vec{g}_2$$

- Applying the equalities to

$$(\vec{g}_2 \times \vec{g}_3) \cdot \vec{g}_1 = \Lambda_1 \vec{g}^1 \cdot \vec{g}_1 = \Lambda_1$$

$$(\vec{g}_3 \times \vec{g}_1) \cdot \vec{g}_2 = \Lambda_2 \vec{g}^2 \cdot \vec{g}_2 = \Lambda_2$$

$$(\vec{g}_1 \times \vec{g}_2) \cdot \vec{g}_3 = \Lambda_3 \vec{g}^3 \cdot \vec{g}_3 = \Lambda_3$$

yields $\Lambda_1 = \Lambda_2 = \Lambda_3 \equiv \Lambda$, which is determined as follows

RECIPROCAL BASIS FOR $\mathcal{R}_0$ - CONTINUED

- First, consider the Cartesian representations of the basis $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$, given by $\vec{g}_k = \frac{\partial X_p}{\partial \xi_k} \hat{i}_p$
- For any three distinct vectors defined by $\vec{a} = a_p \hat{i}_p$, $\vec{b} = b_p \hat{i}_p$, and $\vec{c} = c_p \hat{i}_p$ the product $(\vec{a} \times \vec{b}) \cdot \vec{c}$ has the following determinant representation

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}; \text{ thus,}$$

$$(\vec{g}_1 \times \vec{g}_2) \cdot \vec{g}_3 = \begin{vmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_1} \\ \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_2} \\ \frac{\partial X_1}{\partial \xi_3} & \frac{\partial X_2}{\partial \xi_3} & \frac{\partial X_3}{\partial \xi_3} \end{vmatrix}$$

RECIPROCAL BASIS FOR $\mathcal{R}_0$ - CONTINUED

- Next, consider the general expression for **nonorthogonal coordinates** on $\mathcal{R}_0$ given by

$$dL^2 = (H_1 d\xi_1)^2 + 2H_1 H_2 \cos\Theta_{12} d\xi_1 d\xi_2 + (H_2 d\xi_2)^2 + 2H_1 H_3 \cos\Theta_{13} d\xi_1 d\xi_3 + 2H_2 H_3 \cos\Theta_{23} d\xi_2 d\xi_3 + (H_3 d\xi_3)^2$$

- This expression is re-written in matrix notation as the **quadratic form**

$$dL^2 = \begin{Bmatrix} d\xi_1 & d\xi_2 & d\xi_3 \end{Bmatrix} \begin{bmatrix} (H_1)^2 & H_1 H_2 \cos\Theta_{12} & H_1 H_3 \cos\Theta_{13} \\ H_1 H_2 \cos\Theta_{12} & (H_2)^2 & H_2 H_3 \cos\Theta_{23} \\ H_1 H_3 \cos\Theta_{13} & H_2 H_3 \cos\Theta_{23} & (H_3)^2 \end{bmatrix} \begin{Bmatrix} d\xi_1 \\ d\xi_2 \\ d\xi_3 \end{Bmatrix}$$

- The expression for the square of the arc length also has the Cartesian form $dL^2 = (dX_1)^2 + (dX_2)^2 + (dX_3)^2$

RECIPROCAL BASIS FOR $\mathcal{R}_0$ - CONTINUED

- The Cartesian form is expressed in matrix notation as

$$dL^2 = \left\{ dX_1 \ dX_2 \ dX_3 \right\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{Bmatrix}$$

- Next, the chain rule of differentiation gives

$$\begin{Bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{Bmatrix} = \begin{bmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_1}{\partial \xi_3} \\ \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_3} \\ \frac{\partial X_3}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_3} \end{bmatrix}^T \begin{Bmatrix} d\xi_1 \\ d\xi_2 \\ d\xi_3 \end{Bmatrix}$$

where the matrix superscript denotes transposition

RECIPROCAL BASIS FOR $\mathcal{R}_0$ - CONTINUED

- The Cartesian form is now expressed as

$$dL^2 = \left\{ d\xi_1 \ d\xi_2 \ d\xi_3 \right\} \begin{bmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_1} \\ \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_2} \\ \frac{\partial X_1}{\partial \xi_3} & \frac{\partial X_2}{\partial \xi_3} & \frac{\partial X_3}{\partial \xi_3} \end{bmatrix} \begin{bmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_1} \\ \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_2} \\ \frac{\partial X_1}{\partial \xi_3} & \frac{\partial X_2}{\partial \xi_3} & \frac{\partial X_3}{\partial \xi_3} \end{bmatrix}^T \begin{Bmatrix} d\xi_1 \\ d\xi_2 \\ d\xi_3 \end{Bmatrix}$$

- Comparing this form with the similar previous one yields

$$\begin{bmatrix} (H_1)^2 & H_1 H_2 \cos \Theta_{12} & H_1 H_3 \cos \Theta_{13} \\ H_1 H_2 \cos \Theta_{12} & (H_2)^2 & H_2 H_3 \cos \Theta_{23} \\ H_1 H_3 \cos \Theta_{13} & H_2 H_3 \cos \Theta_{23} & (H_3)^2 \end{bmatrix} = \begin{bmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_1} \\ \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_2} \\ \frac{\partial X_1}{\partial \xi_3} & \frac{\partial X_2}{\partial \xi_3} & \frac{\partial X_3}{\partial \xi_3} \end{bmatrix} \begin{bmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_1} \\ \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_2} \\ \frac{\partial X_1}{\partial \xi_3} & \frac{\partial X_2}{\partial \xi_3} & \frac{\partial X_3}{\partial \xi_3} \end{bmatrix}^T$$

RECIPROCAL BASIS FOR $\mathcal{R}_0$ - CONTINUED

- Noting that the determinant of a matrix is identical to the determinant of its transpose gives

$$\begin{vmatrix} (H_1)^2 & H_1 H_2 \cos \Theta_{12} & H_1 H_3 \cos \Theta_{13} \\ H_1 H_2 \cos \Theta_{12} & (H_2)^2 & H_2 H_3 \cos \Theta_{23} \\ H_1 H_3 \cos \Theta_{13} & H_2 H_3 \cos \Theta_{23} & (H_3)^2 \end{vmatrix} = \begin{vmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_1} \\ \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_2} \\ \frac{\partial X_1}{\partial \xi_3} & \frac{\partial X_2}{\partial \xi_3} & \frac{\partial X_3}{\partial \xi_3} \end{vmatrix}^2$$

- Now, let

$$\mathbf{H}^2 = \begin{vmatrix} (H_1)^2 & H_1 H_2 \cos \Theta_{12} & H_1 H_3 \cos \Theta_{13} \\ H_1 H_2 \cos \Theta_{12} & (H_2)^2 & H_2 H_3 \cos \Theta_{23} \\ H_1 H_3 \cos \Theta_{13} & H_2 H_3 \cos \Theta_{23} & (H_3)^2 \end{vmatrix} \quad \text{to get} \quad \mathbf{H} = \begin{vmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_1} \\ \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_2} \\ \frac{\partial X_1}{\partial \xi_3} & \frac{\partial X_2}{\partial \xi_3} & \frac{\partial X_3}{\partial \xi_3} \end{vmatrix}$$

RECIPROCAL BASIS FOR $\mathcal{R}_0$ - CONTINUED

- Then

$$\left(\vec{g}_1 \times \vec{g}_2 \right) \cdot \vec{g}_3 = \begin{vmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_1} \\ \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_2} \\ \frac{\partial X_1}{\partial \xi_3} & \frac{\partial X_2}{\partial \xi_3} & \frac{\partial X_3}{\partial \xi_3} \end{vmatrix} \quad \text{and} \quad \mathbf{H} = \begin{vmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_1} \\ \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_2} \\ \frac{\partial X_1}{\partial \xi_3} & \frac{\partial X_2}{\partial \xi_3} & \frac{\partial X_3}{\partial \xi_3} \end{vmatrix} \quad \text{give} \quad \Lambda = H$$

and

$$\left(\vec{g}_1 \times \vec{g}_2 \right) \cdot \vec{g}_3 = \left(\vec{g}_2 \times \vec{g}_3 \right) \cdot \vec{g}_1 = \left(\vec{g}_3 \times \vec{g}_1 \right) \cdot \vec{g}_2 = H$$

- The explicit form for H is given by

$$H = H_1 H_2 H_3 \sqrt{\left[\sin^2 \Theta_{12} + \sin^2 \Theta_{13} + \sin^2 \Theta_{23} + 2(\cos \Theta_{12} \cos \Theta_{13} \cos \Theta_{23} - 1) \right]}$$

RECIPROCAL BASIS FOR $\mathcal{R}_0$ - CONTINUED

- From $\vec{g}^m \cdot \vec{g}_n = \delta_n^m$ and $\vec{g}^k = g^{kp} \vec{g}_p$, it follows that $g^{kp}(\vec{g}_p \cdot \vec{g}_n) = \delta_n^m$
- The matrix form of $g^{kp}(\vec{g}_p \cdot \vec{g}_n) = \delta_n^m$ is given by

$$\begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{12} & g^{22} & g^{23} \\ g^{13} & g^{23} & g^{33} \end{bmatrix} \begin{bmatrix} \vec{g}_1 \cdot \vec{g}_1 & \vec{g}_1 \cdot \vec{g}_2 & \vec{g}_1 \cdot \vec{g}_3 \\ \vec{g}_2 \cdot \vec{g}_1 & \vec{g}_2 \cdot \vec{g}_2 & \vec{g}_2 \cdot \vec{g}_3 \\ \vec{g}_3 \cdot \vec{g}_1 & \vec{g}_3 \cdot \vec{g}_2 & \vec{g}_3 \cdot \vec{g}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Thus,
$$\begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{12} & g^{22} & g^{23} \\ g^{13} & g^{23} & g^{33} \end{bmatrix} = \begin{bmatrix} \vec{g}_1 \cdot \vec{g}_1 & \vec{g}_1 \cdot \vec{g}_2 & \vec{g}_1 \cdot \vec{g}_3 \\ \vec{g}_2 \cdot \vec{g}_1 & \vec{g}_2 \cdot \vec{g}_2 & \vec{g}_2 \cdot \vec{g}_3 \\ \vec{g}_3 \cdot \vec{g}_1 & \vec{g}_3 \cdot \vec{g}_2 & \vec{g}_3 \cdot \vec{g}_3 \end{bmatrix}^{-1}$$

RECIPROCAL BASIS FOR $\mathcal{R}_0$ - CONTINUED

- Using $H_j H_k \cos \Theta_{(jk)} = \vec{g}_j \cdot \vec{g}_k$, it follows that

$$\begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{12} & g^{22} & g^{23} \\ g^{13} & g^{23} & g^{33} \end{bmatrix} = \begin{bmatrix} (H_1)^2 & H_1 H_2 \cos \Theta_{12} & H_1 H_3 \cos \Theta_{13} \\ H_1 H_2 \cos \Theta_{12} & (H_2)^2 & H_2 H_3 \cos \Theta_{23} \\ H_1 H_3 \cos \Theta_{13} & H_2 H_3 \cos \Theta_{23} & (H_3)^2 \end{bmatrix}^{-1}$$

- Since $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ form a basis at each point of $\mathcal{R}_0$, and $\vec{g}^k = g^{kp} \vec{g}_p$ yields **three distinct linear combinations** of $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$, it follows that $\{\vec{g}^1, \vec{g}^2, \vec{g}^3\}$ **is also a basis** at each point of $\mathcal{R}_0$

RECIPROCAL BASIS FOR $\mathcal{R}_0$ - CONTINUED

- Now, recalling that $H_3 = 1$ and $\Theta_{13} = \Theta_{23} = \frac{\pi}{2}$ for the (ξ_1, ξ_2, ξ_3) coordinate system based on parallel surfaces, yields $H = H_1 H_2 \sin \Theta_{12}$

- From
$$\frac{\vec{g}_1 \times \vec{g}_2}{A_1 A_2} = \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) + \left(\frac{\xi_3}{r_{12}} \right) \left(\frac{\xi_3}{r_{21}} \right) \csc^2 \theta_{12} \right] \hat{n}$$

and $H_1 H_2 \sin \Theta_{12} = |\vec{g}_1 \times \vec{g}_2|$, it follows that

$$H = A_1 A_2 \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) + \left(\frac{\xi_3}{r_{12}} \right) \left(\frac{\xi_3}{r_{21}} \right) \csc^2 \theta_{12} \right]$$

- Note that the volume of the differential element given by

$$dV = (\vec{g}_1 \times \vec{g}_2) \cdot \vec{g}_3 d\xi_1 d\xi_2 d\xi_3 \quad \text{becomes} \quad dV = H d\xi_1 d\xi_2 d\xi_3$$

RECIPROCAL BASIS FOR $\mathcal{R}_0$ - CONTINUED

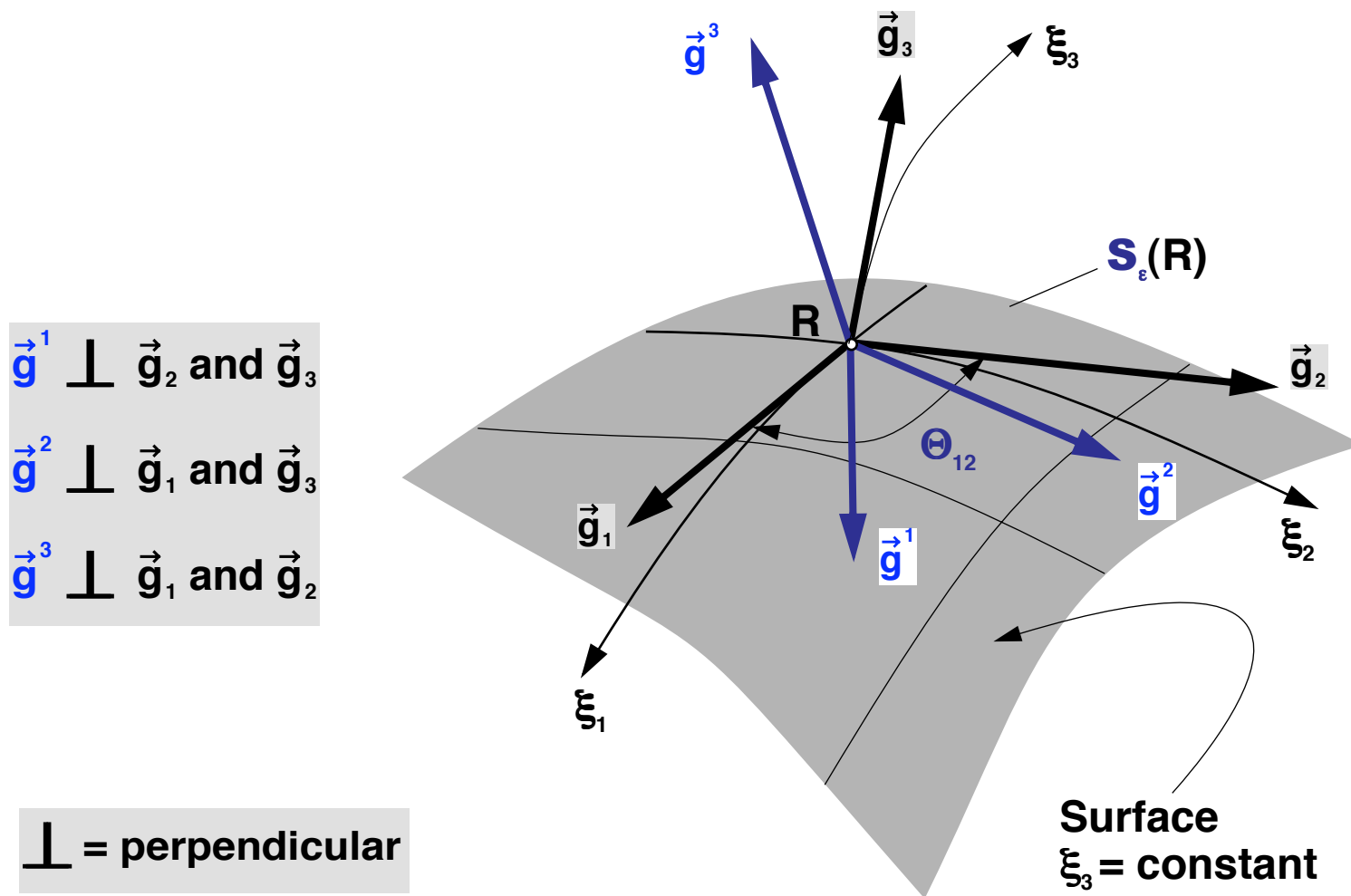
- Additionally,

$$\begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{12} & g^{22} & g^{23} \\ g^{13} & g^{23} & g^{33} \end{bmatrix} = \begin{bmatrix} (H_1)^2 & H_1 H_2 \cos \Theta_{12} & 0 \\ H_1 H_2 \cos \Theta_{12} & (H_2)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \quad \text{which yields}$$

$$g^{13} = g^{23} = 0, \quad g^{33} = 1, \quad \text{and}$$

$$\begin{bmatrix} g^{11} & g^{12} \\ g^{12} & g^{22} \end{bmatrix} = \frac{1}{H^2} \begin{bmatrix} (H_2)^2 & -H_1 H_2 \cos \Theta_{12} \\ -H_1 H_2 \cos \Theta_{12} & (H_1)^2 \end{bmatrix}$$

RECIPROCAL BASIS FOR $\mathcal{R}_0$ - CONCLUDED



MATHEMATICAL DESCRIPTION OF THE DEFORMED-SHELL REFERENCE-SURFACE GEOMETRY

MATHEMATICAL DESCRIPTION OF THE DEFORMED-SHELL REFERENCE-SURFACE GEOMETRY

- **Previously, it was shown that the properties of a parallel surface within the shell space can be given in terms of the properties of a corresponding reference surface**
- **In a similar manner, the deformation of the shell space will be related herein to the deformation of the corresponding reference surface**

THE DEFORMED REFERENCE SURFACE, $\mathcal{S}_t$

- Let $\mathcal{S}_t$ be a smooth surface that is obtained by **displacing and deforming** the initial smooth undeformed reference surface $\mathcal{S}_0$ in some manner
- $\mathcal{S}_t$ is referred to herein as the **deformed reference surface** and it occupies a subset of the same Euclidean space $\mathcal{E}^3$ at time $t > 0$
- The undeformed configuration of $\mathcal{S}_t$ is the surface $\mathcal{S}_0$ and herein it corresponds to time $t = 0$
- The Gaussian coordinates of the reference surface $\mathcal{S}_0$ are presumed to be **nonorthogonal**
- Points of the deformed surface have the Cartesian coordinates (x_1, x_2, x_3) , with respect to the same $\mathcal{O} - X_1 - X_2 - X_3$ coordinate frame

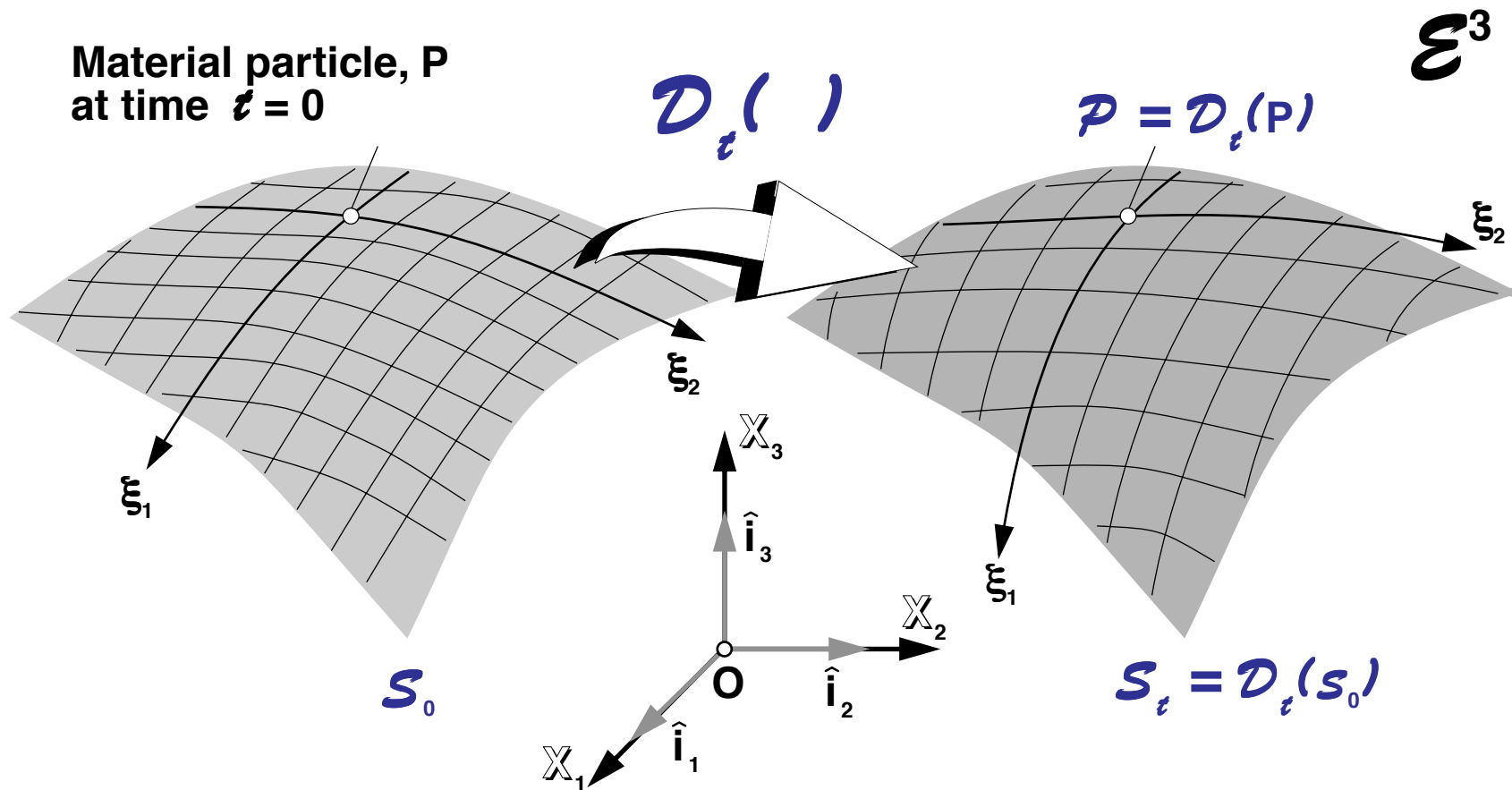
THE DEFORMED REFERENCE SURFACE, $\mathcal{S}_t$ - CONTINUED

- By using the **Lagrangian description** of motion and deformation, it follows that $\mathbf{x}_k = \mathbf{x}_k(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t)$, and as a result, the **deformed surface** has the parametric representation $\mathbf{x}_k = \mathbf{x}_k(\xi_1, \xi_2, t)$
- In addition, the **Gaussian coordinates** (ξ_1, ξ_2) of a point $P \in \mathcal{S}_0$ become coordinates for the image of P in the deformed configuration; that is, $\mathcal{P} \in \mathcal{S}_t$
- As deformation progresses, the **Gaussian coordinate net** associated with the undeformed surface $\mathcal{S}_0$ gets convected through $\mathcal{E}^3$ as part of the deformed image of $\mathcal{S}_0$, that is, the surface $\mathcal{S}_t$
- It is important to observe that the **convected Gaussian coordinates** are generally nonorthogonal, even if they are orthogonal at $t = 0$

THE DEFORMED REFERENCE SURFACE, $\mathcal{S}_t$ - CONTINUED

- This convected nature of the **Gaussian coordinate net** enables one to define vector fields (and tensor fields) associated with the deformed surface in a manner analogous to that for the undeformed surface $\mathcal{S}_0$
- Mathematically, $\mathbf{x}_k = \mathbf{x}_k(\xi_1, \xi_2, t)$ represents an invertible mapping of the undeformed surface $\mathcal{S}_0$ onto the deformed surface $\mathcal{S}_t$
 - This mapping is referred to herein as the **deformation mapping** and is denoted herein by $\mathcal{D}_t(\cdot)$
 - The invertibility of the mapping is inherent because of the physical requirements that every material point P of the undeformed surface $\mathcal{S}_0$ possess only one material-point image in the corresponding deformed configuration at a given time $t > 0$, and that no point of $\mathcal{S}_0$ have more than one image in $\mathcal{S}_t$, and vice versa

THE DEFORMED REFERENCE SURFACE, $\mathcal{S}_t$ - CONCLUDED



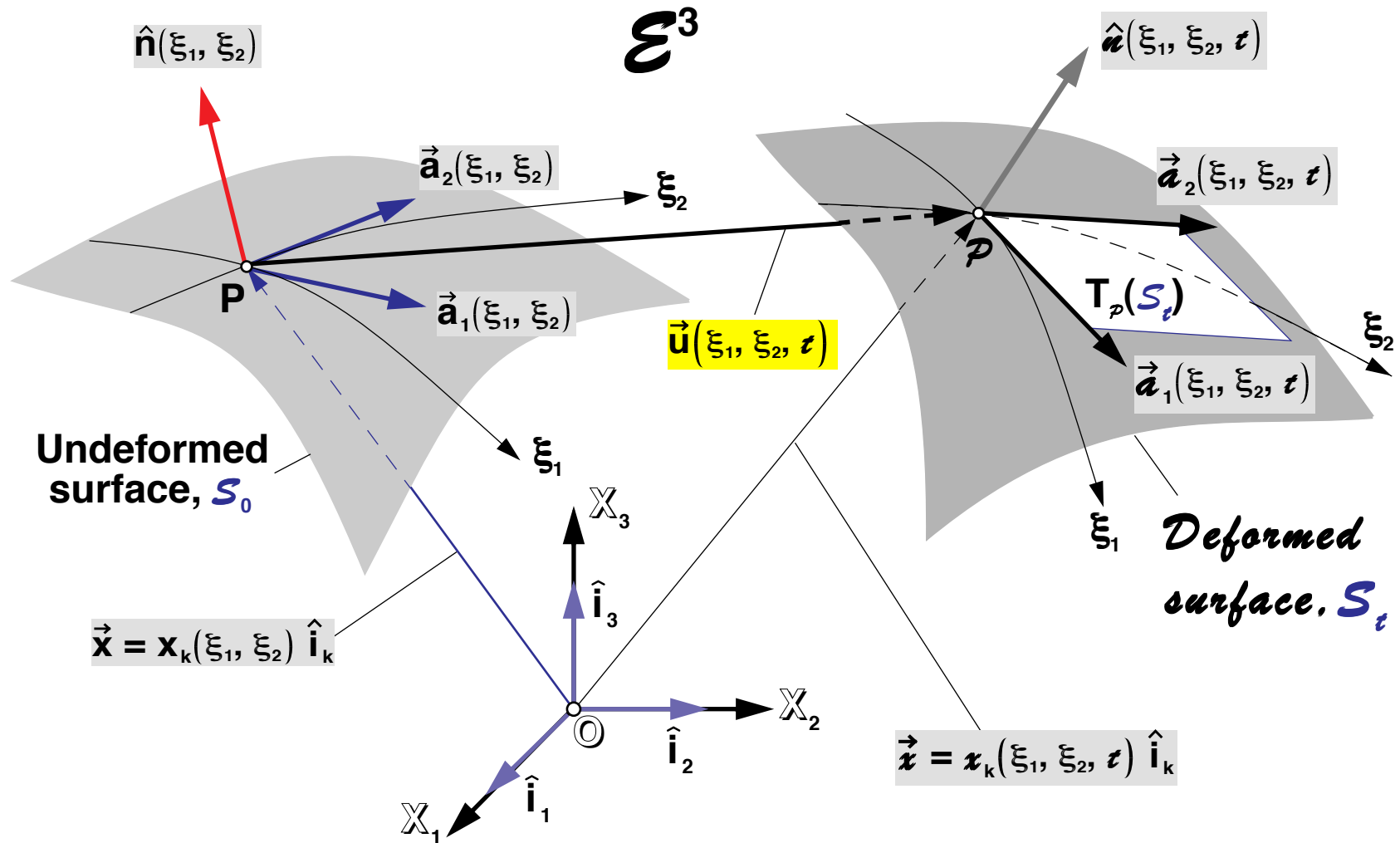
- Note that $\mathcal{D}_0(\cdot)$ is an **identity mapping**, which yields $\mathcal{S}_0 = \mathcal{D}_0(\mathcal{S}_0)$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_t$

- To characterize the geometry of the deformed reference surface, and any associated vector fields, such as material-point velocity and acceleration, it is convenient to introduce the time-dependent position vector $\vec{x} = x_k(\xi_1, \xi_2, t) \hat{i}_k$, with respect to the same $\mathcal{O} - X_1 - X_2 - X_3$ coordinate frame
- This vector locates the deformed image of a generic material point $P \in \mathcal{S}_0$ that is denoted by $\mathcal{P} = \mathcal{D}_t(P) \in \mathcal{S}_t$ and defines the **trajectory** of point $\mathcal{P}$ through $\mathcal{E}^3$
- The undeformed and deformed reference surfaces are related by introducing a vector field $\vec{u}(\xi_1, \xi_2, t)$ that defines the **displacement** of each material point $P \in \mathcal{S}_0$ such that

$$\vec{x}(\xi_1, \xi_2, t) = \vec{x}(\xi_1, \xi_2) + \vec{u}(\xi_1, \xi_2, t) \quad \text{and} \quad \vec{x}(\xi_1, \xi_2, 0) = \vec{x}(\xi_1, \xi_2)$$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED



NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- By direct analogy with the undeformed reference surface $\mathcal{S}_0$, it follows that the natural base-vector fields of $\mathcal{S}_t$, that span the tangent plane, $T_p(\mathcal{S}_t)$, at every point $p \in \mathcal{S}_t$, are given by

$$\vec{a}_1(\xi_1, \xi_2, t) = \frac{\partial \vec{x}}{\partial \xi_1} \quad \text{and} \quad \vec{a}_2(\xi_1, \xi_2, t) = \frac{\partial \vec{x}}{\partial \xi_2}$$

- In terms of the deformation mapping, the base vectors are given by

$$\vec{a}_\alpha(\xi_1, \xi_2, t) = \mathcal{D}_t(\vec{a}_\alpha(\xi_1, \xi_2)), \quad \text{such that} \quad \vec{a}_\alpha(\xi_1, \xi_2, 0) = \vec{a}_\alpha(\xi_1, \xi_2)$$

- The unit-magnitude vector field $\hat{n}(\xi_1, \xi_2, t)$ shown in the previous figure is perpendicular to the tangent plane at $p = \mathcal{D}_t(p) \in \mathcal{S}_t$ and is given by

$$\hat{n}(\xi_1, \xi_2, t) = \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|}, \quad \text{where} \quad \hat{n}(\xi_1, \xi_2, 0) = \hat{n}(\xi_1, \xi_2)$$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- In terms of the **displacement vector**, the base-vector fields of $\mathcal{S}_t$ are expressed as $\vec{a}_\alpha(\xi_1, \xi_2, t) = \vec{a}_\alpha + \frac{\partial \vec{u}}{\partial \xi_\alpha} = A_{(\alpha)} \left(\hat{a}_\alpha + \frac{1}{A_{(\alpha)}} \frac{\partial \vec{u}}{\partial \xi_\alpha} \right)$ where the parentheses around the subscript is used to indicate **suspension of the repeated-index summation convention**
- In general, the **displacement vector field** $\vec{u}(\xi_1, \xi_2, t)$ has several component forms that are convenient for various purposes
- Herein, $\vec{u}(\xi_1, \xi_2, t)$ is associated with points $P \in \mathcal{S}_0$ and is expressed in terms of the unit-magnitude vector fields for $\mathcal{S}_0$ as

$$\vec{u}(\xi_1, \xi_2, t) = u_\alpha(\xi_1, \xi_2, t) \hat{a}_\alpha(\xi_1, \xi_2) + w(\xi_1, \xi_2, t) \hat{n}(\xi_1, \xi_2)$$

- $u_\alpha(\xi_1, \xi_2, t)$ and $w(\xi_1, \xi_2, t)$ are referred to herein as the **tangential displacements** and **normal displacement**, respectively

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- Previously, it was shown herein that **derivatives of a vector field**

$$\vec{V}(\xi_1, \xi_2) = V_\alpha(\xi_1, \xi_2) \hat{a}_\alpha(\xi_1, \xi_2) + V_3(\xi_1, \xi_2) \hat{n}(\xi_1, \xi_2)$$

defined on the undeformed reference surface $\mathcal{S}_0$ are given by

$$\frac{1}{A_{(\beta)}} \frac{\partial \vec{V}}{\partial \xi_\beta} = v^{\langle \alpha \rangle} \Big|_{\langle \beta \rangle} \hat{a}_\alpha + v^{\langle 3 \rangle} \Big|_{\langle \beta \rangle} \hat{n}$$

- Thus, it follows that

$$\frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} = u^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} \hat{a}_1 + u^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} \hat{a}_2 + u^{\langle 3 \rangle} \Big|_{\langle 1 \rangle} \hat{n} \quad \text{and}$$

$$\frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} = u^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} \hat{a}_1 + u^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} \hat{a}_2 + u^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} \hat{n}$$

where, for general reference-surface Gaussian coordinates,

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

$$\mathbf{u}^{(1)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_1}{\partial \xi_1} - \frac{\mathbf{u}_2 \mathbf{csc} \theta_{12}}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\mathbf{csc} \theta_{12}}{\rho_{11}} \left(\mathbf{u}_1 \cos \theta_{12} + \mathbf{u}_2 \right) + w \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right)$$

$$\mathbf{u}^{(2)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_2}{\partial \xi_1} + \frac{\mathbf{csc} \theta_{12}}{\rho_{11}} \left(\mathbf{u}_1 + \mathbf{u}_2 \cos \theta_{12} \right) + \frac{\mathbf{u}_2 \cot \theta_{12}}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} - w \frac{\mathbf{csc} \theta_{12}}{r_{12}}$$

$$\mathbf{u}^{(3)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial w}{\partial \xi_1} - \frac{\mathbf{u}_1}{r_{11}} + \mathbf{u}_2 \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right)$$

$$\mathbf{u}^{(1)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_1}{\partial \xi_2} + \frac{\mathbf{u}_1 \cot \theta_{12}}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} - \frac{\mathbf{csc} \theta_{12}}{\rho_{22}} \left(\mathbf{u}_1 \cos \theta_{12} + \mathbf{u}_2 \right) + w \frac{\mathbf{csc} \theta_{12}}{r_{21}}$$

$$\mathbf{u}^{(2)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_2}{\partial \xi_2} - \frac{\mathbf{u}_1 \mathbf{csc} \theta_{12}}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} + \frac{\mathbf{csc} \theta_{12}}{\rho_{22}} \left(\mathbf{u}_1 + \mathbf{u}_2 \cos \theta_{12} \right) + w \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right)$$

$$\mathbf{u}^{(3)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial w}{\partial \xi_2} - \mathbf{u}_1 \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) - \frac{\mathbf{u}_2}{r_{22}}$$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- For **orthogonal Gaussian coordinates** these expressions reduce to

$$\mathbf{u}^{(1)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_1}{\partial \xi_1} - \frac{\mathbf{u}_2}{\rho_{11}} + \frac{\mathbf{w}}{\mathbf{r}_{11}} \quad \rightarrow \quad \mathbf{u}^{(1)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_1}{\partial \xi_1} + \frac{\mathbf{u}_2}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} + \frac{\mathbf{w}}{\mathbf{r}_{11}}$$

$$\mathbf{u}^{(2)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_2}{\partial \xi_1} + \frac{\mathbf{u}_1}{\rho_{11}} - \frac{\mathbf{w}}{\mathbf{r}_{12}} \quad \rightarrow \quad \mathbf{u}^{(2)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_2}{\partial \xi_1} - \frac{\mathbf{u}_1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} - \frac{\mathbf{w}}{\mathbf{r}_{12}}$$

$$\mathbf{u}^{(1)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_1}{\partial \xi_2} - \frac{\mathbf{u}_2}{\rho_{22}} - \frac{\mathbf{w}}{\mathbf{r}_{12}} \quad \rightarrow \quad \mathbf{u}^{(1)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_1}{\partial \xi_2} - \frac{\mathbf{u}_2}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1} - \frac{\mathbf{w}}{\mathbf{r}_{12}}$$

$$\mathbf{u}^{(2)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_2}{\partial \xi_2} + \frac{\mathbf{u}_1}{\rho_{22}} + \frac{\mathbf{w}}{\mathbf{r}_{22}} \quad \rightarrow \quad \mathbf{u}^{(2)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_2}{\partial \xi_2} + \frac{\mathbf{u}_1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1} + \frac{\mathbf{w}}{\mathbf{r}_{22}}$$

$$\mathbf{u}^{(3)} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{w}}{\partial \xi_1} - \frac{\mathbf{u}_1}{\mathbf{r}_{11}} + \frac{\mathbf{u}_2}{\mathbf{r}_{12}}$$

$$\mathbf{u}^{(3)} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{w}}{\partial \xi_2} + \frac{\mathbf{u}_1}{\mathbf{r}_{12}} - \frac{\mathbf{u}_2}{\mathbf{r}_{22}}$$

NATURAL BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONCLUDED

- Using the general expressions for $\frac{1}{A_{(\alpha)}} \frac{\partial \vec{u}}{\partial \xi_\alpha}$ and the expression

$$\vec{a}_\alpha(\xi_1, \xi_2, t) = \vec{a}_\alpha + \frac{\partial \vec{u}}{\partial \xi_\alpha} = A_{(\alpha)} \left(\hat{a}_\alpha + \frac{1}{A_{(\alpha)}} \frac{\partial \vec{u}}{\partial \xi_\alpha} \right),$$

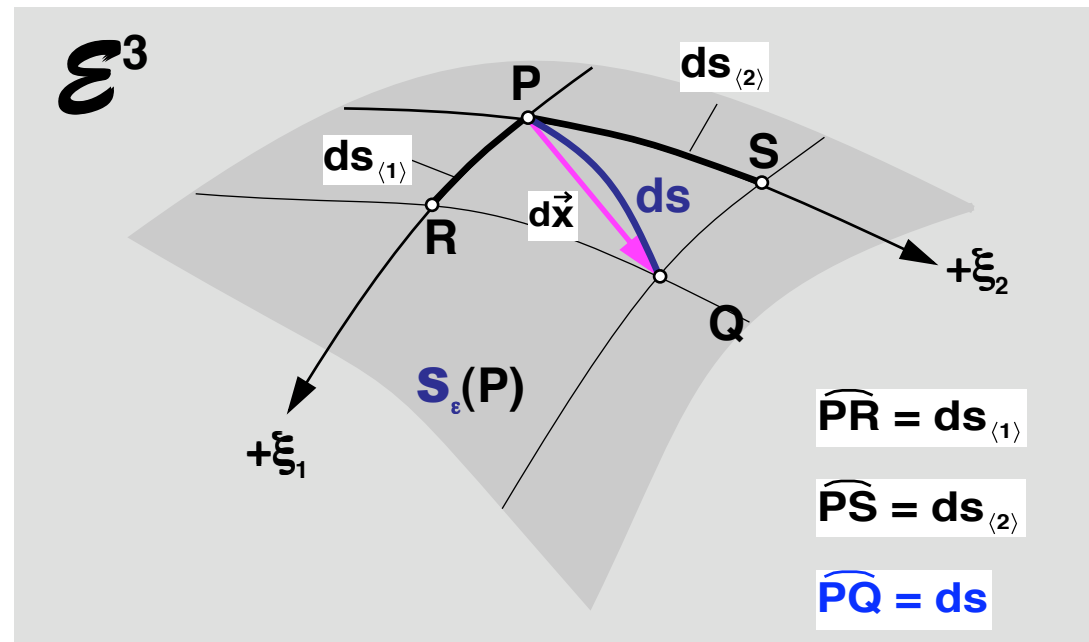
the **base-vector fields of the deformed surface** at time t are given in terms of the displacement components by

$$\frac{\vec{a}_1}{A_1} = \left(1 + u^{(1)} \Big|_{\langle 1 \rangle} \right) \hat{a}_1 + u^{(2)} \Big|_{\langle 1 \rangle} \hat{a}_2 + u^{(3)} \Big|_{\langle 1 \rangle} \hat{n}$$

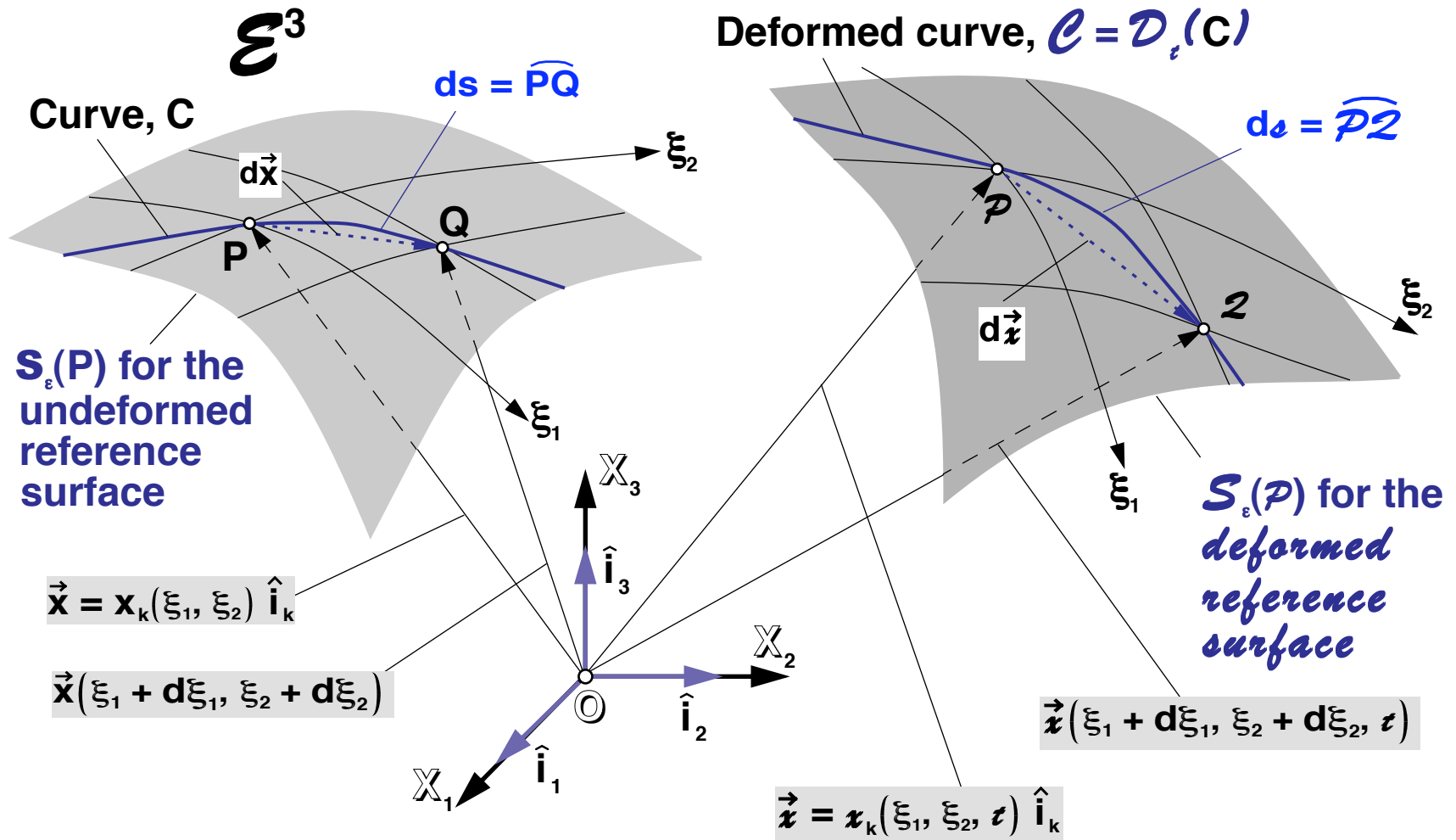
$$\frac{\vec{a}_2}{A_2} = u^{(1)} \Big|_{\langle 2 \rangle} \hat{a}_1 + \left(1 + u^{(2)} \Big|_{\langle 2 \rangle} \right) \hat{a}_2 + u^{(3)} \Big|_{\langle 2 \rangle} \hat{n}$$

METRIC COEFFICIENTS OF $\mathcal{S}_t$

- Recall that $\mathcal{S}_\varepsilon(P)$ denotes a small, *infinitesimal neighborhood* of an arbitrary point P of the undeformed reference surface, $\mathcal{S}_0$
- Likewise, point Q is a “nearby” point in $\mathcal{S}_\varepsilon(P)$ and $\widehat{ds} = \widehat{PQ}$ is a smooth *differential surface arc* connecting points P and Q
- As the reference surface deforms, material points P and Q become $\mathcal{P}$ and $\mathcal{Q}$, respectively
- Likewise, the material curve C becomes the curve $\mathcal{C}$, and the *infinitesimal neighborhood* $\mathcal{S}_\varepsilon(P)$ becomes $\mathcal{S}_\varepsilon(\mathcal{P})$, as shown in the next figure



METRIC COEFFICIENTS OF $\mathcal{S}_t$ - CONTINUED



METRIC COEFFICIENTS OF $\mathcal{S}_t$ - CONTINUED

- Just as the reference-surface coordinates (ξ_1, ξ_2) become convected surface coordinates for the deformed reference surface $\mathcal{S}_t$, the arc-length coordinate s becomes a convected arc-length coordinate for the curve $\mathcal{C} \subset \mathcal{S}_t$ shown in the previous figure
- Thus, the differential $ds = \widehat{PQ}$ becomes the differential $d\mathcal{s} = \widehat{\mathcal{P}\mathcal{Q}}$
- The position vectors to points $\mathcal{P}$ and $\mathcal{Q}$ of the deformed-reference-surface curve $\mathcal{C}$ are given, respectively, by

$$\vec{x} = x_k(\xi_1, \xi_2, t) \hat{i}_k \quad \text{and} \quad \vec{x}(\xi_1 + d\xi_1, \xi_2 + d\xi_2, t) = \vec{x}(\xi_1, \xi_2, t) + d\vec{x}$$

- The arc length $d\mathcal{s}$ of the two infinitesimally close points, $\mathcal{P}$ and $\mathcal{Q}$, is given by the dot product $d\mathcal{s}^2 = d\vec{x} \cdot d\vec{x}$

METRIC COEFFICIENTS OF $\mathcal{S}_t$ - CONTINUED

- Noting that $d\vec{x} = \frac{\partial \vec{x}}{\partial \xi_\alpha} d\xi_\alpha$ and $\vec{a}_\alpha(\xi_1, \xi_2, t) = \frac{\partial \vec{x}}{\partial \xi_\alpha}$, it follows that the arc length $d\mathcal{L}$ is given by $d\mathcal{L}^2 = (\vec{a}_\alpha \cdot \vec{a}_\beta) d\xi_\alpha d\xi_\beta$
- For the undeformed reference surface, the arc length ds is expressed as $ds^2 = (A_1 d\xi_1)^2 + 2A_1 A_2 \cos\theta_{12} d\xi_1 d\xi_2 + (A_2 d\xi_2)^2$, where θ_{12} is the angle between the nonorthogonal Gaussian coordinate curves
- In addition, $A_\alpha(\xi_1, \xi_2) = \sqrt{\vec{a}_\alpha \cdot \vec{a}_{(\alpha)}} = |\vec{a}_\alpha|$ and $\vec{a}_1 \cdot \vec{a}_2 = A_1 A_2 \cos\theta_{12}$
- Again, for the deformed reference surface, it is important to note that the convected-coordinate curves are generally nonorthogonal
- The angle between the **convected** Gaussian coordinate curves is denoted by $\theta_{12} = \theta_{12}(\xi_1, \xi_2, t)$, where $\theta_{12} = \mathcal{D}_t(\theta_{12})$

METRIC COEFFICIENTS OF $\mathcal{S}_t$ - CONTINUED

- Using notation similar to that used for the undeformed reference surface, the arc length $d\mathcal{S}$ is expressed in terms of **Lame' parameters of the deformed reference surface** as

$$d\mathcal{S}^2 = \left(\mathcal{A}_1 d\xi_1 \right)^2 + 2\mathcal{A}_1 \mathcal{A}_2 \cos\theta_{12} d\xi_1 d\xi_2 + \left(\mathcal{A}_2 d\xi_2 \right)^2$$

- Comparison of $d\mathcal{S}^2 = (\vec{a}_\alpha \cdot \vec{a}_\beta) d\xi_\alpha d\xi_\beta$ and the previous equation reveals

$$\mathcal{A}_1 = \sqrt{\vec{a}_1 \cdot \vec{a}_1} = \left| \vec{a}_1 \right|$$

$$\mathcal{A}_2 = \sqrt{\vec{a}_2 \cdot \vec{a}_2} = \left| \vec{a}_2 \right| \quad \text{and}$$

$$\mathcal{A}_1 \mathcal{A}_2 \cos\theta_{12} = \vec{a}_1 \cdot \vec{a}_2$$

METRIC COEFFICIENTS OF $\mathcal{S}_t$ - CONTINUED

- Using $\vec{a}_\alpha(\xi_1, \xi_2, t) = \mathbf{A}_{(\alpha)} \left(\hat{\mathbf{a}}_\alpha + \frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \vec{u}}{\partial \xi_\alpha} \right)$ gives

$$\mathcal{A}_1 = \mathbf{A}_1 \sqrt{1 + 2 \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{u}}{\partial \xi_1} \right) + \left(\frac{1}{\mathbf{A}_1} \frac{\partial \vec{u}}{\partial \xi_1} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{u}}{\partial \xi_1} \right)}$$

$$\mathcal{A}_2 = \mathbf{A}_2 \sqrt{1 + 2 \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{u}}{\partial \xi_2} \right) + \left(\frac{1}{\mathbf{A}_2} \frac{\partial \vec{u}}{\partial \xi_2} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{u}}{\partial \xi_2} \right)}$$

$$\frac{\mathcal{A}_1 \mathcal{A}_2}{\mathbf{A}_1 \mathbf{A}_2} \cos \theta_{12} = \cos \theta_{12} + \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{u}}{\partial \xi_2} \right) + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{u}}{\partial \xi_1} \right) + \left(\frac{1}{\mathbf{A}_1} \frac{\partial \vec{u}}{\partial \xi_1} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{u}}{\partial \xi_2} \right)$$

- With the **Lame' parameters of the deformed reference surface** defined, a set of convected unit-magnitude base-vector field is defined by

$$\hat{\mathbf{a}}_1 \equiv \frac{\vec{a}_1}{|\vec{a}_1|} = \frac{\vec{a}_1}{\mathcal{A}_1} \quad \text{and} \quad \hat{\mathbf{a}}_2 \equiv \frac{\vec{a}_2}{|\vec{a}_2|} = \frac{\vec{a}_2}{\mathcal{A}_2}$$

METRIC COEFFICIENTS OF $\mathcal{S}_t$ - CONTINUED

- Using $\frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_\alpha} = \mathbf{u}^{(1)} \big|_{\langle \alpha \rangle} \hat{\mathbf{a}}_\beta + \mathbf{u}^{(3)} \big|_{\langle \alpha \rangle} \hat{\mathbf{n}}$ gives the following complicated expressions for the **Lame' parameters of the deformed reference surface** in terms of the displacements

$$\left(\frac{\mathcal{A}_1}{\mathbf{A}_1} \right)^2 = 1 + 2 \left(\mathbf{u}^{(1)} \big|_{\langle 1 \rangle} + \mathbf{u}^{(2)} \big|_{\langle 1 \rangle} \cos \theta_{12} \right) + \mathbf{u}^{(1)} \big|_{\langle 1 \rangle} \left[\mathbf{u}^{(1)} \big|_{\langle 1 \rangle} + \mathbf{u}^{(2)} \big|_{\langle 1 \rangle} \cos \theta_{12} \right] \\ + \mathbf{u}^{(2)} \big|_{\langle 1 \rangle} \left[\mathbf{u}^{(2)} \big|_{\langle 1 \rangle} + \mathbf{u}^{(1)} \big|_{\langle 1 \rangle} \cos \theta_{12} \right] + \mathbf{u}^{(3)} \big|_{\langle 1 \rangle} \mathbf{u}^{(3)} \big|_{\langle 1 \rangle}$$

$$\left(\frac{\mathcal{A}_2}{\mathbf{A}_2} \right)^2 = 1 + 2 \left(\mathbf{u}^{(1)} \big|_{\langle 2 \rangle} \cos \theta_{12} + \mathbf{u}^{(2)} \big|_{\langle 2 \rangle} \right) + \mathbf{u}^{(1)} \big|_{\langle 2 \rangle} \left[\mathbf{u}^{(1)} \big|_{\langle 2 \rangle} + \mathbf{u}^{(2)} \big|_{\langle 2 \rangle} \cos \theta_{12} \right] \\ + \mathbf{u}^{(2)} \big|_{\langle 2 \rangle} \left[\mathbf{u}^{(2)} \big|_{\langle 2 \rangle} + \mathbf{u}^{(1)} \big|_{\langle 2 \rangle} \cos \theta_{12} \right] + \mathbf{u}^{(3)} \big|_{\langle 2 \rangle} \mathbf{u}^{(3)} \big|_{\langle 2 \rangle}$$

METRIC COEFFICIENTS OF $\mathcal{S}_t$ - CONCLUDED

- In addition,

$$\begin{aligned} \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathbf{A}_1 \mathbf{A}_2} \cos \theta_{12} = & \mathbf{u}^{(1)} \Big|_{\langle 2 \rangle} + \mathbf{u}^{(2)} \Big|_{\langle 1 \rangle} + \left(1 + \mathbf{u}^{(1)} \Big|_{\langle 1 \rangle} + \mathbf{u}^{(2)} \Big|_{\langle 2 \rangle} \right) \cos \theta_{12} \\ & + \mathbf{u}^{(1)} \Big|_{\langle 1 \rangle} \left[\mathbf{u}^{(1)} \Big|_{\langle 2 \rangle} + \mathbf{u}^{(2)} \Big|_{\langle 2 \rangle} \cos \theta_{12} \right] \\ & + \mathbf{u}^{(2)} \Big|_{\langle 1 \rangle} \left[\mathbf{u}^{(1)} \Big|_{\langle 2 \rangle} \cos \theta_{12} + \mathbf{u}^{(2)} \Big|_{\langle 2 \rangle} \right] + \mathbf{u}^{(3)} \Big|_{\langle 1 \rangle} \mathbf{u}^{(3)} \Big|_{\langle 2 \rangle} \end{aligned}$$

- It is important to remember that although the magnitudes of $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ are constant, the directions are not - thus, $\hat{\mathbf{a}}_\alpha = \hat{\mathbf{a}}_\alpha(\xi_1, \xi_2, t)$

ALTERNATE NORMAL-VECTOR FIELD FOR $\mathcal{S}_t$

- A convenient alternate form of $\hat{n}(\xi_1, \xi_2, t)$ is given by $\hat{n} = \frac{\sqrt{a}}{\sqrt{a}} \vec{m}$ where $\sqrt{a} = A_1 A_2 \sin \theta_{12}$ and $\vec{m} = m_1 \hat{a}_1 + m_2 \hat{a}_2 + m_3 \hat{n}$ is free of metric quantities
- Here, $\sqrt{a} = \left| \vec{a}_1 \times \vec{a}_2 \right|$ is defined such that $\hat{n}(\xi_1, \xi_2, t) = \frac{\vec{a}_1 \times \vec{a}_2}{\sqrt{a}}$
- Note that the orthogonality relations $\vec{a}_1 \cdot \hat{n} = 0$ and $\vec{a}_2 \cdot \hat{n} = 0$ imply the orthogonality relations $\vec{a}_1 \cdot \vec{m} = 0$ and $\vec{a}_2 \cdot \vec{m} = 0$
- Also, from $\hat{n} = \frac{\vec{a}_1 \times \vec{a}_2}{\sqrt{a}}$ and $\hat{n} = \frac{\sqrt{a}}{\sqrt{a}} \vec{m}$ it follows that $\vec{m} = \frac{\vec{a}_1 \times \vec{a}_2}{\sqrt{a}}$

ALTERNATE NORMAL-VECTOR FIELD FOR $\mathcal{S}_t$

CONTINUED

- In addition; $\sqrt{a} = |\vec{a}_1 \times \vec{a}_2|$, $\vec{a}_1 = \mathcal{A}_1 \hat{a}_1$, and $\vec{a}_2 = \mathcal{A}_2 \hat{a}_2$ yield

$$\sqrt{a} = \mathcal{A}_1 \mathcal{A}_2 |\hat{a}_1 \times \hat{a}_2| = \mathcal{A}_1 \mathcal{A}_2 \sin \theta_{12} |\hat{n}| \quad \text{or} \quad \sqrt{a} = \mathcal{A}_1 \mathcal{A}_2 \sin \theta_{12}$$

- Moreover, $\frac{\sqrt{a}}{\mathcal{A}_1 \mathcal{A}_2 \sin \theta_{12}} = \frac{A_1 A_2 \sin \theta_{12}}{\mathcal{A}_1 \mathcal{A}_2 \sin \theta_{12}}$ and $\vec{m} = \frac{\mathcal{A}_1 \mathcal{A}_2}{A_1 A_2 \sin \theta_{12}} (\hat{a}_1 \times \hat{a}_2)$

- Using this equation for $\vec{m}$ with

$$\hat{a}_1 = \frac{A_1}{\mathcal{A}_1} \left[\left(1 + u^{(1)} \right) \hat{a}_1 + u^{(2)} \hat{a}_2 + u^{(3)} \hat{n} \right] \quad \text{and}$$

$$\hat{a}_2 = \frac{A_2}{\mathcal{A}_2} \left[u^{(1)} \hat{a}_1 + \left(1 + u^{(2)} \right) \hat{a}_2 + u^{(3)} \hat{n} \right] \quad \text{gives}$$

ALTERNATE NORMAL-VECTOR FIELD FOR $\mathcal{S}_t$

CONTINUED

$$\begin{aligned} \vec{m} = & \csc\theta_{12} \left[\left(1 + u^{(1)} \Big|_{\langle 1 \rangle} \right) \left(1 + u^{(2)} \Big|_{\langle 2 \rangle} \right) \hat{a}_1 \times \hat{a}_2 + \left(1 + u^{(1)} \Big|_{\langle 1 \rangle} \right) u^{(3)} \Big|_{\langle 2 \rangle} \hat{a}_1 \times \hat{n} \right] \\ & + \csc\theta_{12} \left[u^{(2)} \Big|_{\langle 1 \rangle} u^{(1)} \Big|_{\langle 2 \rangle} \hat{a}_2 \times \hat{a}_1 + u^{(2)} \Big|_{\langle 1 \rangle} u^{(3)} \Big|_{\langle 2 \rangle} \hat{a}_2 \times \hat{n} \right] \\ & + \csc\theta_{12} \left[u^{(3)} \Big|_{\langle 1 \rangle} u^{(1)} \Big|_{\langle 2 \rangle} \hat{n} \times \hat{a}_1 + u^{(3)} \Big|_{\langle 1 \rangle} \left(1 + u^{(2)} \Big|_{\langle 2 \rangle} \right) \hat{n} \times \hat{a}_2 \right] \end{aligned}$$

- Using $\hat{a}_1 \times \hat{a}_2 = \sin\theta_{12} \hat{n}$, $\hat{n} \times \hat{a}_1 = \hat{a}_2 \csc\theta_{12} - \hat{a}_1 \cot\theta_{12}$, and $\hat{a}_2 \times \hat{n} = \hat{a}_1 \csc\theta_{12} - \hat{a}_2 \cot\theta_{12}$ with $\vec{m} = m_1 \hat{a}_1 + m_2 \hat{a}_2 + m_3 \hat{n}$ yields the following results

ALTERNATE NORMAL-VECTOR FIELD FOR $\mathcal{S}_t$

CONCLUDED

$$\begin{aligned} m_1 = \csc\theta_{12}\cot\theta_{12} & \left[\left(1 + u^{(1)}|_{(1)} \right) u^{(3)}|_{(2)} - u^{(3)}|_{(1)} u^{(1)}|_{(2)} \right] \\ & + \csc^2\theta_{12} \left[u^{(2)}|_{(1)} u^{(3)}|_{(2)} - u^{(3)}|_{(1)} \left(1 + u^{(2)}|_{(2)} \right) \right] \end{aligned}$$

$$\begin{aligned} m_2 = \csc\theta_{12}\cot\theta_{12} & \left[u^{(3)}|_{(1)} \left(1 + u^{(2)}|_{(2)} \right) - u^{(2)}|_{(1)} u^{(3)}|_{(2)} \right] \\ & + \csc^2\theta_{12} \left[u^{(3)}|_{(1)} u^{(1)}|_{(2)} - \left(1 + u^{(1)}|_{(1)} \right) u^{(3)}|_{(2)} \right] \end{aligned} \quad \text{and}$$

$$m_3 = \left(1 + u^{(1)}|_{(1)} \right) \left(1 + u^{(2)}|_{(2)} \right) - u^{(2)}|_{(1)} u^{(1)}|_{(2)}$$

- Inspection of the components of $\vec{m}$ given above reveals that they are quadratic in the derivatives of the reference-surface displacements

RECIPROCAL BASE-VECTOR FIELDS OF $\mathcal{S}_t$

- As a prelude to characterizing the geometry of the deformed reference surface, it is convenient to introduce the **"reciprocal" base-vector fields**

$$\hat{\mathbf{a}}^1 \equiv \hat{\mathbf{a}}_2 \times \hat{\mathbf{n}} \quad \text{and} \quad \hat{\mathbf{a}}^2 \equiv \hat{\mathbf{n}} \times \hat{\mathbf{a}}_1$$

for the deformed reference surface such that

$$\hat{\mathbf{a}}^1 \cdot \hat{\mathbf{n}} = 0$$

$$\hat{\mathbf{a}}^2 \cdot \hat{\mathbf{n}} = 0$$

$$\hat{\mathbf{a}}^1 \times \hat{\mathbf{a}}^2 = \hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_2$$

$$\hat{\mathbf{a}}^1 \cdot \hat{\mathbf{a}}_1 = \sin \theta_{12}$$

$$\hat{\mathbf{a}}^2 \cdot \hat{\mathbf{a}}_2 = \sin \theta_{12}$$

$$\hat{\mathbf{a}}^1 \cdot \hat{\mathbf{a}}_2 = \hat{\mathbf{a}}^2 \cdot \hat{\mathbf{a}}_1 = 0$$

- These conditions yield

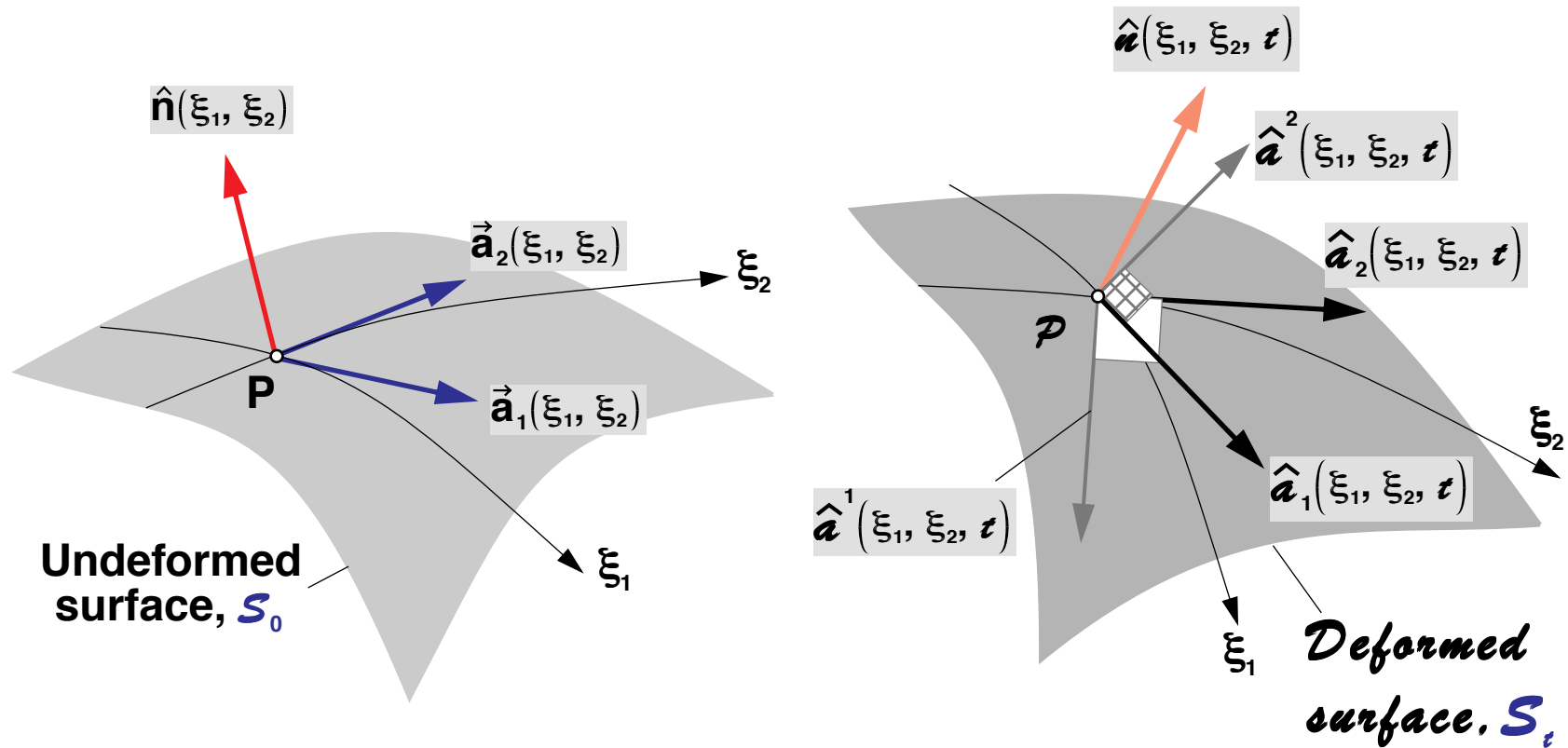
$$\hat{\mathbf{a}}^1 = \hat{\mathbf{a}}_1 \csc \theta_{12} - \hat{\mathbf{a}}_2 \cot \theta_{12}$$

and

$$\hat{\mathbf{a}}^2 = \hat{\mathbf{a}}_2 \csc \theta_{12} - \hat{\mathbf{a}}_1 \cot \theta_{12}$$

RECIPROCAL BASE-VECTOR FIELDS OF $\mathcal{S}_t$

CONCLUDED



$$\hat{a}^1 \perp \hat{a}_2 \text{ and } \hat{a}^1 \perp \hat{n}$$

$$\hat{a}^2 \perp \hat{a}_1 \text{ and } \hat{a}^2 \perp \hat{n}$$

$$\hat{a}^1 \times \hat{a}^2 \parallel \hat{a}_1 \times \hat{a}_2 \parallel \hat{n}$$

DEFORMATION CONTINUITY CONDITION FOR $\mathcal{S}_t$

- A useful relationship between the **convected base-vector fields** is obtained by differentiating the orthogonality conditions $\hat{\mathbf{n}} \cdot \hat{\mathbf{a}}_1 = 0$ and $\hat{\mathbf{n}} \cdot \hat{\mathbf{a}}_2 = 0$ as follows

- $\frac{\partial}{\partial \xi_2} (\hat{\mathbf{n}} \cdot \hat{\mathbf{a}}_1) = 0$ gives $\frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \cdot \hat{\mathbf{a}}_1 = - \hat{\mathbf{n}} \cdot \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2}$

- $\frac{\partial}{\partial \xi_1} (\hat{\mathbf{n}} \cdot \hat{\mathbf{a}}_2) = 0$ gives $\frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \cdot \hat{\mathbf{a}}_2 = - \hat{\mathbf{n}} \cdot \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1}$

- Because $\vec{\mathbf{a}}_1 = \frac{\partial \vec{\mathbf{x}}}{\partial \xi_1}$ and $\vec{\mathbf{a}}_2 = \frac{\partial \vec{\mathbf{x}}}{\partial \xi_2}$, **continuity of deformation** ensures

that $\frac{\partial \vec{\mathbf{a}}_1}{\partial \xi_2} = \frac{\partial \vec{\mathbf{a}}_2}{\partial \xi_1}$, or equivalently, $\frac{\partial}{\partial \xi_2} (\mathcal{A}_1 \hat{\mathbf{a}}_1) = \frac{\partial}{\partial \xi_1} (\mathcal{A}_2 \hat{\mathbf{a}}_2)$

DEFORMATION CONTINUITY CONDITION FOR $\mathcal{S}_t$

CONTINUED

- Expanding $\frac{\partial}{\partial \xi_2}(\mathcal{A}_1 \hat{a}_1) = \frac{\partial}{\partial \xi_1}(\mathcal{A}_2 \hat{a}_2)$ and taking the inner product with $\hat{u}$

gives $\mathcal{A}_1 \left(\hat{u} \cdot \frac{\partial \hat{a}_1}{\partial \xi_2} \right) = \mathcal{A}_2 \left(\hat{u} \cdot \frac{\partial \hat{a}_2}{\partial \xi_1} \right)$ or $\hat{u} \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \hat{a}_1}{\partial \xi_2} = \hat{u} \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \hat{a}_2}{\partial \xi_1}$

- Using $\frac{\partial \hat{u}}{\partial \xi_2} \cdot \hat{a}_1 = -\hat{u} \cdot \frac{\partial \hat{a}_1}{\partial \xi_2}$ and $\frac{\partial \hat{u}}{\partial \xi_1} \cdot \hat{a}_2 = -\hat{u} \cdot \frac{\partial \hat{a}_2}{\partial \xi_1}$ with the last

equation yields the following alternate expression

$$\frac{1}{\mathcal{A}_2} \frac{\partial \hat{u}}{\partial \xi_2} \cdot \hat{a}_1 = \frac{1}{\mathcal{A}_1} \frac{\partial \hat{u}}{\partial \xi_1} \cdot \hat{a}_2$$

DEFORMATION CONTINUITY CONDITION FOR $\mathcal{S}_t$

CONCLUDED

- Previously, it was shown herein that $\frac{1}{A_1} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1} \cdot \hat{\mathbf{n}} = \frac{1}{A_2} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} \cdot \hat{\mathbf{n}}$ for the undeformed reference surface, which yields the identity

$$\frac{1}{r_{12}} + \frac{1}{r_{21}} = \cot \theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$$

- Thus, by direct analogy, $\hat{\mathbf{u}} \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{a}}_1}{\partial \xi_2} = \hat{\mathbf{u}} \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{a}}_2}{\partial \xi_1}$ requires

$$\frac{1}{\kappa_{12}} + \frac{1}{\kappa_{21}} = \cot \theta_{12} \left(\frac{1}{\kappa_{11}} - \frac{1}{\kappa_{22}} \right) \quad \text{where}$$

$\kappa_{\alpha\beta}^{-1}$ are the curvatures and torsions of the deformed reference surface associated with the convected Gaussian-coordinate curves

CURVATURES AND TORSIONS OF $\mathcal{S}_t$

- For the reference surface $\mathcal{S}_0$, the surface curvatures along the Gaussian-coordinate curves have been given as

$$\frac{1}{r_{11}} = \hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \quad \text{and} \quad \frac{1}{r_{22}} = \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2}$$

- Likewise, the surface torsions were given as

$$\frac{1}{r_{12}} = (\hat{\mathbf{a}}_1 \times \hat{\mathbf{n}}) \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} = -\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \quad \frac{1}{r_{21}} = (\hat{\mathbf{a}}_2 \times \hat{\mathbf{n}}) \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} = \hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2}$$

- By direct analogy, the **curvatures** of the **deformed reference surface $\mathcal{S}_t$** along the convected Gaussian-coordinate curves are given

by

$$\frac{1}{r_{11}} = \hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \quad \text{and} \quad \frac{1}{r_{22}} = \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2}$$

CURVATURES AND TORSIONS OF $\mathcal{S}_i$ - CONTINUED

- Similarly, the corresponding **torsions** of the **deformed reference surface** $\mathcal{S}_i$ along the convected Gaussian-coordinate curves are given by

$$\frac{1}{r_{12}} = (\hat{\mathbf{a}}_1 \times \hat{\mathbf{n}}) \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} = -\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \quad \text{and}$$

$$\frac{1}{r_{21}} = (\hat{\mathbf{a}}_2 \times \hat{\mathbf{n}}) \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} = \hat{\mathbf{a}}^1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2}$$

- These equations illustrate the utility of defining the reciprocal base-vector fields

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

- Next, using $\hat{\mathbf{a}} = \frac{\sqrt{\mathbf{a}}}{\sqrt{a}} \vec{m}$ and $\hat{\mathbf{a}}_\alpha \cdot \vec{m} = 0$ with $\frac{1}{r_{11}} = \hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_1}$ and

$$\frac{1}{r_{22}} = \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_2} \text{ gives}$$

$$\frac{1}{r_{11}} = \frac{\sqrt{a}}{\sqrt{a}} \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{m}}{\partial \xi_1} \right) \text{ and } \frac{1}{r_{22}} = \frac{\sqrt{a}}{\sqrt{a}} \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{m}}{\partial \xi_2} \right)$$

- Likewise, the **torsions** are expressed as

$$\frac{1}{r_{12}} = - \frac{\sqrt{a}}{\sqrt{a}} \left(\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{m}}{\partial \xi_1} \right) \text{ and } \frac{1}{r_{21}} = \frac{\sqrt{a}}{\sqrt{a}} \left(\hat{\mathbf{a}}^1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{m}}{\partial \xi_2} \right)$$

CURVATURES AND TORSIONS OF $\mathcal{S}_i$ - CONTINUED

- Moreover, using the following expressions for the convected reciprocal basis for $\mathcal{S}_i$

$$\hat{\mathbf{a}}^1 = \hat{\mathbf{a}}_1 \csc \theta_{12} - \hat{\mathbf{a}}_2 \cot \theta_{12} \quad \text{and} \quad \hat{\mathbf{a}}^2 = \hat{\mathbf{a}}_2 \csc \theta_{12} - \hat{\mathbf{a}}_1 \cot \theta_{12} \quad \text{gives}$$

$$\frac{1}{r_{12}} = \frac{\sqrt{a}}{\sqrt{a}} \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{m}}{\partial \xi_1} \right) \cot \theta_{12} - \frac{\sqrt{a}}{\sqrt{a}} \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{m}}{\partial \xi_1} \right) \csc \theta_{12} \quad \text{and}$$

$$\frac{1}{r_{21}} = \frac{\sqrt{a}}{\sqrt{a}} \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{m}}{\partial \xi_2} \right) \csc \theta_{12} - \frac{\sqrt{a}}{\sqrt{a}} \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{m}}{\partial \xi_2} \right) \cot \theta_{12}$$

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

- The expressions for the **derivatives of** $\vec{m} = m_1 \hat{a}_1 + m_2 \hat{a}_2 + m_3 \hat{n}$ are needed next and are given by

$$\frac{1}{A_1} \frac{\partial \vec{m}}{\partial \xi_1} = m^{(1)} \Big|_{\langle 1 \rangle} \hat{a}_1 + m^{(2)} \Big|_{\langle 1 \rangle} \hat{a}_2 + m^{(3)} \Big|_{\langle 1 \rangle} \hat{n} \quad \text{and}$$

$$\frac{1}{A_2} \frac{\partial \vec{m}}{\partial \xi_2} = m^{(1)} \Big|_{\langle 2 \rangle} \hat{a}_1 + m^{(2)} \Big|_{\langle 2 \rangle} \hat{a}_2 + m^{(3)} \Big|_{\langle 2 \rangle} \hat{n}$$

where, for **general** reference-surface Gaussian coordinates,

$$m^{(1)} \Big|_{\langle 1 \rangle} = \frac{1}{A_1} \frac{\partial m_1}{\partial \xi_1} - \frac{m_2 \csc \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\csc \theta_{12}}{\rho_{11}} \left(m_1 \cos \theta_{12} + m_2 \right) + m_3 \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right)$$

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

$$m^{(2)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_2}{\partial \xi_1} + \frac{\csc \theta_{12}}{\rho_{11}} \left(m_1 + m_2 \cos \theta_{12} \right) + \frac{m_2 \cot \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - m_3 \frac{\csc \theta_{12}}{r_{12}}$$

$$m^{(3)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_3}{\partial \xi_1} - \frac{m_1}{r_{11}} + m_2 \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right)$$

$$m^{(1)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_1}{\partial \xi_2} + \frac{m_1 \cot \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} - \frac{\csc \theta_{12}}{\rho_{22}} \left(m_1 \cos \theta_{12} + m_2 \right) + m_3 \frac{\csc \theta_{12}}{r_{21}}$$

$$m^{(2)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_2}{\partial \xi_2} - \frac{m_1 \csc \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} + \frac{\csc \theta_{12}}{\rho_{22}} \left(m_1 + m_2 \cos \theta_{12} \right) + m_3 \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right)$$

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

$$m^{(3)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_3}{\partial \xi_2} - m_1 \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) - \frac{m_2}{r_{22}}$$

and where for **orthogonal** reference-surface Gaussian coordinates

$$m^{(1)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_1}{\partial \xi_1} - \frac{m_2}{\rho_{11}} + \frac{m_3}{r_{11}}$$

$$m^{(2)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_2}{\partial \xi_1} + \frac{m_1}{\rho_{11}} - \frac{m_3}{r_{12}}$$

$$m^{(3)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_3}{\partial \xi_1} - \frac{m_1}{r_{11}} + \frac{m_2}{r_{12}}$$

$$m^{(1)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_1}{\partial \xi_2} - \frac{m_2}{\rho_{22}} - \frac{m_3}{r_{12}}$$

$$m^{(2)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_2}{\partial \xi_2} + \frac{m_1}{\rho_{22}} + \frac{m_3}{r_{22}}$$

$$m^{(3)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_3}{\partial \xi_2} + \frac{m_1}{r_{12}} - \frac{m_2}{r_{22}}$$

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

- The **curvatures and torsions** for the deformed reference surface $\mathcal{S}_t$ are now given by

$$\frac{1}{r_{11}} = \frac{A_1}{\mathcal{A}_1} \frac{\sqrt{a}}{\sqrt{a}} \left[m^{(1)} \Big|_{\langle 1 \rangle} (\hat{a}_1 \cdot \hat{a}_1) + m^{(2)} \Big|_{\langle 1 \rangle} (\hat{a}_1 \cdot \hat{a}_2) + m^{(3)} \Big|_{\langle 1 \rangle} (\hat{a}_1 \cdot \hat{n}) \right]$$

$$\frac{1}{r_{22}} = \frac{A_2}{\mathcal{A}_2} \frac{\sqrt{a}}{\sqrt{a}} \left[m^{(1)} \Big|_{\langle 2 \rangle} (\hat{a}_2 \cdot \hat{a}_1) + m^{(2)} \Big|_{\langle 2 \rangle} (\hat{a}_2 \cdot \hat{a}_2) + m^{(3)} \Big|_{\langle 2 \rangle} (\hat{a}_2 \cdot \hat{n}) \right]$$

$$\begin{aligned} \frac{1}{r_{12}} = & \frac{A_1 \cot \theta_{12}}{\mathcal{A}_1} \frac{\sqrt{a}}{\sqrt{a}} \left[m^{(1)} \Big|_{\langle 1 \rangle} (\hat{a}_1 \cdot \hat{a}_1) + m^{(2)} \Big|_{\langle 1 \rangle} (\hat{a}_1 \cdot \hat{a}_2) + m^{(3)} \Big|_{\langle 1 \rangle} (\hat{a}_1 \cdot \hat{n}) \right] \\ & - \frac{A_1 \csc \theta_{12}}{\mathcal{A}_1} \frac{\sqrt{a}}{\sqrt{a}} \left[m^{(1)} \Big|_{\langle 1 \rangle} (\hat{a}_2 \cdot \hat{a}_1) + m^{(2)} \Big|_{\langle 1 \rangle} (\hat{a}_2 \cdot \hat{a}_2) + m^{(3)} \Big|_{\langle 1 \rangle} (\hat{a}_2 \cdot \hat{n}) \right] \end{aligned}$$

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

$$\frac{1}{r_{21}} = \frac{A_2 \csc \theta_{12}}{\mathcal{A}_2} \frac{\sqrt{a}}{\sqrt{a}} \left[m^{(1)} \Big|_{(2)} (\hat{a}_1 \cdot \hat{a}_1) + m^{(2)} \Big|_{(2)} (\hat{a}_1 \cdot \hat{a}_2) + m^{(3)} \Big|_{(2)} (\hat{a}_1 \cdot \hat{n}) \right] \\ - \frac{A_2 \cot \theta_{12}}{\mathcal{A}_2} \frac{\sqrt{a}}{\sqrt{a}} \left[m^{(1)} \Big|_{(2)} (\hat{a}_2 \cdot \hat{a}_1) + m^{(2)} \Big|_{(2)} (\hat{a}_2 \cdot \hat{a}_2) + m^{(3)} \Big|_{(2)} (\hat{a}_2 \cdot \hat{n}) \right]$$

- At this point, it is convenient to express $\hat{a}_1 = \frac{\vec{a}_1}{\mathcal{A}_1}$ and $\hat{a}_2 = \frac{\vec{a}_2}{\mathcal{A}_2}$ as

$$\hat{a}_1 = \frac{A_1}{\mathcal{A}_1} \left(\hat{a}_1 + \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \right) \quad \text{and} \quad \hat{a}_2 = \frac{A_2}{\mathcal{A}_2} \left(\hat{a}_2 + \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \right)$$

and to use $\frac{1}{A_{(\alpha)}} \frac{\partial \vec{u}}{\partial \xi_\alpha} = u^{(\beta)} \Big|_{(\alpha)} \hat{a}_\beta + u^{(3)} \Big|_{(\alpha)} \hat{n}$ to get

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

$$\hat{a}_1 = \frac{A_1}{\mathcal{A}_1} \left[\left(1 + u^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} \right) \hat{a}_1 + u^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} \hat{a}_2 + u^{\langle 3 \rangle} \Big|_{\langle 1 \rangle} \hat{n} \right] \quad \text{and}$$

$$\hat{a}_2 = \frac{A_2}{\mathcal{A}_2} \left[u^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} \hat{a}_1 + \left(1 + u^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} \right) \hat{a}_2 + u^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} \hat{n} \right]$$

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

- Then

$$\hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}_1 = \frac{\mathbf{A}_1}{\mathcal{A}_1} \left[1 + \mathbf{u}^{(1)} \Big|_{\langle 1 \rangle} + \mathbf{u}^{(2)} \Big|_{\langle 1 \rangle} \cos \theta_{12} \right]$$

$$\hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}_2 = \frac{\mathbf{A}_1}{\mathcal{A}_1} \left[\left(1 + \mathbf{u}^{(1)} \Big|_{\langle 1 \rangle} \right) \cos \theta_{12} + \mathbf{u}^{(2)} \Big|_{\langle 1 \rangle} \right]$$

$$\hat{\mathbf{a}}_2 \cdot \hat{\mathbf{a}}_1 = \frac{\mathbf{A}_2}{\mathcal{A}_2} \left[\mathbf{u}^{(1)} \Big|_{\langle 2 \rangle} + \left(1 + \mathbf{u}^{(2)} \Big|_{\langle 2 \rangle} \right) \cos \theta_{12} \right]$$

$$\hat{\mathbf{a}}_2 \cdot \hat{\mathbf{a}}_2 = \frac{\mathbf{A}_2}{\mathcal{A}_2} \left[1 + \mathbf{u}^{(2)} \Big|_{\langle 2 \rangle} + \mathbf{u}^{(1)} \Big|_{\langle 2 \rangle} \cos \theta_{12} \right]$$

$$\hat{\mathbf{a}}_1 \cdot \hat{\mathbf{n}} = \frac{\mathbf{A}_1}{\mathcal{A}_1} \mathbf{u}^{(3)} \Big|_{\langle 1 \rangle}$$

$$\hat{\mathbf{a}}_2 \cdot \hat{\mathbf{n}} = \frac{\mathbf{A}_2}{\mathcal{A}_2} \mathbf{u}^{(3)} \Big|_{\langle 2 \rangle}$$

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

- The **curvatures** of the **deformed reference surface** become

$$\frac{1}{r_{11}} = \left(\frac{A_1}{\mathcal{A}_1} \right)^3 \frac{A_2 \sin \theta_{12}}{\mathcal{A}_2 \sin \theta_{12}} \left[m^{(1)} \Big|_{\langle 1 \rangle} \left(1 + u^{(1)} \Big|_{\langle 1 \rangle} + u^{(2)} \Big|_{\langle 1 \rangle} \cos \theta_{12} \right) \right. \\ \left. + m^{(2)} \Big|_{\langle 1 \rangle} \left(\left(1 + u^{(1)} \Big|_{\langle 1 \rangle} \right) \cos \theta_{12} + u^{(2)} \Big|_{\langle 1 \rangle} \right) + m^{(3)} \Big|_{\langle 1 \rangle} u^{(3)} \Big|_{\langle 1 \rangle} \right] \quad \text{and}$$

$$\frac{1}{r_{22}} = \left(\frac{A_2}{\mathcal{A}_2} \right)^3 \frac{A_1 \sin \theta_{12}}{\mathcal{A}_1 \sin \theta_{12}} \left[m^{(1)} \Big|_{\langle 2 \rangle} \left(u^{(1)} \Big|_{\langle 2 \rangle} + \left(1 + u^{(2)} \Big|_{\langle 2 \rangle} \right) \cos \theta_{12} \right) \right. \\ \left. + m^{(2)} \Big|_{\langle 2 \rangle} \left(1 + u^{(2)} \Big|_{\langle 2 \rangle} + u^{(1)} \Big|_{\langle 2 \rangle} \cos \theta_{12} \right) + m^{(3)} \Big|_{\langle 2 \rangle} u^{(3)} \Big|_{\langle 2 \rangle} \right]$$

where $\frac{\sqrt{a}}{\sqrt{\mathcal{a}}} = \frac{A_1 A_2 \sin \theta_{12}}{\mathcal{A}_1 \mathcal{A}_2 \sin \theta_{12}}$ has been used

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

- Likewise, the **torsions** become

$$\begin{aligned} \frac{1}{r_{12}} = & \left(\frac{A_1}{\mathcal{A}_1} \right)^3 \frac{A_2 \sin \theta_{12} \cot \theta_{12}}{\mathcal{A}_2 \sin \theta_{12}} \left[m^{(1)} \Big|_{\langle 1 \rangle} \left(1 + u^{(1)} \Big|_{\langle 1 \rangle} + u^{(2)} \Big|_{\langle 1 \rangle} \cos \theta_{12} \right) \right. \\ & \left. + m^{(2)} \Big|_{\langle 1 \rangle} \left(\left(1 + u^{(1)} \Big|_{\langle 1 \rangle} \right) \cos \theta_{12} + u^{(2)} \Big|_{\langle 1 \rangle} \right) + m^{(3)} \Big|_{\langle 1 \rangle} u^{(3)} \Big|_{\langle 1 \rangle} \right] \\ & - \left(\frac{A_1 A_2}{\mathcal{A}_1 \mathcal{A}_2} \right)^2 \frac{\sin \theta_{12}}{\sin^2 \theta_{12}} \left[m^{(1)} \Big|_{\langle 1 \rangle} \left(u^{(1)} \Big|_{\langle 2 \rangle} + \left(1 + u^{(2)} \Big|_{\langle 2 \rangle} \right) \cos \theta_{12} \right) \right. \\ & \left. + m^{(2)} \Big|_{\langle 1 \rangle} \left(1 + u^{(2)} \Big|_{\langle 2 \rangle} + u^{(1)} \Big|_{\langle 2 \rangle} \cos \theta_{12} \right) + m^{(3)} \Big|_{\langle 1 \rangle} u^{(3)} \Big|_{\langle 2 \rangle} \right] \end{aligned}$$

and

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

$$\begin{aligned} \frac{1}{r_{21}} = & \left(\frac{\mathbf{A}_1 \mathbf{A}_2}{\mathcal{A}_1 \mathcal{A}_2} \right)^2 \frac{\sin \theta_{12}}{\sin^2 \theta_{12}} \left[m^{(1)} \Big|_{\langle 2 \rangle} \left(1 + u^{(1)} \Big|_{\langle 1 \rangle} + u^{(2)} \Big|_{\langle 1 \rangle} \cos \theta_{12} \right) \right. \\ & \left. + m^{(2)} \Big|_{\langle 2 \rangle} \left(\left(1 + u^{(1)} \Big|_{\langle 1 \rangle} \right) \cos \theta_{12} + u^{(2)} \Big|_{\langle 1 \rangle} \right) + m^{(3)} \Big|_{\langle 2 \rangle} u^{(3)} \Big|_{\langle 1 \rangle} \right] \\ & - \left(\frac{\mathbf{A}_2}{\mathcal{A}_2} \right)^3 \frac{\mathbf{A}_1 \sin \theta_{12} \cot \theta_{12}}{\mathcal{A}_1 \sin \theta_{12}} \left[m^{(1)} \Big|_{\langle 2 \rangle} \left(u^{(1)} \Big|_{\langle 2 \rangle} + \left(1 + u^{(2)} \Big|_{\langle 2 \rangle} \right) \cos \theta_{12} \right) \right. \\ & \left. + m^{(2)} \Big|_{\langle 2 \rangle} \left(1 + u^{(2)} \Big|_{\langle 2 \rangle} + u^{(1)} \Big|_{\langle 2 \rangle} \cos \theta_{12} \right) + m^{(3)} \Big|_{\langle 2 \rangle} u^{(3)} \Big|_{\langle 2 \rangle} \right] \end{aligned}$$

where it is recalled that the **Lame' parameters of the deformed surface** and the corresponding trigonometric functions of θ_{12} are complicated irrational functions of the displacements

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

- The **changes in surface curvature**, $K_{11}^o(\xi_1, \xi_2, t)$ and $K_{22}^o(\xi_1, \xi_2, t)$, and the **change in surface torsion**, $K_{12}^o(\xi_1, \xi_2, t)$, caused by deformation are defined by

$$K_{11}^o \equiv \frac{1}{\kappa_{11}} - \frac{1}{r_{11}}$$

$$K_{22}^o \equiv \frac{1}{\kappa_{22}} - \frac{1}{r_{22}}$$

$$K_{12}^o \equiv - \left[\frac{1}{2} \left(\frac{1}{\kappa_{12}} - \frac{1}{\kappa_{21}} \right) - \frac{1}{2} \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) \right]$$

- Examination of the previously derived expressions for $\kappa_{\alpha\beta}^{-1}$ indicates that the changes in surface curvature and torsion are **complicated nonlinear functions** of the surface displacement field
 - Thus, these expressions appear to have very limited practical value
 - These expressions also have **no implicit limitations on the magnitude of the displacements** that may occur during deformation

CURVATURES AND TORSIONS OF $\mathcal{S}_i$ - CONTINUED

- For orthogonal Gaussian coordinates the **change in surface torsion** reduces to

$$K_{12}^o \equiv - \left[\frac{1}{2} \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) - \frac{1}{r_{12}} \right]$$

- In addition, the components of $\vec{m}$ reduce to

$$m_1 = u^{(2)} \Big|_{(1)} u^{(3)} \Big|_{(2)} - u^{(3)} \Big|_{(1)} \left(1 + u^{(2)} \Big|_{(2)} \right)$$

$$m_2 = u^{(3)} \Big|_{(1)} u^{(1)} \Big|_{(2)} - u^{(3)} \Big|_{(2)} \left(1 + u^{(1)} \Big|_{(1)} \right)$$

$$m_3 = \left(1 + u^{(1)} \Big|_{(1)} \right) \left(1 + u^{(2)} \Big|_{(2)} \right) - u^{(2)} \Big|_{(1)} u^{(1)} \Big|_{(2)}$$

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

- Moreover, the **curvatures** reduce to

$$\frac{1}{r_{11}} = \left(\frac{A_1}{\mathcal{A}_1} \right)^3 \frac{A_2}{\mathcal{A}_2 \sin \theta_{12}} \left[\left(1 + u^{(1)} \Big|_{\langle 1 \rangle} \right) m^{(1)} \Big|_{\langle 1 \rangle} + u^{(2)} \Big|_{\langle 1 \rangle} m^{(2)} \Big|_{\langle 1 \rangle} + u^{(3)} \Big|_{\langle 1 \rangle} m^{(3)} \Big|_{\langle 1 \rangle} \right]$$

$$\frac{1}{r_{22}} = \left(\frac{A_2}{\mathcal{A}_2} \right)^3 \frac{A_1}{\mathcal{A}_1 \sin \theta_{12}} \left[u^{(1)} \Big|_{\langle 2 \rangle} m^{(1)} \Big|_{\langle 2 \rangle} + \left(1 + u^{(2)} \Big|_{\langle 2 \rangle} \right) m^{(2)} \Big|_{\langle 2 \rangle} + u^{(3)} \Big|_{\langle 2 \rangle} m^{(3)} \Big|_{\langle 2 \rangle} \right]$$

and the **torsions** reduce to

$$\begin{aligned} \frac{1}{r_{12}} = & \left(\frac{A_1}{\mathcal{A}_1} \right)^3 \frac{A_2}{\mathcal{A}_2} \csc \theta_{12} \cot \theta_{12} \left[m^{(1)} \Big|_{\langle 1 \rangle} \left(1 + u^{(1)} \Big|_{\langle 1 \rangle} \right) + m^{(2)} \Big|_{\langle 1 \rangle} u^{(2)} \Big|_{\langle 1 \rangle} + m^{(3)} \Big|_{\langle 1 \rangle} u^{(3)} \Big|_{\langle 1 \rangle} \right] \\ & - \left(\frac{A_1 A_2}{\mathcal{A}_1 \mathcal{A}_2} \right)^2 \csc^2 \theta_{12} \left[m^{(1)} \Big|_{\langle 1 \rangle} u^{(1)} \Big|_{\langle 2 \rangle} + m^{(2)} \Big|_{\langle 1 \rangle} \left(1 + u^{(2)} \Big|_{\langle 2 \rangle} \right) + m^{(3)} \Big|_{\langle 1 \rangle} u^{(3)} \Big|_{\langle 2 \rangle} \right] \end{aligned}$$

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONTINUED

$$\frac{1}{r_{21}} = \left(\frac{A_1 A_2}{\mathcal{A}_1 \mathcal{A}_2} \right)^2 \csc^2 \theta_{12} \left[m^{(1)} \Big|_{\langle 2 \rangle} \left(1 + u^{(1)} \Big|_{\langle 1 \rangle} \right) + m^{(2)} \Big|_{\langle 2 \rangle} u^{(2)} \Big|_{\langle 1 \rangle} + m^{(3)} \Big|_{\langle 2 \rangle} u^{(3)} \Big|_{\langle 1 \rangle} \right] \\ - \left(\frac{A_2}{\mathcal{A}_2} \right)^3 \frac{A_1}{\mathcal{A}_1} \csc \theta_{12} \cot \theta_{12} \left[m^{(1)} \Big|_{\langle 2 \rangle} u^{(1)} \Big|_{\langle 2 \rangle} + m^{(2)} \Big|_{\langle 2 \rangle} \left(1 + u^{(2)} \Big|_{\langle 2 \rangle} \right) + m^{(3)} \Big|_{\langle 2 \rangle} u^{(3)} \Big|_{\langle 2 \rangle} \right]$$

- Furthermore, the **Lame' parameters of the deformed surface** reduce to

$$\mathcal{A}_1 = A_1 \sqrt{1 + 2 u^{(1)} \Big|_{\langle 1 \rangle} + u^{(1)} \Big|_{\langle 1 \rangle} u^{(1)} \Big|_{\langle 1 \rangle} + u^{(2)} \Big|_{\langle 1 \rangle} u^{(2)} \Big|_{\langle 1 \rangle} + u^{(3)} \Big|_{\langle 1 \rangle} u^{(3)} \Big|_{\langle 1 \rangle}}$$

$$\mathcal{A}_2 = A_2 \sqrt{1 + 2 u^{(2)} \Big|_{\langle 2 \rangle} + u^{(1)} \Big|_{\langle 2 \rangle} u^{(1)} \Big|_{\langle 2 \rangle} + u^{(2)} \Big|_{\langle 2 \rangle} u^{(2)} \Big|_{\langle 2 \rangle} + u^{(3)} \Big|_{\langle 2 \rangle} u^{(3)} \Big|_{\langle 2 \rangle}}$$

- In addition,

$$\frac{\mathcal{A}_1 \mathcal{A}_2}{A_1 A_2} \cos \theta_{12} = u^{(1)} \Big|_{\langle 2 \rangle} + u^{(2)} \Big|_{\langle 1 \rangle} + u^{(1)} \Big|_{\langle 1 \rangle} u^{(1)} \Big|_{\langle 2 \rangle} \\ + u^{(2)} \Big|_{\langle 1 \rangle} u^{(2)} \Big|_{\langle 2 \rangle} + u^{(3)} \Big|_{\langle 1 \rangle} u^{(3)} \Big|_{\langle 2 \rangle}$$

CURVATURES AND TORSIONS OF $\mathcal{S}_t$ - CONCLUDED

- To obtain practical expressions that **characterize the geometry of the shell reference surface**, it is first noted that deformation can be partitioned in **strain and rotation** of material line segments, areas, and volumes
- Thus, simplification of the previously derived deformed-surface expressions can be obtained by **examining practical limitations** on the magnitudes of the strains and rotations

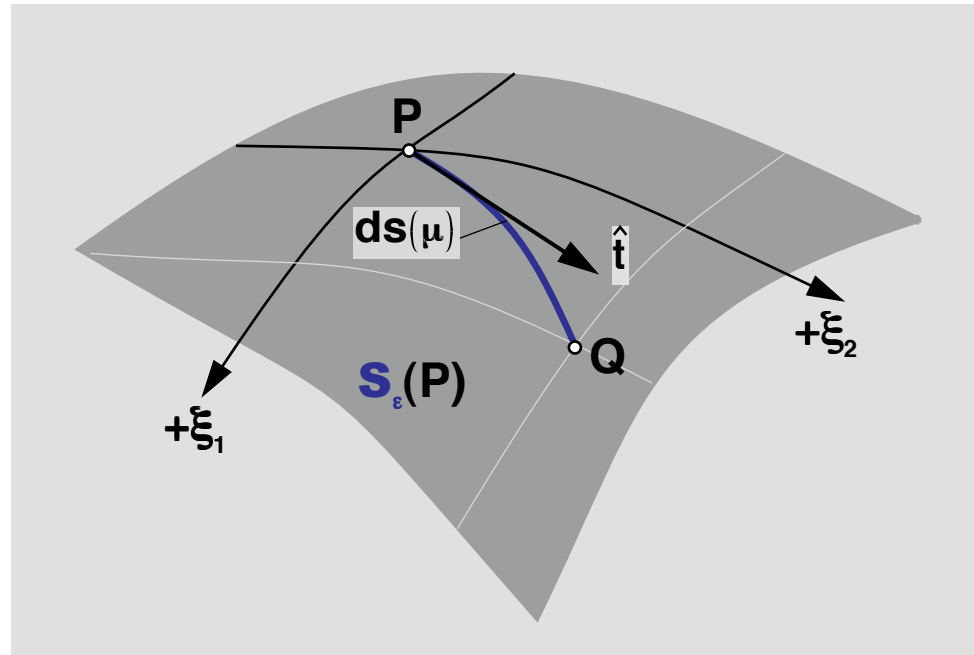
CHARACTERIZATION OF REFERENCE-SURFACE DEFORMATIONS

ELONGATION AND SHEARING OF $\mathcal{S}_0$

- Two primitive aspects of deformation are **elongation** and **shearing**
- Consider the differential reference-surface arc $ds(\mu) = \widehat{PQ}$ shown in the figure
- This arc is part of a surface curve, defined parametrically by

$$\vec{\mathbf{x}}(\mu) = \mathbf{x}_k(\xi_1(\mu), \xi_2(\mu)) \hat{\mathbf{i}}_k$$

where μ is a parameter

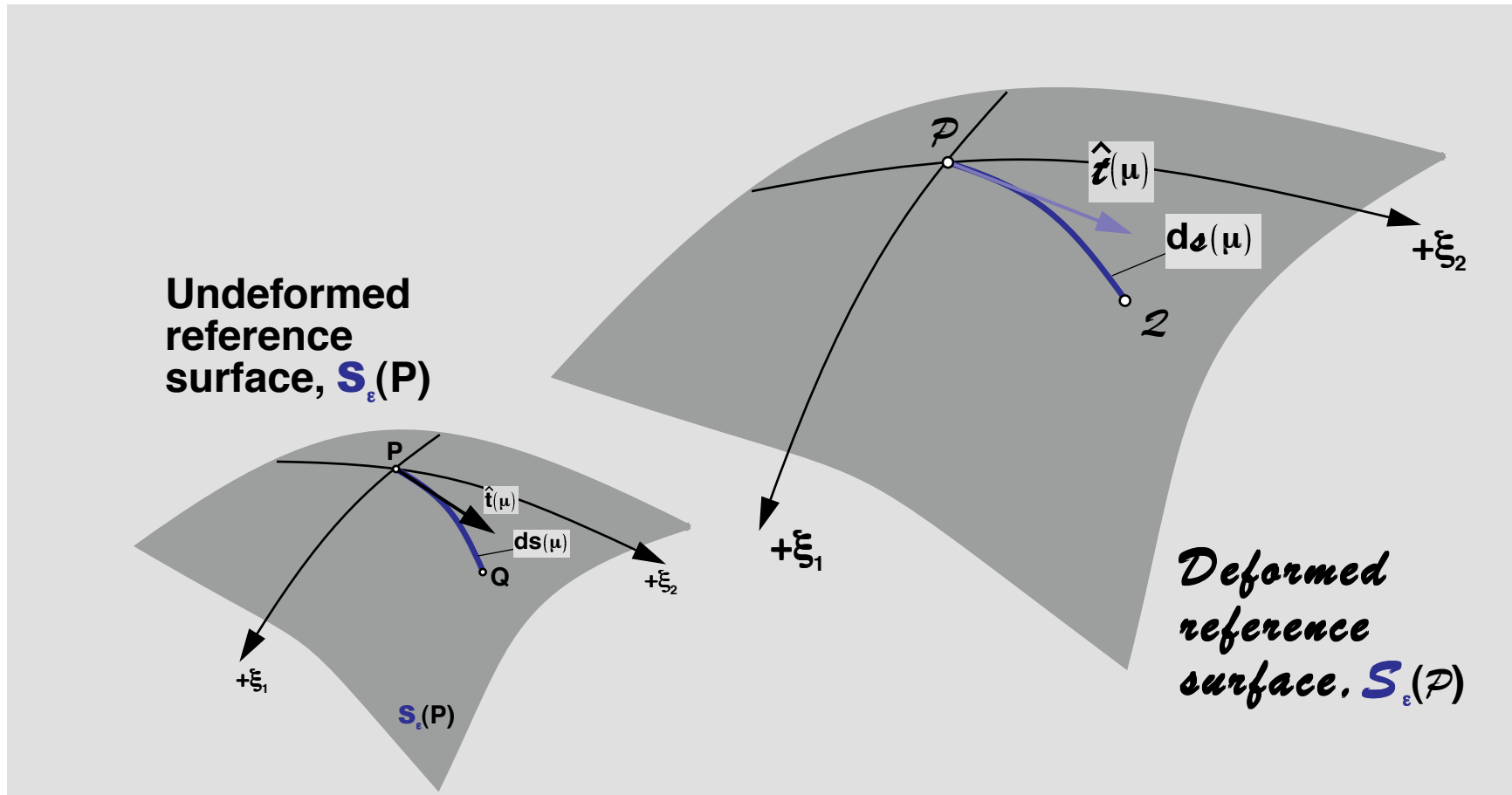


- The corresponding unit-magnitude tangent vector is denoted by $\hat{\mathbf{t}}$

ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONTINUED

- Other surface curves are given by different parametrizations of the surface coordinates (ξ_1, ξ_2)
- During deformation, the material points forming the reference-surface region in the neighborhood of point P , $\mathcal{S}_\varepsilon(P)$, generally change their spatial orientation and their relative positions
- As a result, the arc of material points $\widehat{PQ}$ also generally undergoes a rigid-body motion and either **elongates or contracts** (negative elongation) into the deformed image $\widehat{\mathcal{P}2}$ shown in the next figure
- The differential arc length of $\widehat{\mathcal{P}2}$ is denoted by $d\mathcal{L}(\mu)$ and the corresponding unit-magnitude tangent vector is denoted by $\hat{\mathbf{t}}(\mu)$

ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONTINUED



ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONTINUED

- The **elongation** $e_{\hat{\mathbf{t}}(\mu)}(\xi_1(\mu), \xi_2(\mu), t)$ of the differential arc $\widehat{PQ}$, as it

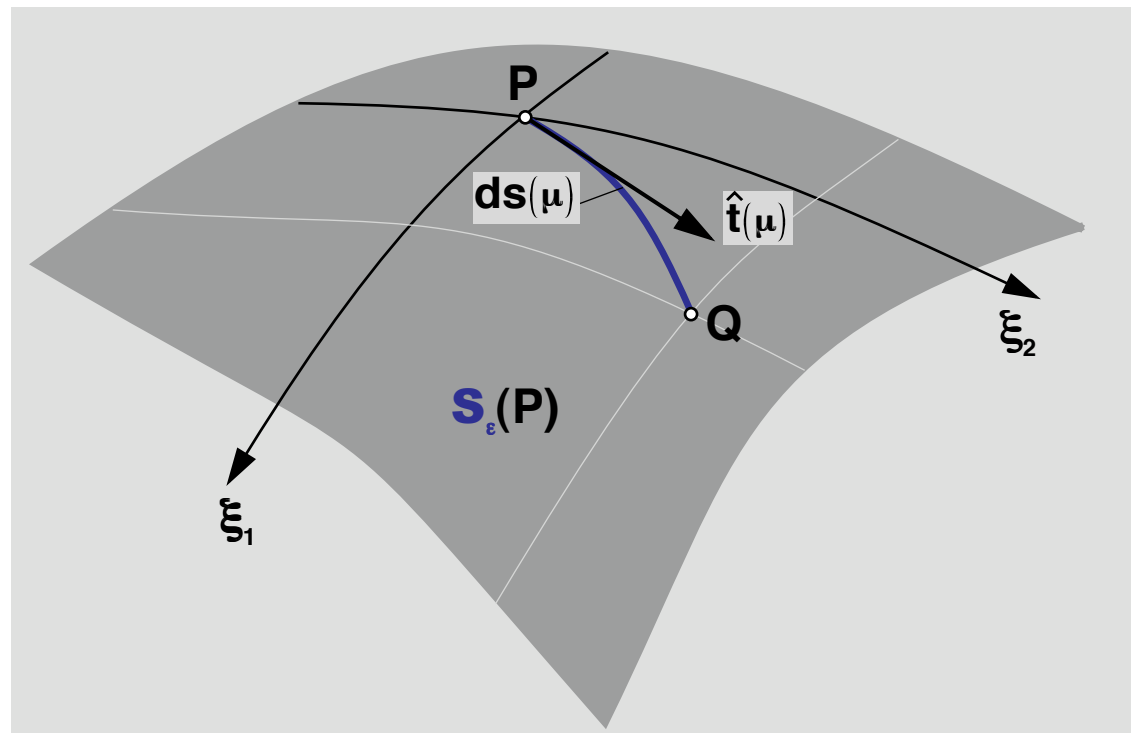
deforms into the differential arc $\widehat{PQ}$, is defined as

$$e_{\hat{\mathbf{t}}(\mu)} \equiv \frac{d\ell - ds}{ds}$$

- Note that in this definition, the **unit-magnitude vector field**

$$\hat{\mathbf{t}}(\xi_1(\mu), \xi_2(\mu)) = \hat{\mathbf{t}}(\mu) \equiv \frac{d\vec{\mathbf{x}}}{ds}$$

defines the initial orientation of $\widehat{PQ}$ with respect to the Gaussian coordinates of the point P



ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONTINUED

- From the definition $\mathbf{e}_{\hat{\mathbf{t}}_{(\mu)}} = \frac{d\mathcal{L} - ds}{ds}$, it follows that

$$d\mathcal{L} = \left(1 + \mathbf{e}_{\hat{\mathbf{t}}_{(\mu)}}\right) ds \quad \text{and} \quad \left(1 + \mathbf{e}_{\hat{\mathbf{t}}_{(\mu)}}\right)^2 - 1 = \frac{d\mathcal{L}^2 - ds^2}{ds^2}$$

- Using $d\mathcal{L}^2 = (\vec{\mathbf{a}}_\alpha \cdot \vec{\mathbf{a}}_\beta) d\xi_\alpha d\xi_\beta$ and $ds^2 = (\vec{\mathbf{a}}_\alpha \cdot \vec{\mathbf{a}}_\beta) d\xi_\alpha d\xi_\beta$ gives

$$\left(1 + \mathbf{e}_{\hat{\mathbf{t}}_{(\mu)}}\right)^2 - 1 = \left[(\vec{\mathbf{a}}_\alpha \cdot \vec{\mathbf{a}}_\beta) - (\vec{\mathbf{a}}_\alpha \cdot \vec{\mathbf{a}}_\beta) \right] \frac{d\xi_\alpha}{ds} \frac{d\xi_\beta}{ds}$$

- Noting that $\hat{\mathbf{t}}_{(\mu)} \equiv \frac{d\vec{\mathbf{x}}}{ds} = \frac{\partial \vec{\mathbf{x}}}{\partial \xi_\alpha} \frac{d\xi_\alpha}{ds} = \vec{\mathbf{a}}_\alpha \frac{d\xi_\alpha}{ds} = \mathbf{A}_{(\alpha)} \hat{\mathbf{a}}_\alpha \frac{d\xi_\alpha}{ds}$ gives

$$\mathbf{A}_{(\alpha)} \frac{d\xi_\alpha}{ds} = \left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^\alpha \right) \csc \theta_{12} \quad \text{and} \quad \hat{\mathbf{t}}_{(\mu)} = \left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^\alpha \right) \csc \theta_{12} \hat{\mathbf{a}}_\alpha$$

ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONTINUED

- Thus,
$$\left(1 + e_{\hat{\mathbf{t}}_{(\mu)}}\right)^2 - 1 = \left[\left(\vec{\mathbf{a}}_{\alpha} \cdot \vec{\mathbf{a}}_{\beta}\right) - \left(\hat{\mathbf{a}}_{\alpha} \cdot \hat{\mathbf{a}}_{\beta}\right) \right] \frac{\left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^{\alpha}\right)\left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^{\beta}\right)}{\mathbf{A}_{(\alpha)}\mathbf{A}_{(\beta)} \sin^2 \theta_{12}}$$

- The **three independent quantities** given by

$$\frac{1}{\mathbf{A}_{(\alpha)}\mathbf{A}_{(\beta)}} \left[\left(\vec{\mathbf{a}}_{\alpha} \cdot \vec{\mathbf{a}}_{\beta}\right) - \left(\hat{\mathbf{a}}_{\alpha} \cdot \hat{\mathbf{a}}_{\beta}\right) \right] = \frac{\vec{\mathbf{a}}_{\alpha} \cdot \vec{\mathbf{a}}_{\beta}}{\mathbf{A}_{(\alpha)}\mathbf{A}_{(\beta)}} - \hat{\mathbf{a}}_{\alpha} \cdot \hat{\mathbf{a}}_{\beta} \equiv \mathbf{2}\varepsilon_{\alpha\beta}^o \left(\xi_1, \xi_2, \boldsymbol{\tau} \right)$$

completely define the elongation $e_{\hat{\mathbf{t}}_{(\mu)}}(\xi_1, \xi_2, \boldsymbol{\tau})$; that is,

$$e_{\hat{\mathbf{t}}_{(\mu)}} = \sqrt{1 + 2\varepsilon_{\alpha\beta}^o \left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^{\alpha} \right) \left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^{\beta} \right) \csc^2 \theta_{12}} - 1$$

- Also,
$$\left(1 + e_{\hat{\mathbf{t}}_{(\mu)}}\right)^2 - 1 = \frac{d\boldsymbol{\ell}^2 - ds^2}{ds^2} \text{ gives } \frac{d\boldsymbol{\ell}^2 - ds^2}{ds^2} = 2\varepsilon_{\alpha\beta}^o \frac{\left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^{\alpha}\right)\left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^{\beta}\right)}{\sin^2 \theta_{12}}$$

ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONTINUED

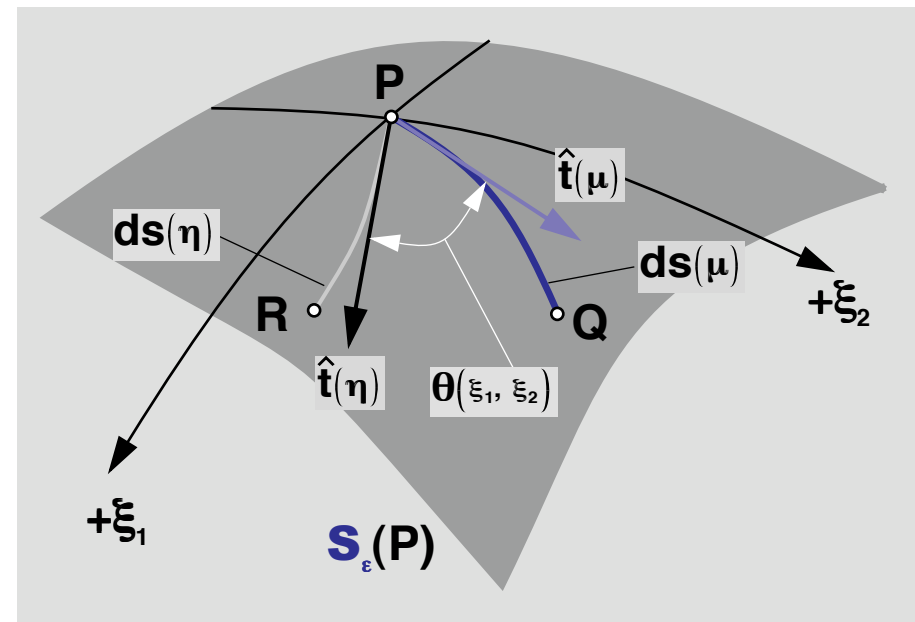
- Now consider the intersecting differential arcs associated with the **two distinct surface curves** passing through point P of the undeformed reference surface, as shown in the figure

- The arc $ds(\mu) = \widehat{PQ}$ is defined parametrically by

$$\vec{x}(\mu) = \mathbf{x}_k(\xi_1(\mu), \xi_2(\mu)) \hat{\mathbf{i}}_k$$

- The arc $ds(\eta) = \widehat{PR}$ is defined parametrically by

$$\vec{x}(\eta) = \mathbf{x}_k(\xi_1(\eta), \xi_2(\eta)) \hat{\mathbf{i}}_k$$



- The vectors $\hat{\mathbf{t}}(\mu)$ and $\hat{\mathbf{t}}(\eta)$ have unit magnitude and are tangent to the arcs $\widehat{PQ}$ and $\widehat{PR}$, respectively, and are presumed to be distinct

ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONTINUED

- The angle $\theta(\xi_1, \xi_2)$ between $\hat{\mathbf{t}}_{(\mu)}$ and $\hat{\mathbf{t}}_{(\eta)}$ is given by

$$\cos\theta = \hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{t}}_{(\eta)} \quad \text{and} \quad \sin\theta = (\hat{\mathbf{t}}_{(\eta)} \times \hat{\mathbf{t}}_{(\mu)}) \cdot \hat{\mathbf{n}}$$

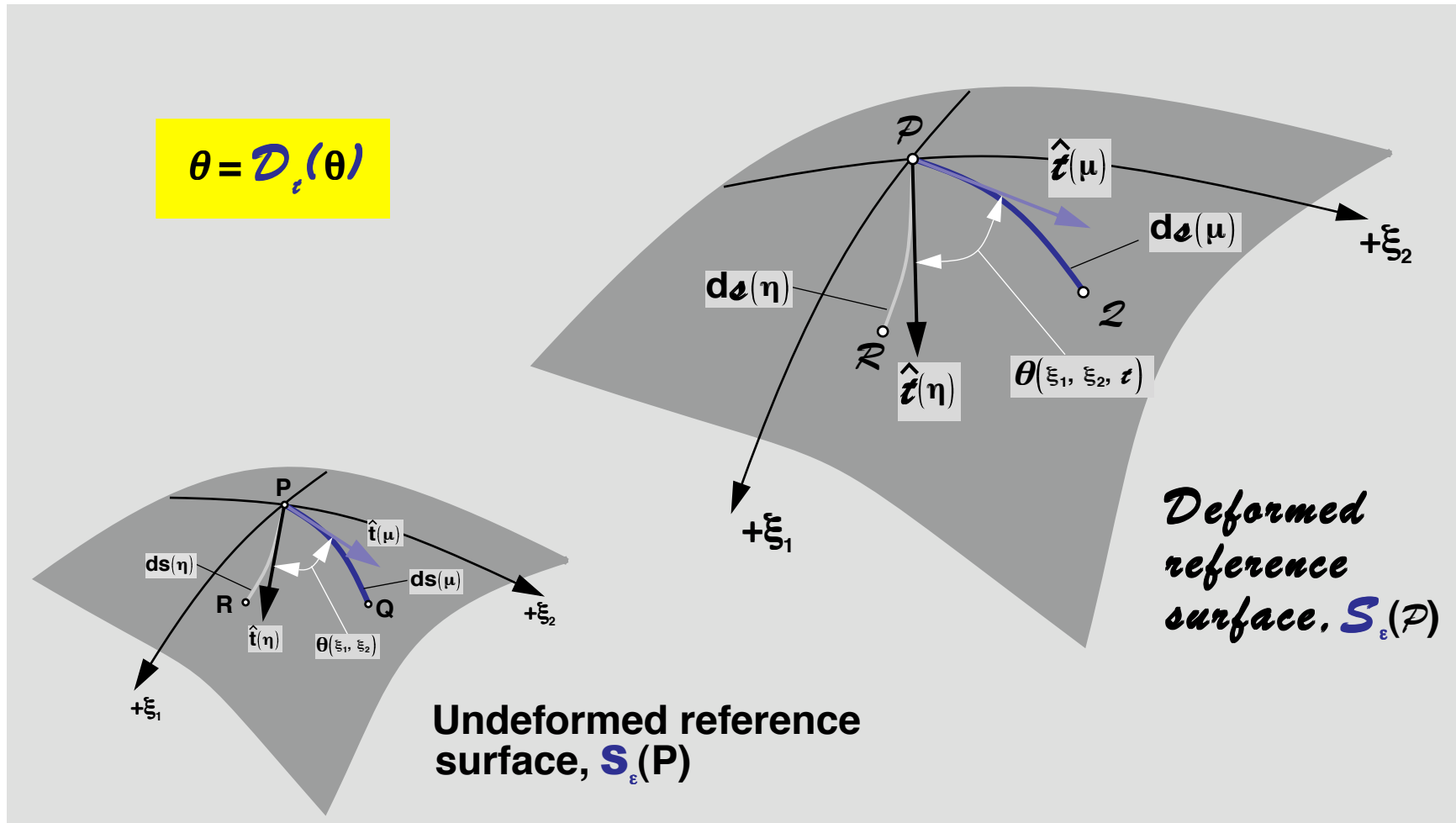
- As the reference surface deforms, the angle $\theta(\xi_1, \xi_2)$ generally changes into a different angle, given by $\theta(\xi_1, \xi_2, \tau)$

- That is, $\theta = \mathcal{D}_\tau(\theta)$ such that $\theta(\xi_1, \xi_2, 0) = \theta(\xi_1, \xi_2)$

- The arc $ds_{(\mu)} = \widehat{PQ}$ deforms into the arc $d\boldsymbol{\mathcal{A}}_{(\mu)} = \widehat{\mathcal{P}\mathcal{Q}}$ and the arc $ds_{(\eta)} = \widehat{PR}$ deforms into the arc $d\boldsymbol{\mathcal{A}}_{(\eta)} = \widehat{\mathcal{P}\mathcal{R}}$

- Unit-magnitude vector fields that are tangent to the deformed arcs $\widehat{\mathcal{P}\mathcal{Q}}$ and $\widehat{\mathcal{P}\mathcal{R}}$ are given by $\hat{\boldsymbol{\mathcal{t}}}_{(\mu)} = \frac{d\vec{\boldsymbol{\mathcal{x}}}(\mu)}{d\boldsymbol{\mathcal{A}}_{(\mu)}}$ and $\hat{\boldsymbol{\mathcal{t}}}_{(\eta)} = \frac{d\vec{\boldsymbol{\mathcal{x}}}(\eta)}{d\boldsymbol{\mathcal{A}}_{(\eta)}}$, respectively

ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONTINUED



ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONTINUED

- Using the previously derived relations $A_{(\alpha)} \frac{d\xi_\alpha}{ds(\mu)} = \hat{\mathbf{t}}(\mu) \cdot \hat{\mathbf{a}}^\alpha \csc\theta_{12}$ and

$d\mathcal{L}(\mu) = (1 + e_{\hat{\mathbf{t}}(\mu)})ds(\mu)$, it follows by direct analogy that

$$A_{(\alpha)} \frac{d\xi_\alpha}{ds(\eta)} = \hat{\mathbf{t}}(\eta) \cdot \hat{\mathbf{a}}^\alpha \csc\theta_{12} \quad \text{and} \quad d\mathcal{L}(\eta) = (1 + e_{\hat{\mathbf{t}}(\eta)})ds(\eta)$$

- Therefore,

$$\hat{\mathbf{z}}(\mu) = \frac{d\vec{\mathbf{x}}(\mu)}{d\mathcal{L}(\mu)} = \frac{\partial \vec{\mathbf{x}}(\mu)}{\partial \xi_\alpha} \frac{d\xi_\alpha}{d\mathcal{L}(\mu)} = \frac{\vec{\mathbf{a}}_\alpha}{1 + e_{\hat{\mathbf{t}}(\mu)}} \frac{d\xi_\alpha}{ds(\mu)} = \frac{\vec{\mathbf{a}}_\alpha (\hat{\mathbf{t}}(\mu) \cdot \hat{\mathbf{a}}^\alpha) \csc\theta_{12}}{A_{(\alpha)} [1 + e_{\hat{\mathbf{t}}(\mu)}]}$$

$$\hat{\mathbf{z}}(\eta) = \frac{d\vec{\mathbf{x}}(\eta)}{d\mathcal{L}(\eta)} = \frac{\partial \vec{\mathbf{x}}(\eta)}{\partial \xi_\alpha} \frac{d\xi_\alpha}{d\mathcal{L}(\eta)} = \frac{\vec{\mathbf{a}}_\alpha}{1 + e_{\hat{\mathbf{t}}(\eta)}} \frac{d\xi_\alpha}{ds(\eta)} = \frac{\vec{\mathbf{a}}_\alpha (\hat{\mathbf{t}}(\eta) \cdot \hat{\mathbf{a}}^\alpha) \csc\theta_{12}}{A_{(\alpha)} [1 + e_{\hat{\mathbf{t}}(\eta)}]}$$

ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONTINUED

- Next, noting that $\cos\theta = \hat{\mathbf{z}}^{(\mu)} \cdot \hat{\mathbf{z}}^{(\eta)}$ gives

$$\cos\theta = \frac{(\hat{\mathbf{t}}^{(\mu)} \cdot \hat{\mathbf{a}}^\alpha)(\hat{\mathbf{t}}^{(\eta)} \cdot \hat{\mathbf{a}}^\beta) \csc^2\theta_{12}}{\mathbf{A}_{(\alpha)}\mathbf{A}_{(\beta)}[1 + e_{\hat{\mathbf{t}}^{(\mu)}}][1 + e_{\hat{\mathbf{t}}^{(\eta)}}]} (\vec{\mathbf{a}}_\alpha \cdot \vec{\mathbf{a}}_\beta)$$

- Also, from the definition $\frac{\vec{\mathbf{a}}_\alpha \cdot \vec{\mathbf{a}}_\beta}{\mathbf{A}_{(\alpha)}\mathbf{A}_{(\beta)}} - \hat{\mathbf{a}}_\alpha \cdot \hat{\mathbf{a}}_\beta \equiv 2\varepsilon_{\alpha\beta}^o(\xi_1, \xi_2, \epsilon)$ and from

$$\hat{\mathbf{a}}_\alpha \cdot \hat{\mathbf{a}}_\beta = \delta_{\alpha\beta} + (1 - \delta_{\alpha\beta})\cos\theta_{12} \quad \text{it follows that}$$

$$(\vec{\mathbf{a}}_\alpha \cdot \vec{\mathbf{a}}_\beta) = \left[2\varepsilon_{\alpha\beta}^o + \delta_{\alpha\beta} + (1 - \delta_{\alpha\beta})\cos\theta_{12} \right] \mathbf{A}_{(\alpha)}\mathbf{A}_{(\beta)}$$

- Therefore,
$$\cos\theta = \frac{(\hat{\mathbf{t}}^{(\mu)} \cdot \hat{\mathbf{a}}^\alpha)(\hat{\mathbf{t}}^{(\eta)} \cdot \hat{\mathbf{a}}^\beta) \csc^2\theta_{12}}{[1 + e_{\hat{\mathbf{t}}^{(\mu)}}][1 + e_{\hat{\mathbf{t}}^{(\eta)}}]} \left[2\varepsilon_{\alpha\beta}^o + \delta_{\alpha\beta} + (1 - \delta_{\alpha\beta})\cos\theta_{12} \right]$$

ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONTINUED

- Now, using

$$\mathbf{e}_{\hat{\mathbf{t}}(\mu)} = \sqrt{1 + 2\epsilon_{\alpha\beta}^{\circ} (\hat{\mathbf{t}}(\mu) \cdot \hat{\mathbf{a}}^{\alpha})(\hat{\mathbf{t}}(\mu) \cdot \hat{\mathbf{a}}^{\beta}) \csc^2 \theta_{12}} - 1 \quad \text{and}$$

$$\mathbf{e}_{\hat{\mathbf{t}}(\eta)} = \sqrt{1 + 2\epsilon_{\alpha\beta}^{\circ} (\hat{\mathbf{t}}(\eta) \cdot \hat{\mathbf{a}}^{\alpha})(\hat{\mathbf{t}}(\eta) \cdot \hat{\mathbf{a}}^{\beta}) \csc^2 \theta_{12}} - 1$$

gives

$$\cos \theta = \frac{(\hat{\mathbf{t}}(\mu) \cdot \hat{\mathbf{a}}^{\alpha})(\hat{\mathbf{t}}(\eta) \cdot \hat{\mathbf{a}}^{\beta}) \left[2\epsilon_{\alpha\beta}^{\circ} + \delta_{\alpha\beta} + (1 - \delta_{\alpha\beta}) \cos \theta_{12} \right]}{\sqrt{1 + 2\epsilon_{\sigma\chi}^{\circ} (\hat{\mathbf{t}}(\mu) \cdot \hat{\mathbf{a}}^{\sigma})(\hat{\mathbf{t}}(\mu) \cdot \hat{\mathbf{a}}^{\chi})} \sqrt{1 + 2\epsilon_{\lambda\rho}^{\circ} (\hat{\mathbf{t}}(\eta) \cdot \hat{\mathbf{a}}^{\lambda})(\hat{\mathbf{t}}(\eta) \cdot \hat{\mathbf{a}}^{\rho})}}$$

- $\cos \theta = \hat{\mathbf{t}}(\mu) \cdot \hat{\mathbf{t}}(\eta)$ and the previous expression show that the three independent quantities $\epsilon_{\alpha\beta}^{\circ}$ also completely defined the change in angle
- The **shear** or scissoring of the arcs, $\gamma^{\circ}(\xi_1, \xi_2, \epsilon)$, is defined by $\gamma^{\circ} = \theta - \theta$

ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONTINUED

- For the **special case** where $\hat{\mathbf{t}}_{(\eta)} = \hat{\mathbf{a}}_1$ and $\hat{\mathbf{t}}_{(\mu)} = \hat{\mathbf{a}}_2$, the two differential arcs of the undeformed reference surface are aligned with the ξ_1 and ξ_2 Gaussian-coordinate curves and are denoted by $ds_{\langle 1 \rangle}$ and $ds_{\langle 2 \rangle}$, respectively
- During deformation, $ds_{\langle 1 \rangle}$ and $ds_{\langle 2 \rangle}$ deform into $d\boldsymbol{\ell}_{\langle 1 \rangle}$ and $d\boldsymbol{\ell}_{\langle 2 \rangle}$, respectively, as shown in the next figure
- The angle between intersecting convected differential arcs is given by

$$\cos \theta_{12} = \frac{\sin^2 \theta_{12} \left[2\varepsilon_{12}^o + \cos \theta_{12} \right]}{\sqrt{1 + 2\varepsilon_{11}^o \sin^2 \theta_{12}} \sqrt{1 + 2\varepsilon_{22}^o \sin^2 \theta_{12}}}$$

ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONTINUED

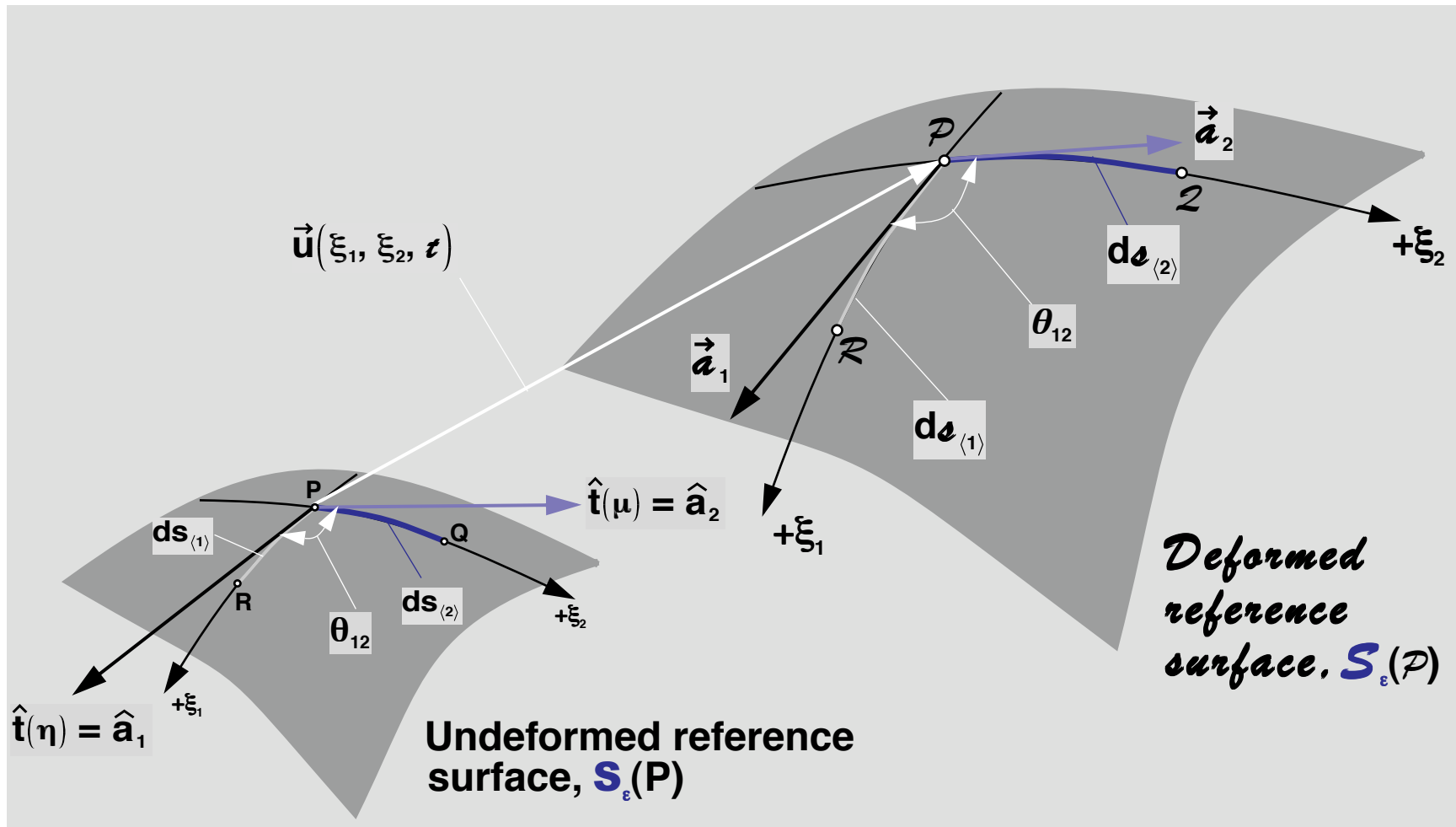
- Likewise, the elongations of two differential arcs are obtained from

$$e_{\hat{\mathbf{t}}(\mu)} = \sqrt{1 + 2\varepsilon_{\alpha\beta}^o \left(\hat{\mathbf{t}}(\mu) \cdot \hat{\mathbf{a}}^\alpha \right) \left(\hat{\mathbf{t}}(\mu) \cdot \hat{\mathbf{a}}^\beta \right) \csc^2 \theta_{12}} - 1$$

and are given by

$$e_{\hat{\mathbf{a}}_1} \equiv e_1 = \sqrt{1 + 2\varepsilon_{11}^o} - 1 \quad \text{and} \quad e_{\hat{\mathbf{a}}_2} \equiv e_2 = \sqrt{1 + 2\varepsilon_{22}^o} - 1$$

ELONGATION AND SHEARING OF $\mathcal{S}_0$ - CONCLUDED



GREEN-LAGRANGE REFERENCE-SURFACE STRAINS

- The previous analysis shows that the changes in lengths of material line segments and changes in angles between intersecting material line segments are **specified entirely by the three independent quantities** $\varepsilon_{\alpha\beta}^o$

- These three quantities are defined by
$$\frac{d\boldsymbol{\ell}^2 - ds^2}{ds^2} = 2\varepsilon_{\alpha\beta}^o (\hat{\mathbf{t}} \cdot \hat{\mathbf{a}}_\alpha)(\hat{\mathbf{t}} \cdot \hat{\mathbf{a}}_\beta)$$
 and constitute the **physical components** of the Green-Lagrange strain tensor, referred to herein as simply the **Green-Lagrange strains**

- The **Green-Lagrange strains** for the shell reference surface are also given by

$$2\varepsilon_{\alpha\beta}^o (\mathbf{A}_{(\alpha)} d\xi_\alpha) (\mathbf{A}_{(\beta)} d\xi_\beta) = d\boldsymbol{\ell}^2 - ds^2$$

where parentheses around the subscript is used to indicate suspension of the repeated-index summation convention

GREEN-LAGRANGE REFERENCE-SURFACE STRAINS

CONTINUED

- Previously, it was shown that

$$ds^2 = (\mathbf{A}_1 d\xi_1)^2 + 2\mathbf{A}_1 \mathbf{A}_2 \cos\theta_{12} d\xi_1 d\xi_2 + (\mathbf{A}_2 d\xi_2)^2 \quad \text{and}$$

$$d\mathcal{L}^2 = (\mathcal{A}_1 d\xi_1)^2 + 2\mathcal{A}_1 \mathcal{A}_2 \cos\theta_{12} d\xi_1 d\xi_2 + (\mathcal{A}_2 d\xi_2)^2$$

where $\theta_{12} = \theta_{12}(\xi_1, \xi_2, t)$ is the angle between the **convected** Gaussian-coordinate curves

- Thus,

$$\begin{aligned} d\mathcal{L}^2 - ds^2 = & \left(\mathcal{A}_1^2 - \mathbf{A}_1^2 \right) d\xi_1 d\xi_1 + \left(\mathcal{A}_2^2 - \mathbf{A}_2^2 \right) d\xi_2 d\xi_2 \\ & + 2 \left(\mathcal{A}_1 \mathcal{A}_2 \cos\theta_{12} - \mathbf{A}_1 \mathbf{A}_2 \cos\theta_{12} \right) d\xi_1 d\xi_2 \end{aligned}$$

- Comparing the previous equation with the strain definition

$$2\varepsilon_{\alpha\beta}^0 (\mathbf{A}_{(\alpha)} d\xi_{\alpha}) (\mathbf{A}_{(\beta)} d\xi_{\beta}) = d\mathcal{L}^2 - ds^2 \quad \text{gives}$$

GREEN-LAGRANGE REFERENCE-SURFACE STRAINS

CONTINUED

$$2\varepsilon_{11}^o = \left(\frac{\mathcal{A}_1}{\mathbf{A}_1} \right)^2 - 1$$

$$2\varepsilon_{22}^o = \left(\frac{\mathcal{A}_2}{\mathbf{A}_2} \right)^2 - 1$$

$$2\varepsilon_{12}^o = \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathbf{A}_1 \mathbf{A}_2} \cos \theta_{12} - \cos \theta_{12}$$

- Next, the strains are expressed in terms of the **displacement-vector field** $\vec{\mathbf{u}}(\xi_1, \xi_2, t)$ of the reference surface by using the previously derived relationships

$$\mathcal{A}_1 = \mathbf{A}_1 \sqrt{1 + 2 \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) + \left(\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right)}$$

$$\mathcal{A}_2 = \mathbf{A}_2 \sqrt{1 + 2 \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) + \left(\frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right)}$$

$$\frac{\mathcal{A}_1 \mathcal{A}_2}{\mathbf{A}_1 \mathbf{A}_2} \cos \theta_{12} = \cos \theta_{12} + \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) + \left(\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right)$$

GREEN-LAGRANGE REFERENCE-SURFACE STRAINS

CONTINUED

- Therefore,

$$\epsilon_{11}^{\circ} = \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) + \frac{1}{2} \left(\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right)$$

$$\epsilon_{22}^{\circ} = \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) + \frac{1}{2} \left(\frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right)$$

$$2\epsilon_{12}^{\circ} = \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) + \left(\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right)$$

or, in indicial notation,

$$2\epsilon_{\alpha\beta}^{\circ} = \left(\hat{\mathbf{a}}_{\alpha} \cdot \frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_{\beta}} \right) + \left(\frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_{\alpha}} \cdot \hat{\mathbf{a}}_{\beta} \right) + \left(\frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_{\alpha}} \cdot \frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_{\beta}} \right)$$

- In addition, it follows that

$$\mathcal{A}_1 = \mathbf{A}_1 \sqrt{1 + 2\epsilon_{11}^{\circ}}$$

$$\mathcal{A}_2 = \mathbf{A}_2 \sqrt{1 + 2\epsilon_{22}^{\circ}}$$

$$\mathcal{A}_1 \mathcal{A}_2 \cos \theta_{12} = \mathbf{A}_1 \mathbf{A}_2 (2\epsilon_{12}^{\circ} + \cos \theta_{12})$$

GREEN-LAGRANGE REFERENCE-SURFACE STRAINS

CONTINUED

- In addition, using $\frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_\alpha} = \mathbf{u}^{(\beta)} \big|_{\langle \alpha \rangle} \hat{\mathbf{a}}_\beta + \mathbf{u}^{(3)} \big|_{\langle \alpha \rangle} \hat{\mathbf{n}}$ gives

$$\begin{aligned} \varepsilon_{11}^{\circ} = & \mathbf{u}^{(1)} \big|_{\langle 1 \rangle} + \mathbf{u}^{(2)} \big|_{\langle 1 \rangle} \cos \theta_{12} + \frac{1}{2} \mathbf{u}^{(1)} \big|_{\langle 1 \rangle} \left[\mathbf{u}^{(1)} \big|_{\langle 1 \rangle} + \mathbf{u}^{(2)} \big|_{\langle 1 \rangle} \cos \theta_{12} \right] \\ & + \frac{1}{2} \mathbf{u}^{(2)} \big|_{\langle 1 \rangle} \left[\mathbf{u}^{(2)} \big|_{\langle 1 \rangle} + \mathbf{u}^{(1)} \big|_{\langle 1 \rangle} \cos \theta_{12} \right] + \frac{1}{2} \mathbf{u}^{(3)} \big|_{\langle 1 \rangle} \mathbf{u}^{(3)} \big|_{\langle 1 \rangle} \end{aligned}$$

$$\begin{aligned} \varepsilon_{22}^{\circ} = & \mathbf{u}^{(2)} \big|_{\langle 2 \rangle} + \mathbf{u}^{(1)} \big|_{\langle 2 \rangle} \cos \theta_{12} + \frac{1}{2} \mathbf{u}^{(1)} \big|_{\langle 2 \rangle} \left[\mathbf{u}^{(1)} \big|_{\langle 2 \rangle} + \mathbf{u}^{(2)} \big|_{\langle 2 \rangle} \cos \theta_{12} \right] \\ & + \frac{1}{2} \mathbf{u}^{(2)} \big|_{\langle 2 \rangle} \left[\mathbf{u}^{(2)} \big|_{\langle 2 \rangle} + \mathbf{u}^{(1)} \big|_{\langle 2 \rangle} \cos \theta_{12} \right] + \frac{1}{2} \mathbf{u}^{(3)} \big|_{\langle 2 \rangle} \mathbf{u}^{(3)} \big|_{\langle 2 \rangle} \end{aligned}$$

$$\begin{aligned} 2\varepsilon_{12}^{\circ} = & \mathbf{u}^{(1)} \big|_{\langle 2 \rangle} + \mathbf{u}^{(2)} \big|_{\langle 1 \rangle} + \left(\mathbf{u}^{(2)} \big|_{\langle 2 \rangle} + \mathbf{u}^{(1)} \big|_{\langle 1 \rangle} \right) \cos \theta_{12} + \mathbf{u}^{(1)} \big|_{\langle 1 \rangle} \left[\mathbf{u}^{(1)} \big|_{\langle 2 \rangle} + \mathbf{u}^{(2)} \big|_{\langle 2 \rangle} \cos \theta_{12} \right] \\ & + \mathbf{u}^{(2)} \big|_{\langle 1 \rangle} \left[\mathbf{u}^{(1)} \big|_{\langle 2 \rangle} \cos \theta_{12} + \mathbf{u}^{(2)} \big|_{\langle 2 \rangle} \right] + \mathbf{u}^{(3)} \big|_{\langle 1 \rangle} \mathbf{u}^{(3)} \big|_{\langle 2 \rangle} \end{aligned}$$

GREEN-LAGRANGE REFERENCE-SURFACE STRAINS

CONTINUED

where, for **general** reference-surface Gaussian coordinates,

$$u^{(1)} \Big|_{\langle 1 \rangle} = \frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} - \frac{u_2 \csc \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\csc \theta_{12}}{\rho_{11}} (u_1 \cos \theta_{12} + u_2) + w \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right)$$

$$u^{(2)} \Big|_{\langle 1 \rangle} = \frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} + \frac{\csc \theta_{12}}{\rho_{11}} (u_1 + u_2 \cos \theta_{12}) + \frac{u_2 \cot \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - w \frac{\csc \theta_{12}}{r_{12}}$$

$$u^{(3)} \Big|_{\langle 1 \rangle} = \frac{1}{A_1} \frac{\partial w}{\partial \xi_1} - \frac{u_1}{r_{11}} + u_2 \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right)$$

$$u^{(1)} \Big|_{\langle 2 \rangle} = \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{u_1 \cot \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} - \frac{\csc \theta_{12}}{\rho_{22}} (u_1 \cos \theta_{12} + u_2) + w \frac{\csc \theta_{12}}{r_{21}}$$

$$u^{(2)} \Big|_{\langle 2 \rangle} = \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} - \frac{u_1 \csc \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} + \frac{\csc \theta_{12}}{\rho_{22}} (u_1 + u_2 \cos \theta_{12}) + w \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right)$$

$$u^{(3)} \Big|_{\langle 2 \rangle} = \frac{1}{A_2} \frac{\partial w}{\partial \xi_2} - u_1 \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) - \frac{u_2}{r_{22}}$$

GREEN-LAGRANGE REFERENCE-SURFACE STRAINS

CONTINUED

- The corresponding indicial form is

$$2\varepsilon_{\alpha\beta}^{\circ} = u^{\langle\delta\rangle} \Big|_{\langle\beta\rangle} (\hat{a}_{\alpha} \cdot \hat{a}_{\delta}) + u^{\langle\delta\rangle} \Big|_{\langle\alpha\rangle} (\hat{a}_{\delta} \cdot \hat{a}_{\beta}) + u^{\langle\mu\rangle} \Big|_{\langle\alpha\rangle} u^{\langle\delta\rangle} \Big|_{\langle\beta\rangle} (\hat{a}_{\mu} \cdot \hat{a}_{\delta}) + u^{\langle3\rangle} \Big|_{\langle\alpha\rangle} u^{\langle3\rangle} \Big|_{\langle\beta\rangle}$$

- For **orthogonal** reference-surface Gaussian coordinates, the strains expressions reduce to

$$\varepsilon_{11}^{\circ} = u^{\langle1\rangle} \Big|_{\langle1\rangle} + \frac{1}{2} \left[\left(u^{\langle1\rangle} \Big|_{\langle1\rangle} \right)^2 + \left(u^{\langle2\rangle} \Big|_{\langle1\rangle} \right)^2 + \left(u^{\langle3\rangle} \Big|_{\langle1\rangle} \right)^2 \right]$$

$$\varepsilon_{22}^{\circ} = u^{\langle2\rangle} \Big|_{\langle2\rangle} + \frac{1}{2} \left[\left(u^{\langle1\rangle} \Big|_{\langle2\rangle} \right)^2 + \left(u^{\langle2\rangle} \Big|_{\langle2\rangle} \right)^2 + \left(u^{\langle3\rangle} \Big|_{\langle2\rangle} \right)^2 \right]$$

$$2\varepsilon_{12}^{\circ} = u^{\langle1\rangle} \Big|_{\langle2\rangle} + u^{\langle2\rangle} \Big|_{\langle1\rangle} + u^{\langle1\rangle} \Big|_{\langle1\rangle} u^{\langle1\rangle} \Big|_{\langle2\rangle} + u^{\langle2\rangle} \Big|_{\langle1\rangle} u^{\langle2\rangle} \Big|_{\langle2\rangle} + u^{\langle3\rangle} \Big|_{\langle1\rangle} u^{\langle3\rangle} \Big|_{\langle2\rangle} \quad \text{or}$$

$$2\varepsilon_{\alpha\beta}^{\circ} = u^{\langle\alpha\rangle} \Big|_{\langle\beta\rangle} + u^{\langle\beta\rangle} \Big|_{\langle\alpha\rangle} + u^{\langle\rho\rangle} \Big|_{\langle\alpha\rangle} u^{\langle\rho\rangle} \Big|_{\langle\beta\rangle} + u^{\langle3\rangle} \Big|_{\langle\alpha\rangle} u^{\langle3\rangle} \Big|_{\langle\beta\rangle}$$

GREEN-LAGRANGE REFERENCE-SURFACE STRAINS

CONCLUDED

where, for **orthogonal** reference-surface Gaussian coordinates,

$$\mathbf{u}^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_1}{\partial \xi_1} - \frac{\mathbf{u}_2}{\rho_{11}} + \frac{\mathbf{w}}{\mathbf{r}_{11}}$$



$$\mathbf{u}^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_1}{\partial \xi_1} + \frac{\mathbf{u}_2}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} + \frac{\mathbf{w}}{\mathbf{r}_{11}}$$

$$\mathbf{u}^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_2}{\partial \xi_1} + \frac{\mathbf{u}_1}{\rho_{11}} - \frac{\mathbf{w}}{\mathbf{r}_{12}}$$



$$\mathbf{u}^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_2}{\partial \xi_1} - \frac{\mathbf{u}_1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} - \frac{\mathbf{w}}{\mathbf{r}_{12}}$$

$$\mathbf{u}^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_1}{\partial \xi_2} - \frac{\mathbf{u}_2}{\rho_{22}} - \frac{\mathbf{w}}{\mathbf{r}_{12}}$$



$$\mathbf{u}^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_1}{\partial \xi_2} - \frac{\mathbf{u}_2}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1} - \frac{\mathbf{w}}{\mathbf{r}_{12}}$$

$$\mathbf{u}^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_2}{\partial \xi_2} + \frac{\mathbf{u}_1}{\rho_{22}} + \frac{\mathbf{w}}{\mathbf{r}_{22}}$$



$$\mathbf{u}^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_2}{\partial \xi_2} + \frac{\mathbf{u}_1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1} + \frac{\mathbf{w}}{\mathbf{r}_{22}}$$

$$\mathbf{u}^{\langle 3 \rangle} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{w}}{\partial \xi_1} - \frac{\mathbf{u}_1}{\mathbf{r}_{11}} + \frac{\mathbf{u}_2}{\mathbf{r}_{12}}$$

$$\mathbf{u}^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{w}}{\partial \xi_2} + \frac{\mathbf{u}_1}{\mathbf{r}_{12}} - \frac{\mathbf{u}_2}{\mathbf{r}_{22}}$$

“SMALL” GREEN-LAGRANGE STRAINS

- Up to this point of the present study, the size of the reference-surface strains that occur during deformation have been unrestricted
- However, a broad range of engineering shell structures exhibit nonlinear deformations for which the magnitude of the strains are very small compared to unity
- The size of the corresponding reference-surface strains are characterized by $|\epsilon_{\alpha\beta}^{\circ}| \ll 1$ and described herein as **“small” strains**
- As will be seen subsequently, it turns out that $|\epsilon_{\alpha\beta}^{\circ}| \ll 1$ provides a means for simplifying the equations characterizing reference-surface deformation by converting irrational quantities into rational quantities

“SMALL” GREEN-LAGRANGE STRAINS - CONTINUED

- For the very important, practical class of deformations that exhibit **“small” strains**, the Green-Lagrange strains are described as terms of second order, denoted by the symbol $\mathcal{O}(\varepsilon^2)$, where ε indicates that the order symbol applies to strain-related quantities
 - In particular, $\mathcal{O}(\varepsilon^2)$ means that the strain magnitudes are **approximately 10^2 times smaller** than unity; i. e., $|\varepsilon_{\alpha\beta}^o| \ll 1$
 - Hence, for any “small” strain quantity γ , **binomial series** are used to get $\sqrt{1 + \gamma} = 1 + \frac{1}{2}\gamma + \mathcal{O}(\varepsilon^4)$ and $\frac{1}{\sqrt{1 + \gamma}} = 1 - \frac{1}{2}\gamma + \mathcal{O}(\varepsilon^4)$,
where $\mathcal{O}(\varepsilon^4)$ signifies terms **10^4 times smaller** than unity
- Thus, for **“small” strains**

$$\begin{array}{ll}
 \mathcal{A}_1 = A_1 \sqrt{1 + 2\varepsilon_{11}^o} & \rightarrow \mathcal{A}_1 = A_1 \left(1 + \varepsilon_{11}^o + \mathcal{O}(\varepsilon^4) \right) \quad \text{and} \\
 \mathcal{A}_2 = A_2 \sqrt{1 + 2\varepsilon_{22}^o} & \rightarrow \mathcal{A}_2 = A_2 \left(1 + \varepsilon_{22}^o + \mathcal{O}(\varepsilon^4) \right)
 \end{array}$$

“SMALL” GREEN-LAGRANGE STRAINS - CONTINUED

- In addition, $\mathcal{A}_1 \mathcal{A}_2 \cos \theta_{12} = \mathbf{A}_1 \mathbf{A}_2 \left(2\varepsilon_{12}^{\circ} + \cos \theta_{12} \right)$ becomes

$$\cos \theta_{12} = 2\varepsilon_{12}^{\circ} + \cos \theta_{12} \left(1 - \varepsilon_{11}^{\circ} - \varepsilon_{22}^{\circ} \right) + \mathcal{O}(\varepsilon^4)$$

- Moreover, the elongations along the coordinate curves become

$$\mathbf{e}_1 = \varepsilon_{11}^{\circ} + \mathcal{O}(\varepsilon^4) \quad \text{and} \quad \mathbf{e}_2 = \varepsilon_{22}^{\circ} + \mathcal{O}(\varepsilon^4)$$

- Likewise, $\hat{\mathbf{a}}_1 = \frac{\mathbf{A}_1}{\mathcal{A}_1} \left(\hat{\mathbf{a}}_1 + \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right)$ and $\hat{\mathbf{a}}_2 = \frac{\mathbf{A}_2}{\mathcal{A}_2} \left(\hat{\mathbf{a}}_2 + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right)$ become

$$\hat{\mathbf{a}}_1 = \left(\hat{\mathbf{a}}_1 + \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \left(1 - \varepsilon_{11}^{\circ} + \mathcal{O}(\varepsilon^4) \right) \quad \text{and} \quad \hat{\mathbf{a}}_2 = \left(\hat{\mathbf{a}}_2 + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \left(1 - \varepsilon_{22}^{\circ} + \mathcal{O}(\varepsilon^4) \right)$$

“SMALL” GREEN-LAGRANGE STRAINS - CONTINUED

- For a positive-valued $\mathcal{O}(\varepsilon^2)$ change in angle (**shear**) between the convected coordinate curves during deformation equal to γ_{12}° , the resulting angle is given by $\theta_{12} = \theta_{12}^\circ - \gamma_{12}^\circ$

- Application of trigonometric identities and trigonometric series to $\sin \gamma_{12}^\circ$ and $\cos \gamma_{12}^\circ$ yields the results

$$\sin \theta_{12} = \sin \theta_{12}^\circ - \cos \theta_{12}^\circ \gamma_{12}^\circ + \mathcal{O}(\varepsilon^4) \quad \text{and} \quad \cos \theta_{12} = \cos \theta_{12}^\circ + \sin \theta_{12}^\circ \gamma_{12}^\circ + \mathcal{O}(\varepsilon^6)$$

- Combining the last expression with

$$\cos \theta_{12} = 2\varepsilon_{12}^\circ + \cos \theta_{12}^\circ (1 - \varepsilon_{11}^\circ - \varepsilon_{22}^\circ) + \mathcal{O}(\varepsilon^4) \quad \text{yields the shearing strain as}$$

$$\gamma_{12}^\circ = 2\varepsilon_{12}^\circ \csc \theta_{12}^\circ - \cot \theta_{12}^\circ (\varepsilon_{11}^\circ + \varepsilon_{22}^\circ) + \mathcal{O}(\varepsilon^4) \quad \text{for “small” strains}$$

“SMALL” GREEN-LAGRANGE STRAINS - CONTINUED

- Then, substituting $\gamma_{12}^{\circ} = 2\varepsilon_{12}^{\circ} \csc\theta_{12} - \cot\theta_{12}(\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}) + \mathcal{O}(\varepsilon^4)$ into $\sin\theta_{12} = \sin\theta_{12} - \cos\theta_{12} \gamma_{12}^{\circ} + \mathcal{O}(\varepsilon^4)$ gives

$$\sin\theta_{12} = \sin\theta_{12} - \cot\theta_{12} \left[2\varepsilon_{12}^{\circ} - \cos\theta_{12}(\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}) \right] + \mathcal{O}(\varepsilon^4)$$

for “small” strains

- In addition, $\left| \hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^{\alpha} \right| \leq 1$ and, as a result, a binomial series expansion gives

$$\sqrt{1 + 2\varepsilon_{\alpha\beta}^{\circ} \left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^{\alpha} \right) \left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^{\beta} \right) \csc^2\theta_{12}} = 1 + \varepsilon_{\alpha\beta}^{\circ} \left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^{\alpha} \right) \left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^{\beta} \right) \csc^2\theta_{12} + \mathcal{O}(\varepsilon^4)$$

“SMALL” GREEN-LAGRANGE STRAINS - CONTINUED

- Thus, the general form of the **elongation** is given by

$$\mathbf{e}_{\hat{\mathbf{t}}(\mu)} = \varepsilon_{\alpha\beta}^{\circ} \left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^{\alpha} \right) \left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^{\beta} \right) \csc^2 \theta_{12} + \mathcal{O}(\varepsilon^4)$$

for “small” strains

- For a positive-valued change in the angle (**shear**) $\theta(\xi_1, \xi_2, t)$ between intersecting differential surface arcs at a point, that occurs during deformation, the shearing $\gamma^{\circ}(\xi_1, \xi_2, t)$, is defined by $\gamma^{\circ} = \theta - \theta$

- Thus, $\cos \theta = \cos \theta \cos \gamma^{\circ} + \sin \theta \sin \gamma^{\circ} = \cos \theta + \sin \theta \gamma^{\circ} + \mathcal{O}(\varepsilon^4)$

for “small” strains, which yields

$$\gamma^{\circ} = \frac{\cos \theta - \cos \theta}{\sin \theta} + \mathcal{O}(\varepsilon^4)$$

“SMALL” GREEN-LAGRANGE STRAINS - CONTINUED

- Likewise, the previous expression

$$\cos \theta = \frac{(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^\alpha)(\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^\beta) \csc^2 \theta_{12} \left[2\varepsilon_{\alpha\beta}^\circ + \delta_{\alpha\beta} + (1 - \delta_{\alpha\beta}) \cos \theta_{12} \right]}{\left[1 + e_{\hat{\mathbf{t}}_{(\mu)}} \right] \left[1 + e_{\hat{\mathbf{t}}_{(\eta)}} \right]}$$

is expressed as

$$\cos \theta = \frac{(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^\alpha)(\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^\beta) \left[2\varepsilon_{\alpha\beta}^\circ + \delta_{\alpha\beta} + (1 - \delta_{\alpha\beta}) \cos \theta_{12} \right] \csc^2 \theta_{12}}{\left[1 + \varepsilon_{\sigma\chi}^\circ (\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^\sigma)(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^\chi) \csc^2 \theta_{12} \right] \left[1 + \varepsilon_{\lambda\rho}^\circ (\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^\lambda)(\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^\rho) \csc^2 \theta_{12} \right]} + \mathcal{O}(\varepsilon^4)$$

- Furthermore, $\cos \theta = \hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{t}}_{(\eta)}$ and $\sin \theta = (\hat{\mathbf{t}}_{(\eta)} \times \hat{\mathbf{t}}_{(\mu)}) \cdot \hat{\mathbf{n}}$, and

$$\hat{\mathbf{t}}_{(\mu)} = \left[\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^\alpha \right] \csc \theta_{12} \hat{\mathbf{a}}_\alpha \quad \text{and} \quad \hat{\mathbf{t}}_{(\eta)} = \left[\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^\beta \right] \csc \theta_{12} \hat{\mathbf{a}}_\beta$$

are used with $\hat{\mathbf{a}}_\alpha \cdot \hat{\mathbf{a}}_\beta = \delta_{\alpha\beta} + (1 - \delta_{\alpha\beta}) \cos \theta_{12}$ to get

“SMALL” GREEN-LAGRANGE STRAINS - CONTINUED

$$\cos\theta = \left[\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^\alpha \right] \left[\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^\beta \right] \left(\delta_{\alpha\beta} + (1 - \delta_{\alpha\beta}) \cos\theta_{12} \right) \csc^2\theta_{12} \quad \text{and}$$

$$\sin\theta = \left(\left[\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^1 \right] \left[\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^2 \right] - \left[\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^2 \right] \left[\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^1 \right] \right) \csc\theta_{12}$$

- Thus, for “small” strains,

$$\gamma^\circ = \frac{\cos\theta - \left[\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^\alpha \right] \left[\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^\beta \right] \left(\delta_{\alpha\beta} + (1 - \delta_{\alpha\beta}) \cos\theta_{12} \right) \csc^2\theta_{12}}{\left(\left[\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^1 \right] \left[\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^2 \right] - \left[\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^2 \right] \left[\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^1 \right] \right) \csc\theta_{12}} + \mathcal{O}(\varepsilon^4)$$

where

$$\cos\theta = \frac{\left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^\alpha \right) \left(\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^\beta \right) \left[2\varepsilon_{\alpha\beta}^\circ + \delta_{\alpha\beta} + (1 - \delta_{\alpha\beta}) \cos\theta_{12} \right] \csc^2\theta_{12}}{\left[1 + \varepsilon_{\sigma\chi}^\circ \left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^\sigma \right) \left(\hat{\mathbf{t}}_{(\mu)} \cdot \hat{\mathbf{a}}^\chi \right) \csc^2\theta_{12} \right] \left[1 + \varepsilon_{\lambda\rho}^\circ \left(\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^\lambda \right) \left(\hat{\mathbf{t}}_{(\eta)} \cdot \hat{\mathbf{a}}^\rho \right) \csc^2\theta_{12} \right]} + \mathcal{O}(\varepsilon^4)$$

“SMALL” GREEN-LAGRANGE STRAINS - CONCLUDED

- In many practical cases, known values of the linearized strains and rotations provide **useful measures of the degree of nonlinearity** in given problem
- Thus, it is useful to express the **nonlinear Green-Lagrange strains** in terms of the linearized strain and rotation measures
- In addition, for most engineering applications, nonlinear and buckling analyses need only **consider the class of deformations** in which the strains are $\mathcal{O}(\varepsilon^2)$ and the rotations are $\mathcal{O}(\varepsilon)$
 - Thus, products of strains and products of strains and rotations can be neglected
 - The nonlinear deformation state of interest is presumed to be “near” the linear deformation state and can be assessed adequately in terms of the strains and rotations of the linear state

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS

- For **infinitesimal displacements**, characteristic of linear behavior, the magnitude of the displacement gradients are such that $\left| \frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \vec{u}}{\partial \xi_\alpha} \right| \ll 1$, which is small enough for their **products to be negligible**
- For this class of deformations, linearization of

$$2\varepsilon_{\alpha\beta}^{\circ} = \left(\hat{\mathbf{a}}_{\alpha} \cdot \frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \vec{u}}{\partial \xi_{\beta}} \right) + \left(\frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \vec{u}}{\partial \xi_{\alpha}} \cdot \hat{\mathbf{a}}_{\beta} \right) + \left(\frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \vec{u}}{\partial \xi_{\alpha}} \cdot \frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \vec{u}}{\partial \xi_{\beta}} \right)$$

results in $\varepsilon_{\alpha\beta}^{\circ}(\xi_1, \xi_2, \boldsymbol{\tau}) = \mathbf{e}_{\alpha\beta}^{\circ}(\xi_1, \xi_2, \boldsymbol{\tau}) + \mathcal{O}(\varepsilon^4)$, where

$$2\mathbf{e}_{\alpha\beta}^{\circ} \equiv \left(\hat{\mathbf{a}}_{\alpha} \cdot \frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \vec{u}}{\partial \xi_{\beta}} \right) + \left(\frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \vec{u}}{\partial \xi_{\alpha}} \cdot \hat{\mathbf{a}}_{\beta} \right)$$

- Here, $\mathbf{e}_{\alpha\beta}^{\circ}$ are presumed to be “small” quantities of $\mathcal{O}(\varepsilon^2)$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- In explicit form,

$$\mathbf{e}_{11}^{\circ} \equiv \hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1}$$

$$\mathbf{e}_{22}^{\circ} \equiv \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2}$$

$$2\mathbf{e}_{12}^{\circ} \equiv \hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} + \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1}$$

- Now, consider **linearization** of the unit-magnitude convected base-vector fields obtained from

$$\vec{\mathbf{a}}_{\alpha}(\xi_1, \xi_2, \boldsymbol{\varepsilon}) = \mathbf{A}_{(\alpha)} \left(\hat{\mathbf{a}}_{\alpha} + \frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_{\alpha}} \right)$$

- First, $\boldsymbol{\varepsilon}_{\alpha\beta}^{\circ} = \mathbf{e}_{\alpha\beta}^{\circ} + \mathcal{O}(\varepsilon^4)$ yields

$$\mathcal{A}_1 = \mathbf{A}_1 (1 + \mathbf{e}_{11}^{\circ}) + \mathcal{O}(\varepsilon^4) \quad \text{and} \quad \mathcal{A}_2 = \mathbf{A}_2 (1 + \mathbf{e}_{22}^{\circ}) + \mathcal{O}(\varepsilon^4)$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- Likewise, $\cos \theta_{12} = 2\mathbf{e}_{12}^{\circ} + \cos \theta_{12} (1 - \mathbf{e}_{11}^{\circ} - \mathbf{e}_{22}^{\circ}) + \mathcal{O}(\varepsilon^4)$ and

$$\sin \theta_{12} = \sin \theta_{12} - \cot \theta_{12} \left[2\mathbf{e}_{12}^{\circ} - \cos \theta_{12} (\mathbf{e}_{11}^{\circ} + \mathbf{e}_{22}^{\circ}) \right] + \mathcal{O}(\varepsilon^4)$$

- Next, **unit-magnitude convected base-vector fields** are defined by

$$\hat{\mathbf{a}}_1 = \frac{\vec{\mathbf{a}}_1}{|\vec{\mathbf{a}}_1|} = \frac{\vec{\mathbf{a}}_1}{\mathcal{A}_1} \quad \text{and} \quad \hat{\mathbf{a}}_2 = \frac{\vec{\mathbf{a}}_2}{|\vec{\mathbf{a}}_2|} = \frac{\vec{\mathbf{a}}_2}{\mathcal{A}_2} \quad \text{which yield}$$

$$\hat{\mathbf{a}}_1 = \left(\hat{\mathbf{a}}_1 + \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \left(1 + \mathbf{e}_{11}^{\circ} + \mathcal{O}(\varepsilon^4) \right)^{-1} \quad \text{or} \quad \hat{\mathbf{a}}_1 = \left(\hat{\mathbf{a}}_1 + \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \left(1 - \mathbf{e}_{11}^{\circ} + \mathcal{O}(\varepsilon^4) \right)$$

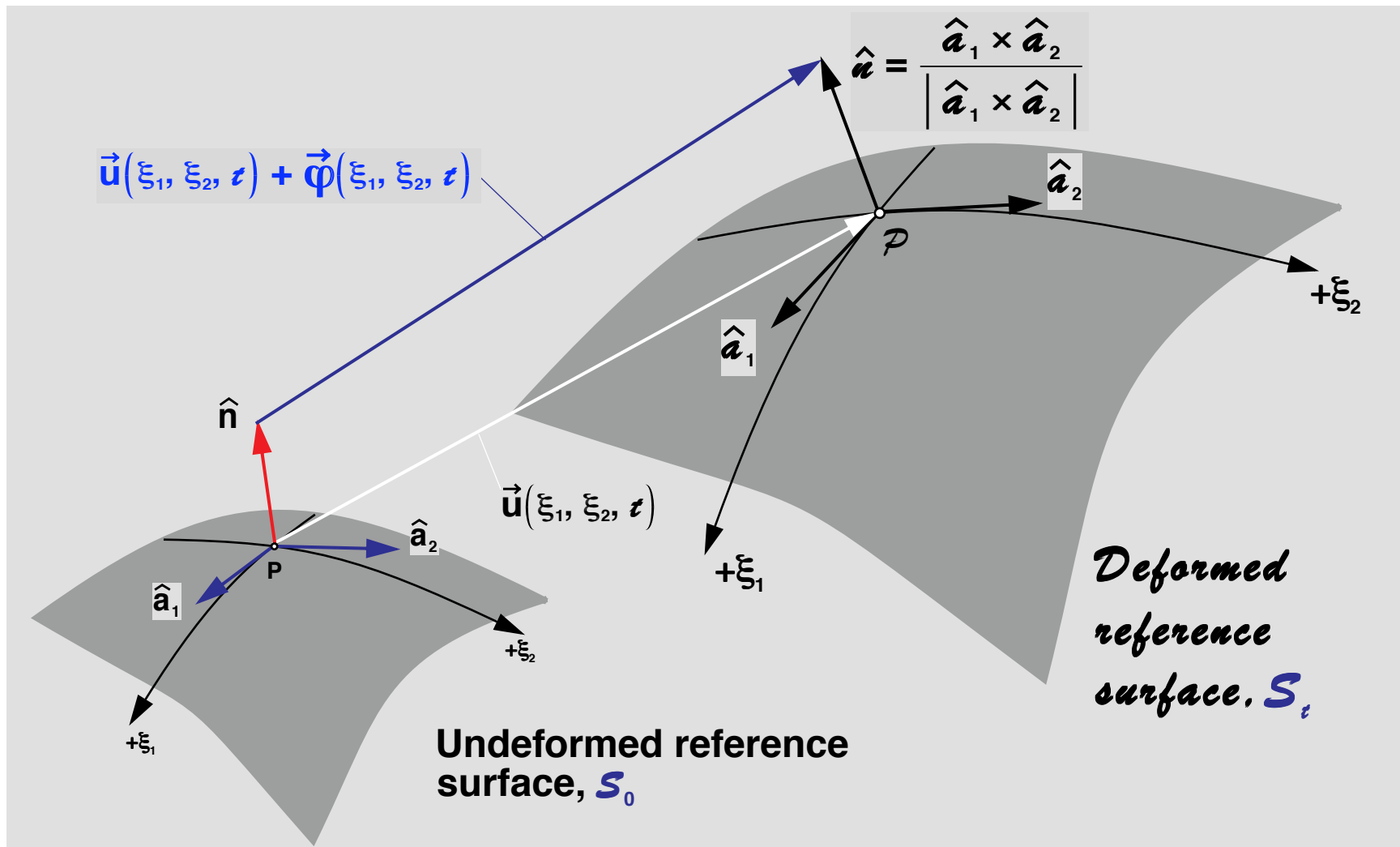
and

$$\hat{\mathbf{a}}_2 = \left(\hat{\mathbf{a}}_2 + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \left(1 + \mathbf{e}_{22}^{\circ} + \mathcal{O}(\varepsilon^4) \right)^{-1} \quad \text{or} \quad \hat{\mathbf{a}}_2 = \left(\hat{\mathbf{a}}_2 + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \left(1 - \mathbf{e}_{22}^{\circ} + \mathcal{O}(\varepsilon^4) \right)$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- As the shell deforms, the vector $\hat{n}(\xi_1, \xi_2)$ at point P of the reference surface **translates** to point $\mathcal{P}$ of the corresponding deformed surface, and $\hat{n}(\xi_1, \xi_2)$ undergoes a **rigid-body rotation** into $\hat{n}(\xi_1, \xi_2, t)$, as shown in the next figure
- This rotation corresponds to the rotation of the tangent plane at point P into the tangent plane at point $\mathcal{P}$
- Now, consider a rigid-body rotation of $\hat{n}(\xi_1, \xi_2)$ into $\hat{n}(\xi_1, \xi_2, t)$ that is the same order of magnitude as the strains $e_{\alpha\beta}^o$
- Thus, for the **difference vector** $\vec{\varphi}(\xi_1, \xi_2, t)$, defined by $\vec{\varphi} \equiv \hat{n} - \hat{n}$, it follows that $|\vec{\varphi}|$ is $\mathcal{O}(\varepsilon^2)$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED



LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- By using $\hat{\mathbf{u}} = \vec{\varphi} + \hat{\mathbf{n}}$ and $\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = 1$, it also follows that $\vec{\varphi} \cdot \hat{\mathbf{n}} = -\frac{1}{2}|\vec{\varphi}|^2$ and is negligible

- Therefore, the **linearized difference vector** $\vec{\varphi}$ is given by

$$\vec{\varphi} = \varphi_1 \hat{\mathbf{a}}_1 + \varphi_2 \hat{\mathbf{a}}_2$$

- Note that $\hat{\mathbf{a}}_1 \perp \hat{\mathbf{a}}_2$ indicates that the component $\varphi_1 \hat{\mathbf{a}}_1$ corresponds to a right-handed rotation of amount φ_1 about the reciprocal base vector $\hat{\mathbf{a}}_2$
- Likewise, the component $\varphi_2 \hat{\mathbf{a}}_2$ corresponds to a right-handed rotation of amount φ_2 about the reciprocal base vector $-\hat{\mathbf{a}}_1$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- By introducing the **linear rotation vector field** $\vec{\omega}(\xi_1, \xi_2, t)$ given by

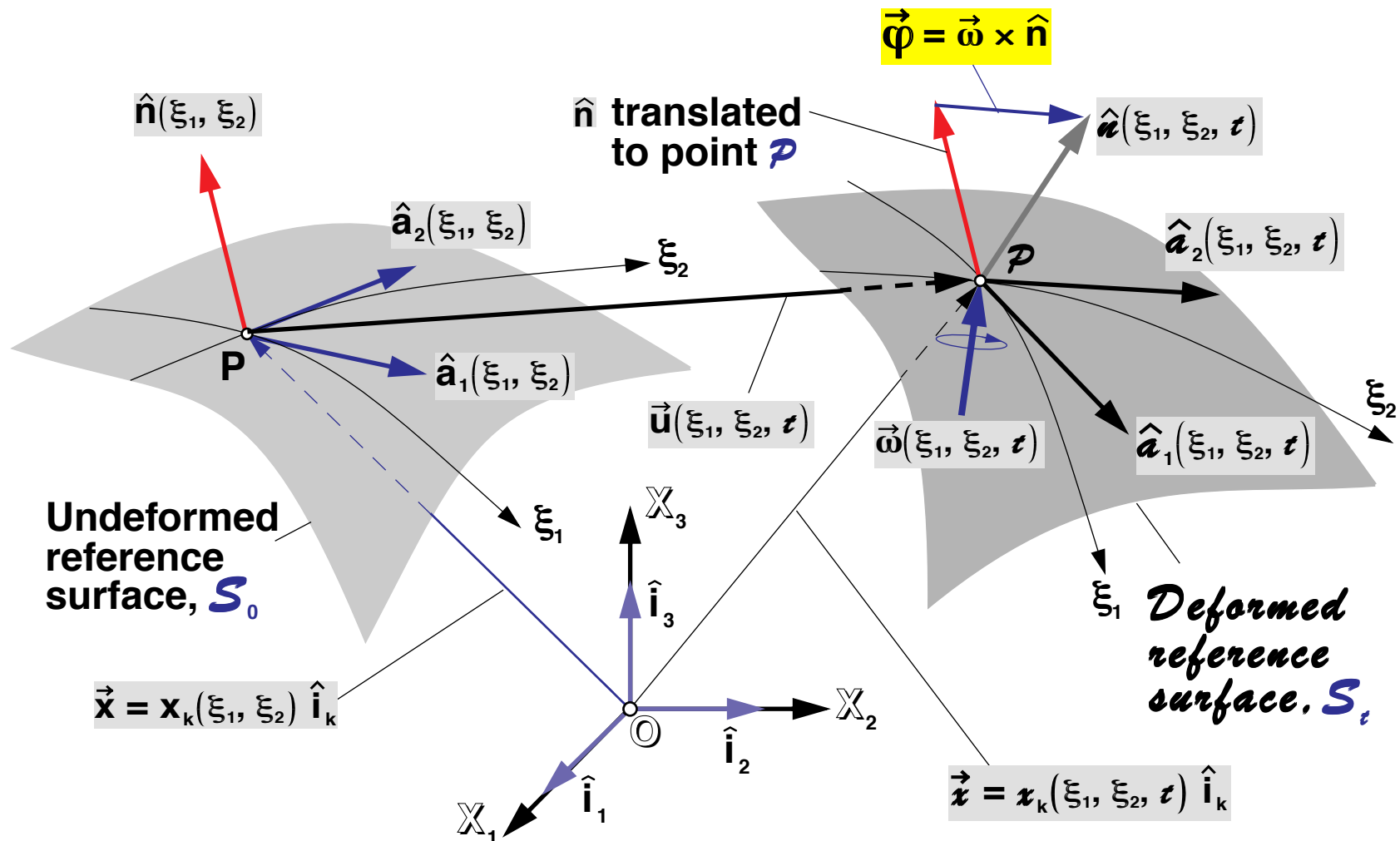
$$\vec{\omega} = -\varphi_2 \hat{a}^1 + \varphi_1 \hat{a}^2 + \varphi \hat{n}$$

and using $\hat{a}^1 \times \hat{n} = -\hat{a}_2$ and $\hat{a}^2 \times \hat{n} = \hat{a}_1$, it is found that $\vec{\varphi} = \vec{\omega} \times \hat{n}$,

as shown in the next figure, and that $\hat{n} = \hat{n} + (\vec{\omega} \times \hat{n})$

- An important rotation appearing in $\vec{\omega}(\xi_1, \xi_2, t)$ is the rotation about the normal vector $\hat{n}$, given by the component $\varphi(\xi_1, \xi_2, t)$
- The definition of $\varphi(\xi_1, \xi_2, t)$ is obtained by noting that the rotation field of a continuum undergoing infinitesimal deformations is characterized by the curl of the displacement vector field; that is, $\vec{\nabla} \times \vec{u}$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED



LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- To obtain the expression for the **surface gradient operator**, $\vec{\nabla}$, consider the differentiable function $F(\xi_1, \xi_2)$
- The chain rule of differentiation gives $dF = \frac{\partial F}{\partial \xi_1} d\xi_1 + \frac{\partial F}{\partial \xi_2} d\xi_2$
- Noting that the gradient operator has the form $\vec{\nabla} = \vec{a} \frac{1}{A_1} \frac{\partial}{\partial \xi_1} + \vec{b} \frac{1}{A_2} \frac{\partial}{\partial \xi_2}$, and that it must satisfy $dF = \vec{\nabla} F \cdot d\vec{x}$, where

$$d\vec{x} = \frac{\partial \vec{x}}{\partial \xi_1} d\xi_1 + \frac{\partial \vec{x}}{\partial \xi_2} d\xi_2 = A_1 \hat{a}_1 d\xi_1 + A_2 \hat{a}_2 d\xi_2,$$

yields $\vec{a} = \frac{\hat{a}^1}{\sin\theta_{12}}$ and $\vec{b} = \frac{\hat{a}^2}{\sin\theta_{12}}$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- Therefore, the **surface gradient operator** is given by

$$\vec{\nabla} = \csc\theta_{12} \left(\hat{\mathbf{a}}^1 \frac{1}{A_1} \frac{\partial}{\partial \xi_1} + \hat{\mathbf{a}}^2 \frac{1}{A_2} \frac{\partial}{\partial \xi_2} \right)$$

- In addition, $\vec{\nabla} \times \vec{\mathbf{u}} = \csc\theta_{12} \left(\hat{\mathbf{a}}^1 \times \frac{1}{A_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} + \hat{\mathbf{a}}^2 \times \frac{1}{A_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right)$

- The dot products $\hat{\mathbf{a}}^\alpha \cdot \hat{\mathbf{a}}_\beta = \delta_\beta^\alpha \sin\theta_{12}$ and $\hat{\mathbf{a}}^\alpha \cdot \hat{\mathbf{n}} = 0$ are used next to get

$$\frac{1}{A_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} = \left(\hat{\mathbf{a}}^1 \cdot \frac{1}{A_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \csc\theta_{12} \hat{\mathbf{a}}_1 + \left(\hat{\mathbf{a}}^2 \cdot \frac{1}{A_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \csc\theta_{12} \hat{\mathbf{a}}_2 + \left(\hat{\mathbf{n}} \cdot \frac{1}{A_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \hat{\mathbf{n}}$$

$$\frac{1}{A_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} = \left(\hat{\mathbf{a}}^1 \cdot \frac{1}{A_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \csc\theta_{12} \hat{\mathbf{a}}_1 + \left(\hat{\mathbf{a}}^2 \cdot \frac{1}{A_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \csc\theta_{12} \hat{\mathbf{a}}_2 + \left(\hat{\mathbf{n}} \cdot \frac{1}{A_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \hat{\mathbf{n}}$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- The cross-product terms are given by

$$\hat{\mathbf{a}}^1 \times \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} = \left(\hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \cot \theta_{12} \hat{\mathbf{n}} + \left(\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \csc \theta_{12} \hat{\mathbf{n}} - \left(\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \hat{\mathbf{a}}_2$$

$$\hat{\mathbf{a}}^2 \times \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} = - \left(\hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \csc \theta_{12} \hat{\mathbf{n}} - \left(\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \cot \theta_{12} \hat{\mathbf{n}} + \left(\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \hat{\mathbf{a}}_1$$

- Using these equations, $\vec{\nabla} \times \vec{\mathbf{u}}$ is expressed in component form as

$$\vec{\nabla} \times \vec{\mathbf{u}} = \left(\vec{\nabla} \times \vec{\mathbf{u}} \right)^1 \hat{\mathbf{a}}_1 + \left(\vec{\nabla} \times \vec{\mathbf{u}} \right)^2 \hat{\mathbf{a}}_2 + \left(\vec{\nabla} \times \vec{\mathbf{u}} \right)^3 \hat{\mathbf{n}}$$

where

$$\left(\vec{\nabla} \times \vec{\mathbf{u}} \right)^1 = \left(\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \csc \theta_{12}$$

$$\left(\vec{\nabla} \times \vec{\mathbf{u}} \right)^2 = - \left(\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \csc \theta_{12}$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

$$\begin{aligned} \left(\vec{\nabla} \times \vec{u} \right)^3 = & \left[\left(\hat{a}^2 \cdot \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \right) - \left(\hat{a}^1 \cdot \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \right) \right] \csc^2 \theta_{12} \\ & + \left[\left(\hat{a}^1 \cdot \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \right) - \left(\hat{a}^2 \cdot \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \right) \right] \csc \theta_{12} \cot \theta_{12} \end{aligned}$$

- By using

$$\hat{a}^1 = \hat{a}_1 \csc \theta_{12} - \hat{a}_2 \cot \theta_{12} = \hat{a}_2 \times \hat{n} \quad \text{and} \quad \hat{a}^2 = \hat{a}_2 \csc \theta_{12} - \hat{a}_1 \cot \theta_{12} = \hat{n} \times \hat{a}_1$$

it is also found that

$$\left(\vec{\nabla} \times \vec{u} \right)^3 = \left[\left(\hat{a}_2 \cdot \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi} \right) - \left(\hat{a}_1 \cdot \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \right) \right] \csc \theta_{12}$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- To obtain a form for $\vec{\nabla} \times \vec{u}$ that is similar to $\vec{\omega} = -\varphi_2 \hat{a}^1 + \varphi_1 \hat{a}^2 + \varphi \hat{n}$, $\vec{\nabla} \times \vec{u}$ is expressed as

$$\vec{\nabla} \times \vec{u} = (\vec{\nabla} \times \vec{u})_1 \hat{a}^1 + (\vec{\nabla} \times \vec{u})_2 \hat{a}^2 + (\vec{\nabla} \times \vec{u})^3 \hat{n} \quad \text{where}$$

$$(\vec{\nabla} \times \vec{u})_1 = (\vec{\nabla} \times \vec{u})^1 \csc\theta_{12} + (\vec{\nabla} \times \vec{u})^2 \cot\theta_{12} \quad \text{and}$$

$$(\vec{\nabla} \times \vec{u})_2 = (\vec{\nabla} \times \vec{u})^1 \cot\theta_{12} + (\vec{\nabla} \times \vec{u})^2 \csc\theta_{12}$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- Using

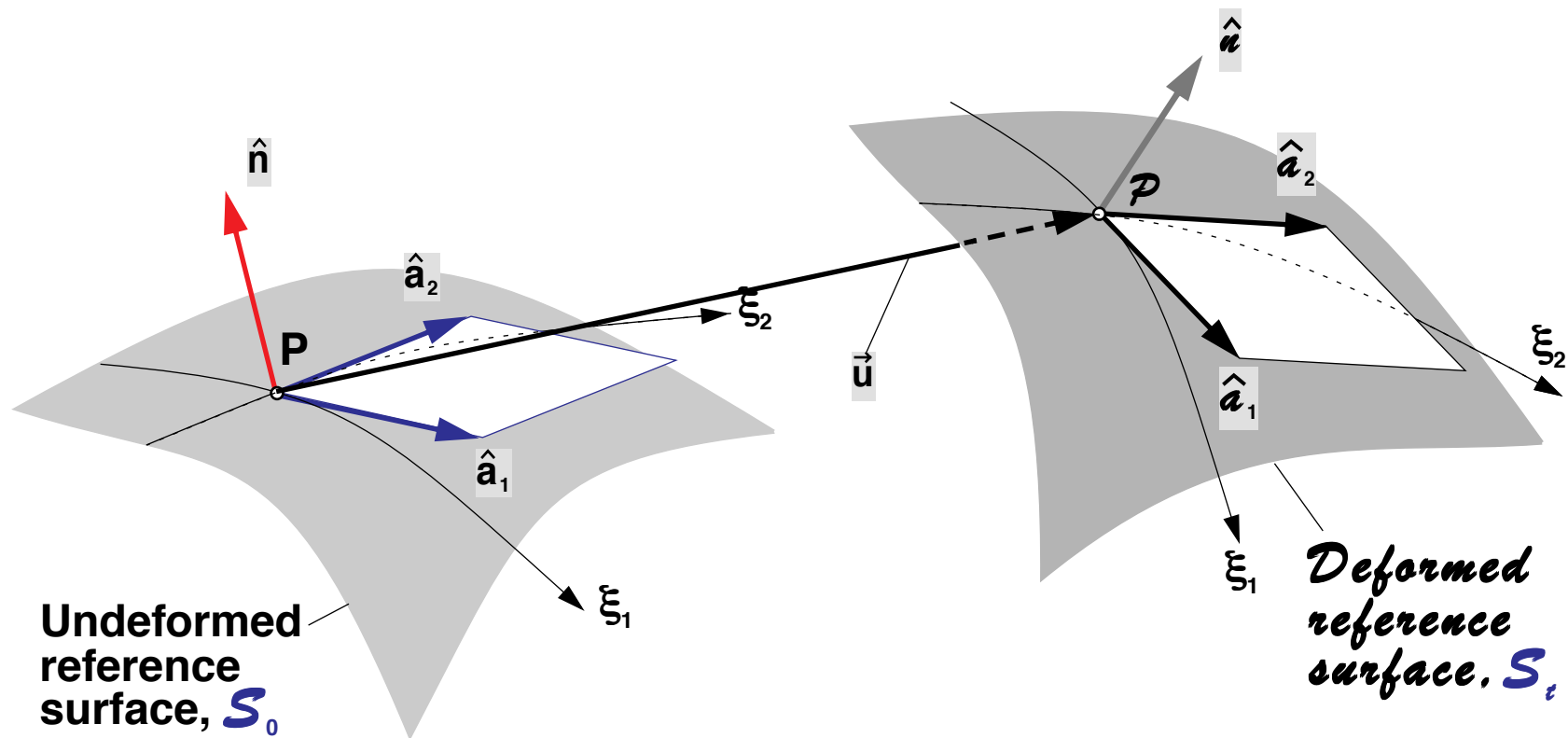
$$\left(\vec{\nabla} \times \vec{u}\right)^1 = \left(\hat{n} \cdot \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2}\right) \csc \theta_{12} \quad \left(\vec{\nabla} \times \vec{u}\right)^2 = - \left(\hat{n} \cdot \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1}\right) \csc \theta_{12} \quad \text{gives}$$

$$\left(\vec{\nabla} \times \vec{u}\right)_1 = \left(\hat{n} \cdot \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2}\right) \csc^2 \theta_{12} - \left(\hat{n} \cdot \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1}\right) \csc \theta_{12} \cot \theta_{12} \quad \text{and}$$

$$\left(\vec{\nabla} \times \vec{u}\right)_2 = \left(\hat{n} \cdot \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2}\right) \csc \theta_{12} \cot \theta_{12} - \left(\hat{n} \cdot \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1}\right) \csc^2 \theta_{12}$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

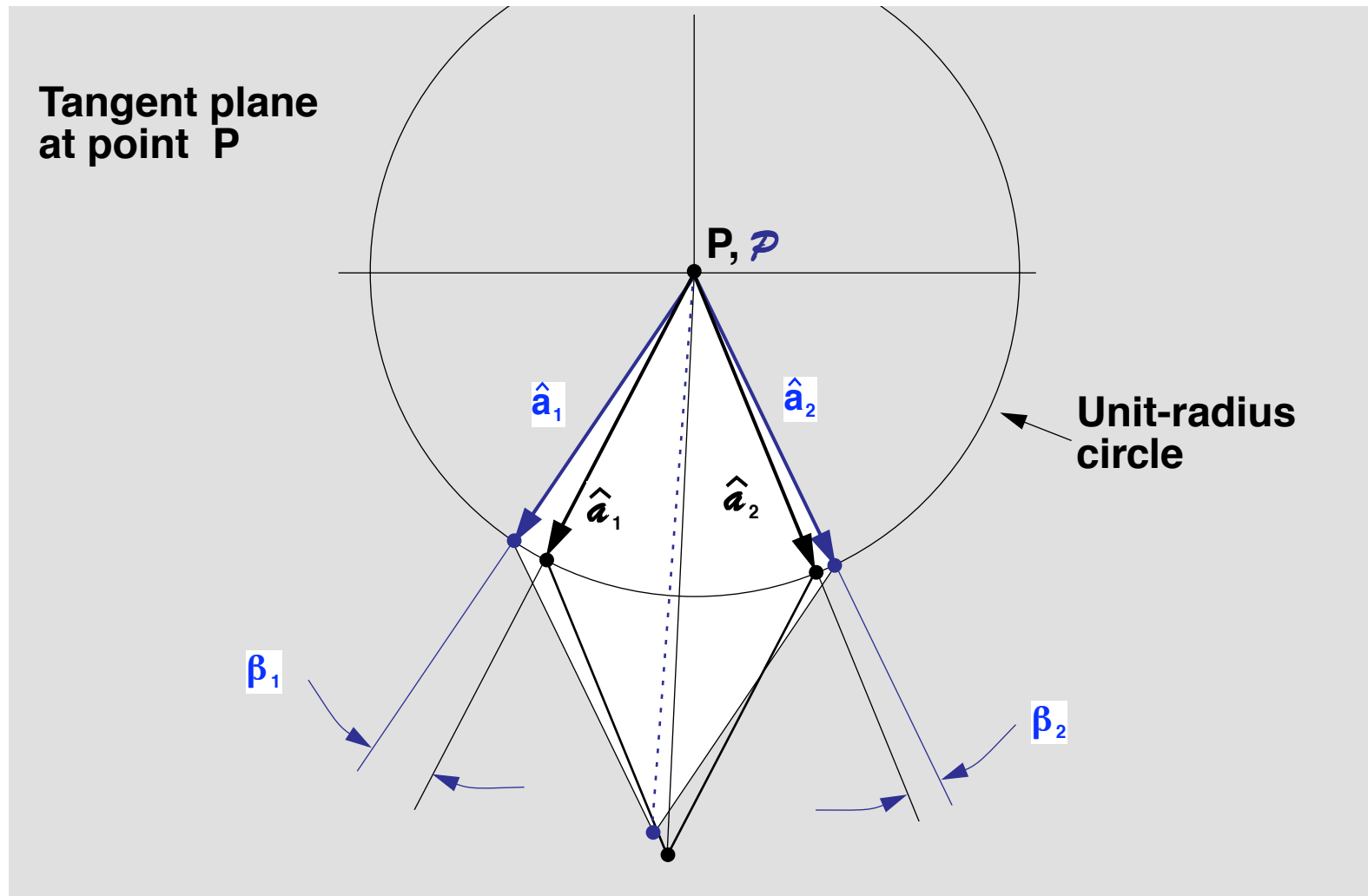
- Now, consider the deformation and rotation of the reference surface



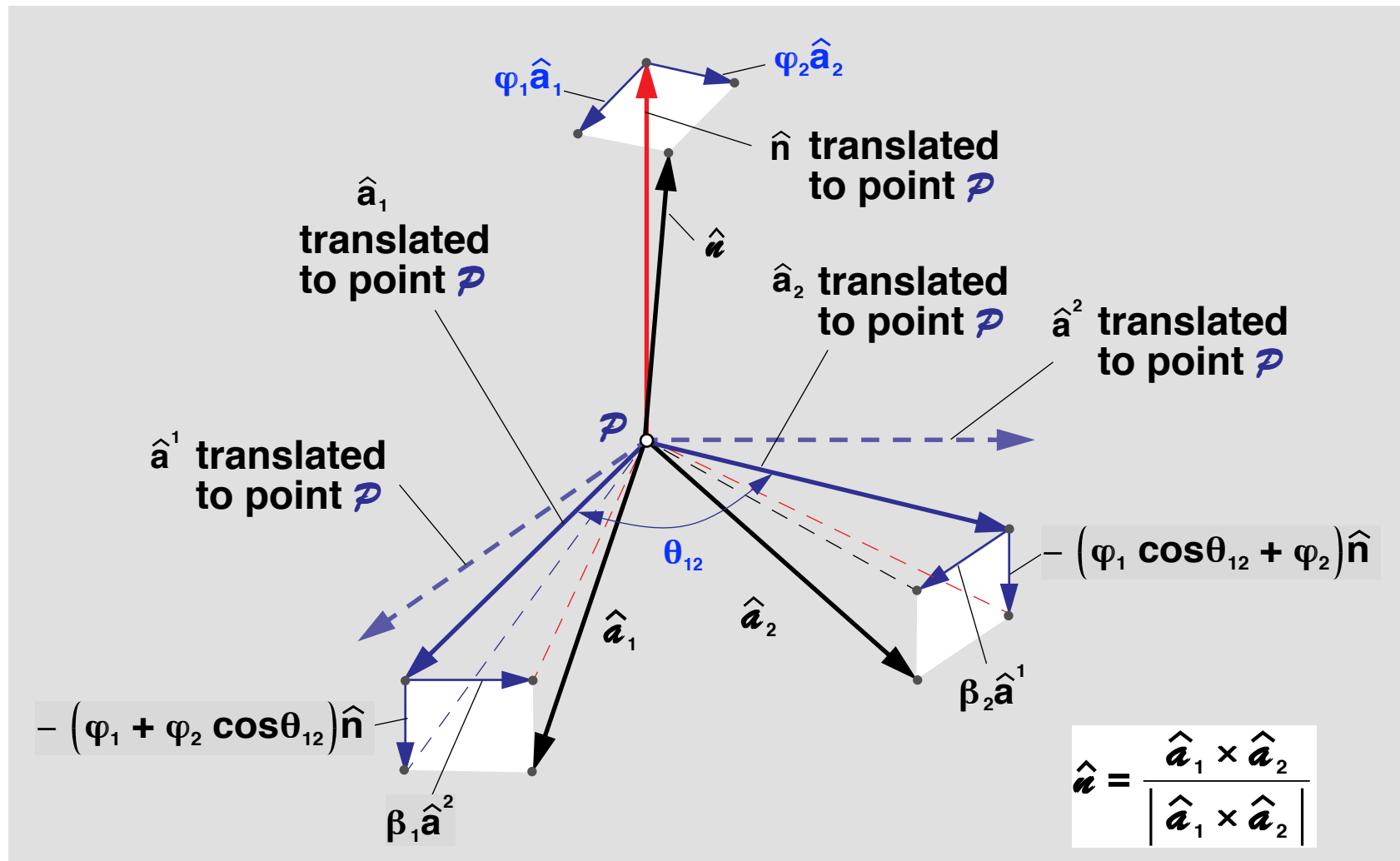
LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- As the reference surface deforms, the unit-magnitude vectors $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ are convected into the unit-magnitude vectors $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$, which also rotate as the reference surface deforms
- For the class of very small deformations with $\left| \frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_\alpha} \right| \ll 1$, the rotation of $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ is represented by the rotation of the corresponding tangent plane and the rotation of each vector within the tangent plane that is caused by deformation
 - The rotations φ_1 and φ_2 quantify the **rotation of the tangent plane** as the reference surface deforms
- Let β_1 and β_2 denote the rotations of $\hat{\mathbf{a}}_1(\xi_1, \xi_2, t)$ and $\hat{\mathbf{a}}_2(\xi_1, \xi_2, t)$ **within the tangent plane**, respectively, as depicted in the next figure

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED



LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED



LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- In the previous two figures, the base vectors $\{\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{n}}\}$, associated with point $\mathbf{P} \in \mathcal{S}_0$, have been translated to the corresponding point of the deformed surface, $\mathcal{P} = \mathcal{D}(\mathbf{P}) \in \mathcal{S}_t$
- The positive directions of β_1 and β_2 shown in the figures correspond to a decrease in the angle between $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ as they are convected into $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$, consistent with a positive-valued shearing deformation
- Like φ_1 and φ_2 , for $\left| \frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_\alpha} \right| \ll 1$, the rotations β_1 and β_2 are also presumed to be of $\mathcal{O}(\varepsilon^2)$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- For rotations that are $\mathcal{O}(\varepsilon^2)$, it is shown in textbooks on rigid-body dynamics that the rotation of a vector $\vec{a}$ into a vector $\vec{b}$ is given by $\vec{b} = \vec{a} + \vec{r} \times \vec{a}$, where $\vec{r}$ is a rotation vector with $|\vec{r}| \ll 1$

- Thus, the rotation of $\hat{a}_1$ into $\hat{a}_1$ is given by

$$\hat{a}_1 = \hat{a}_1 + \left(\varphi_1 \hat{a}^2 - \varphi_2 \hat{a}^1 + \beta_1 \hat{n} \right) \times \hat{a}_1$$

- Similarly, the rotation of $\hat{a}_2$ into $\hat{a}_2$ is given by

$$\hat{a}_2 = \hat{a}_2 + \left(\varphi_1 \hat{a}^2 - \varphi_2 \hat{a}^1 - \beta_2 \hat{n} \right) \times \hat{a}_2$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- Upon performing the cross-product operations, the convected base-vector fields are obtained as

$$\begin{aligned}\hat{\mathbf{a}}_1 &= \hat{\mathbf{a}}_1 + \beta_1 \hat{\mathbf{a}}^2 - (\varphi_1 + \varphi_2 \cos\theta_{12}) \hat{\mathbf{n}} + \vec{\mathcal{O}}(\epsilon^3) \\ \hat{\mathbf{a}}_2 &= \beta_2 \hat{\mathbf{a}}^1 + \hat{\mathbf{a}}_2 - (\varphi_1 \cos\theta_{12} + \varphi_2) \hat{\mathbf{n}} + \vec{\mathcal{O}}(\epsilon^3) \\ \hat{\mathbf{a}} &= \varphi_1 \hat{\mathbf{a}}_1 + \varphi_2 \hat{\mathbf{a}}_2 + \hat{\mathbf{n}} + \vec{\mathcal{O}}(\epsilon^3)\end{aligned}$$

- Here, $\vec{\mathcal{O}}(\epsilon^3) \equiv \mathcal{O}(\epsilon^3)(\hat{\mathbf{a}}_1 + \hat{\mathbf{a}}_2 + \hat{\mathbf{n}})$ is introduced for convenience
- From these expressions, it follows that

$$\sin\theta_{12} - \beta_1 \cos\theta_{12} + \mathcal{O}(\epsilon^3) = \hat{\mathbf{a}}^1 \cdot \hat{\mathbf{a}}_1$$

$$\beta_1 + \mathcal{O}(\epsilon^3) = \hat{\mathbf{a}}^2 \cdot \hat{\mathbf{a}}_1$$

$$\sin\theta_{12} - \beta_2 \cos\theta_{12} + \mathcal{O}(\epsilon^3) = \hat{\mathbf{a}}^2 \cdot \hat{\mathbf{a}}_2$$

$$\beta_2 + \mathcal{O}(\epsilon^3) = \hat{\mathbf{a}}^1 \cdot \hat{\mathbf{a}}_2$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- Likewise,

$$\varphi_1 + \varphi_2 \cos\theta_{12} + \mathcal{O}(\varepsilon^3) = -\hat{\mathbf{n}} \cdot \hat{\mathbf{a}}_1$$

$$\varphi_1 \cos\theta_{12} + \varphi_2 + \mathcal{O}(\varepsilon^3) = -\hat{\mathbf{n}} \cdot \hat{\mathbf{a}}_2$$

- Using $\hat{\mathbf{a}}_1 = \left(\hat{\mathbf{a}}_1 + \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \left(1 - \mathbf{e}_{11}^o + \mathcal{O}(\varepsilon^4) \right)$ with $\beta_1 + \mathcal{O}(\varepsilon^3) = \hat{\mathbf{a}}^2 \cdot \hat{\mathbf{a}}_1$ gives

$$\left[1 - \mathbf{e}_{11}^o + \mathcal{O}(\varepsilon^4) \right]^{-1} \left[\beta_1 + \mathcal{O}(\varepsilon^3) \right] = \hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1}$$

- Applying the binomial series $\frac{1}{1 - \gamma} = 1 + \gamma + \mathcal{O}(\varepsilon^4)$ to the left-hand-side of the last equation gives

$$\left[1 + \mathbf{e}_{11}^o + \mathcal{O}(\varepsilon^4) \right] \left[\beta_1 + \mathcal{O}(\varepsilon^3) \right] = \hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1}$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- Similarly, using $\hat{\mathbf{a}}_2 = \left(\hat{\mathbf{a}}_2 + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \left(1 - \mathbf{e}_{22}^o + \mathcal{O}(\epsilon^4) \right)$ gives

$$\left[1 + \mathbf{e}_{22}^o + \mathcal{O}(\epsilon^4) \right] \left[\beta_2 + \mathcal{O}(\epsilon^3) \right] = \hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2}$$

$$\left[1 + \mathbf{e}_{11}^o + \mathcal{O}(\epsilon^4) \right] \left[\varphi_1 + \varphi_2 \cos \theta_{12} + \mathcal{O}(\epsilon^3) \right] = -\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1}$$

$$\left[1 + \mathbf{e}_{22}^o + \mathcal{O}(\epsilon^4) \right] \left[\varphi_1 \cos \theta_{12} + \varphi_2 + \mathcal{O}(\epsilon^3) \right] = -\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2}$$

$$\left[1 + \mathbf{e}_{11}^o + \mathcal{O}(\epsilon^4) \right] \left[\sin \theta_{12} - \beta_1 \cos \theta_{12} + \mathcal{O}(\epsilon^3) \right] = \sin \theta_{12} + \hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1}$$

$$\left[1 + \mathbf{e}_{22}^o + \mathcal{O}(\epsilon^4) \right] \left[\sin \theta_{12} - \beta_2 \cos \theta_{12} + \mathcal{O}(\epsilon^3) \right] = \sin \theta_{12} + \hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2}$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- Noting that **products of rotations and strains** are at most $\mathcal{O}(\varepsilon^4)$, the following expressions are obtained

$$\hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} = \mathbf{e}_{11}^o \sin \theta_{12} - \beta_1 \cos \theta_{12} + \mathcal{O}(\varepsilon^3)$$

$$\hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} = \beta_2 + \mathcal{O}(\varepsilon^3)$$

$$\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} = \beta_1 + \mathcal{O}(\varepsilon^3)$$

$$\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} = \mathbf{e}_{22}^o \sin \theta_{12} - \beta_2 \cos \theta_{12} + \mathcal{O}(\varepsilon^3)$$

$$\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} = - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) + \mathcal{O}(\varepsilon^3)$$

$$\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} = - \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) + \mathcal{O}(\varepsilon^3)$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- The expressions

$$\frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} = \left(\hat{a}^1 \cdot \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \right) \csc \theta_{12} \hat{a}_1 + \left(\hat{a}^2 \cdot \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \right) \csc \theta_{12} \hat{a}_2 + \left(\hat{n} \cdot \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \right) \hat{n}$$

$$\frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} = \left(\hat{a}^1 \cdot \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \right) \csc \theta_{12} \hat{a}_1 + \left(\hat{a}^2 \cdot \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \right) \csc \theta_{12} \hat{a}_2 + \left(\hat{n} \cdot \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \right) \hat{n}$$

are used next to get

$$\frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} = \left(\mathbf{e}_{11}^o - \beta_1 \cot \theta_{12} \right) \hat{a}_1 + \beta_1 \csc \theta_{12} \hat{a}_2 - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \hat{n} + \vec{\mathcal{O}}(\varepsilon^3)$$

$$\frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} = \beta_2 \csc \theta_{12} \hat{a}_1 + \left(\mathbf{e}_{22}^o - \beta_2 \cot \theta_{12} \right) \hat{a}_2 - \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) \hat{n} + \vec{\mathcal{O}}(\varepsilon^3)$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- Next, these expressions are used to obtain

$$\left(\vec{\nabla} \times \vec{u}\right)_1 = -\varphi_2 + \mathcal{O}(\varepsilon^3) \quad \left(\vec{\nabla} \times \vec{u}\right)_2 = \varphi_1 + \mathcal{O}(\varepsilon^3)$$

$$\left(\vec{\nabla} \times \vec{u}\right)^3 = \beta_1 - \beta_2 + \left(\mathbf{e}_{11}^o - \mathbf{e}_{22}^o\right) \cot \theta_{12} + \mathcal{O}(\varepsilon^3)$$

- Thus, $\left(\vec{\nabla} \times \vec{u}\right)_1$ and $\left(\vec{\nabla} \times \vec{u}\right)_2$ define the rotation of $\hat{n}$ into $\hat{n}$
- For the convection of $\hat{a}_1$ and $\hat{a}_2$ into $\hat{a}_1$ and $\hat{a}_2$, $\left(\vec{\nabla} \times \vec{u}\right)^3$ represents twice the average value of the rotation about $\hat{n}$ that occurs during deformation; therefore,

$$\varphi \equiv \frac{1}{2} \left(\vec{\nabla} \times \vec{u}\right) \cdot \hat{n}$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- The next step is to eliminate the parameters β_1 and β_2 from the previous expressions for the displacement derivatives

- First, note that $\beta_1 + \beta_2 + \mathcal{O}(\epsilon^3) = \hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} + \hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2}$

- Using $\hat{\mathbf{a}}^1 = \hat{\mathbf{a}}_1 \csc\theta_{12} - \hat{\mathbf{a}}_2 \cot\theta_{12}$ and $\hat{\mathbf{a}}^2 = \hat{\mathbf{a}}_2 \csc\theta_{12} - \hat{\mathbf{a}}_1 \cot\theta_{12}$ gives

$$\beta_1 + \beta_2 + \mathcal{O}(\epsilon^3) =$$

$$\left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} + \hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \csc\theta_{12} - \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} + \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \cot\theta_{12}$$

- Then, using the definitions for the linearized strains yields

$$\beta_1 + \beta_2 = 2\mathbf{e}_{12}^o \csc\theta_{12} - (\mathbf{e}_{11}^o + \mathbf{e}_{22}^o) \cot\theta_{12} + \mathcal{O}(\epsilon^3)$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- Likewise, $2\varphi = \left(\vec{\nabla} \times \vec{u} \right)^3 = \beta_1 - \beta_2 + \left(\mathbf{e}_{11}^o - \mathbf{e}_{22}^o \right) \cot\theta_{12} + \mathcal{O}(\varepsilon^3)$ gives

$$\beta_1 - \beta_2 = 2\varphi - \left(\mathbf{e}_{11}^o - \mathbf{e}_{22}^o \right) \cot\theta_{12} + \mathcal{O}(\varepsilon^3)$$

- Therefore,

$$\beta_1 = \varphi + \mathbf{e}_{12}^o \csc\theta_{12} - \mathbf{e}_{11}^o \cot\theta_{12} + \mathcal{O}(\varepsilon^3)$$

$$\beta_2 = \mathbf{e}_{12}^o \csc\theta_{12} - \varphi - \mathbf{e}_{22}^o \cot\theta_{12} + \mathcal{O}(\varepsilon^3)$$

- In addition,

$$\vec{\nabla} \times \vec{u} = -\varphi_2 \hat{\mathbf{a}}^1 + \varphi_1 \hat{\mathbf{a}}^2 + 2\varphi \hat{\mathbf{n}} + \mathcal{O}(\varepsilon^3)$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- Using

$$\beta_1 = \left(\mathbf{e}_{12}^{\circ} + \varphi \sin\theta_{12} \right) \csc\theta_{12} - \mathbf{e}_{11}^{\circ} \cot\theta_{12} + \mathcal{O}(\varepsilon^3) \quad \text{and}$$

$$\beta_2 = \left(\mathbf{e}_{12}^{\circ} - \varphi \sin\theta_{12} \right) \csc\theta_{12} - \mathbf{e}_{22}^{\circ} \cot\theta_{12} + \mathcal{O}(\varepsilon^3) \quad \text{gives}$$

$$\begin{aligned} \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} &= \left[\mathbf{e}_{11}^{\circ} \csc\theta_{12} - \left(\mathbf{e}_{12}^{\circ} + \varphi \sin\theta_{12} \right) \cot\theta_{12} \right] \csc\theta_{12} \hat{\mathbf{a}}_1 \\ &+ \left[\left(\mathbf{e}_{12}^{\circ} + \varphi \sin\theta_{12} \right) \csc\theta_{12} - \mathbf{e}_{11}^{\circ} \cot\theta_{12} \right] \csc\theta_{12} \hat{\mathbf{a}}_2 - \left(\varphi_1 + \varphi_2 \cos\theta_{12} \right) \hat{\mathbf{n}} + \vec{\mathcal{O}}(\varepsilon^3) \end{aligned}$$

$$\begin{aligned} \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} &= \left[\left(\mathbf{e}_{12}^{\circ} - \varphi \sin\theta_{12} \right) \csc\theta_{12} - \mathbf{e}_{22}^{\circ} \cot\theta_{12} \right] \csc\theta_{12} \hat{\mathbf{a}}_1 \\ &+ \left[\mathbf{e}_{22}^{\circ} \csc\theta_{12} - \left(\mathbf{e}_{12}^{\circ} - \varphi \sin\theta_{12} \right) \cot\theta_{12} \right] \csc\theta_{12} \hat{\mathbf{a}}_2 - \left(\varphi_1 \cos\theta_{12} + \varphi_2 \right) \hat{\mathbf{n}} + \vec{\mathcal{O}}(\varepsilon^3) \end{aligned}$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- Expressions for the **linearized rotations** in terms of the displacements are obtained by using the previously derived general expressions for the components of the derivatives of a vector field given by

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{V}}}{\partial \xi_1} = \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\mathbf{V}_1 + \mathbf{V}_2 \cos \theta_{12} \right) - \frac{\mathbf{V}_2 \sin \theta_{12}}{\rho_{11}} + \frac{\mathbf{V}_3}{\mathbf{r}_{11}}$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{V}}}{\partial \xi_1} = \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{V}_1}{\partial \xi_1} + \frac{\mathbf{V}_3}{\mathbf{r}_{11}} \right) \cos \theta_{12} + \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{V}_2}{\partial \xi_1} + \left(\frac{\mathbf{V}_1}{\rho_{11}} - \frac{\mathbf{V}_3}{\mathbf{r}_{12}} \right) \sin \theta_{12}$$

$$\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{V}}}{\partial \xi_1} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{V}_3}{\partial \xi_1} - \frac{\mathbf{V}_1}{\mathbf{r}_{11}} + \mathbf{V}_2 \left(\frac{\sin \theta_{12}}{\mathbf{r}_{12}} - \frac{\cos \theta_{12}}{\mathbf{r}_{11}} \right)$$

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{V}}}{\partial \xi_2} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{V}_1}{\partial \xi_2} + \left(\frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{V}_2}{\partial \xi_2} + \frac{\mathbf{V}_3}{\mathbf{r}_{22}} \right) \cos \theta_{12} + \left(\frac{\mathbf{V}_3}{\mathbf{r}_{21}} - \frac{\mathbf{V}_2}{\rho_{22}} \right) \sin \theta_{12}$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{V}}}{\partial \xi_2} = \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} \left(\mathbf{V}_2 + \mathbf{V}_1 \cos \theta_{12} \right) + \frac{\mathbf{V}_1 \sin \theta_{12}}{\rho_{22}} + \frac{\mathbf{V}_3}{\mathbf{r}_{22}}$$

$$\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{V}}}{\partial \xi_2} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{V}_3}{\partial \xi_2} - \mathbf{V}_1 \left(\frac{\sin \theta_{12}}{\mathbf{r}_{21}} + \frac{\cos \theta_{12}}{\mathbf{r}_{22}} \right) - \frac{\mathbf{V}_2}{\mathbf{r}_{22}}$$

along with the identity $\frac{1}{\mathbf{r}_{12}} + \frac{1}{\mathbf{r}_{21}} = \cot \theta_{12} \left(\frac{1}{\mathbf{r}_{11}} - \frac{1}{\mathbf{r}_{22}} \right)$

- In particular, using these expressions for the dot products with

$$\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} = - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) + \mathcal{O}(\varepsilon^3)$$

$$\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} = - \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) + \mathcal{O}(\varepsilon^3) \quad \text{gives}$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

$$\varphi_1 = \left(\frac{u_1}{r_{11}} - \frac{1}{A_1} \frac{\partial w}{\partial \xi_1} \right) \csc^2 \theta_{12} - u_1 \left(\frac{1}{r_{21}} + \frac{\cot \theta_{12}}{r_{22}} \right) \cot \theta_{12} + \left(\frac{\cot \theta_{12}}{A_2} \frac{\partial w}{\partial \xi_2} + \frac{u_2}{r_{21}} \right) \csc \theta_{12} + \mathcal{O}(\varepsilon^3)$$

$$\varphi_2 = \left(\frac{u_2}{r_{22}} - \frac{1}{A_2} \frac{\partial w}{\partial \xi_2} \right) \csc^2 \theta_{12} + u_2 \left(\frac{1}{r_{12}} - \frac{\cot \theta_{12}}{r_{11}} \right) \cot \theta_{12} + \left(\frac{\cot \theta_{12}}{A_1} \frac{\partial w}{\partial \xi_1} - \frac{u_1}{r_{12}} \right) \csc \theta_{12} + \mathcal{O}(\varepsilon^3)$$

- Similarly, using the expressions for the dot products with

$$2\varphi = \left(\vec{\nabla} \times \vec{u} \right)^3 = \left[\left(\hat{a}_2 \cdot \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi} \right) - \left(\hat{a}_1 \cdot \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \right) \right] \csc \theta_{12} \text{ gives}$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

$$2\varphi = \left(\frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} \right) \cot \theta_{12} + \left(\frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} \right) \csc \theta_{12} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}} + \mathcal{O}(\epsilon^3)$$

where

$$\frac{1}{\rho_{11}} = \frac{\csc \theta_{12}}{A_1 A_2} \left(\frac{\partial}{\partial \xi_1} [A_2 \cos \theta_{12}] - \frac{\partial A_1}{\partial \xi_2} \right) \quad \frac{1}{\rho_{22}} = - \frac{\csc \theta_{12}}{A_1 A_2} \left(\frac{\partial}{\partial \xi_2} [A_1 \cos \theta_{12}] - \frac{\partial A_2}{\partial \xi_1} \right)$$

- Moreover, the **linearized Green-Lagrange strains** are obtained in terms of the displacements by using the dot product expressions with

$$e_{11}^{\circ} \equiv \hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1}$$

$$e_{22}^{\circ} \equiv \hat{a}_2 \cdot \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2}$$

$$2e_{12}^{\circ} \equiv \hat{a}_1 \cdot \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} + \hat{a}_2 \cdot \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \quad \text{to get}$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

$$\mathbf{e}_{11}^{\circ} = \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} (\mathbf{u}_1 + \mathbf{u}_2 \cos \theta_{12}) - \frac{\mathbf{u}_2 \sin \theta_{12}}{\rho_{11}} + \frac{\mathbf{w}}{\mathbf{r}_{11}} + \mathcal{O}(\varepsilon^3)$$

$$\mathbf{e}_{22}^{\circ} = \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} (\mathbf{u}_2 + \mathbf{u}_1 \cos \theta_{12}) + \frac{\mathbf{u}_1 \sin \theta_{12}}{\rho_{22}} + \frac{\mathbf{w}}{\mathbf{r}_{22}} + \mathcal{O}(\varepsilon^3)$$

$$\begin{aligned} 2\mathbf{e}_{12}^{\circ} = & \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_1}{\partial \xi_2} + \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_2}{\partial \xi_1} + \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_1}{\partial \xi_1} + \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_2}{\partial \xi_2} \right) \cos \theta_{12} \\ & + \left(\frac{\mathbf{u}_1}{\rho_{11}} - \frac{\mathbf{u}_2}{\rho_{22}} \right) \sin \theta_{12} + \mathbf{w} \left(\frac{1}{\mathbf{r}_{11}} + \frac{1}{\mathbf{r}_{22}} \right) \cos \theta_{12} + \mathbf{w} \left(\frac{1}{\mathbf{r}_{21}} - \frac{1}{\mathbf{r}_{12}} \right) \sin \theta_{12} + \mathcal{O}(\varepsilon^3) \end{aligned}$$

- These expressions are in complete agreement with the corresponding tensor equations given in:
 - Sanders, J. L.: Nonlinear Theories for Thin Shells. Quarterly Journal of Applied Mathematics, vol. 21, no. 1, 1963, pp. 21-36.

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONTINUED

- For the special case of **orthogonal** reference-surface Gaussian coordinates, the **linearized rotations** and **linearized Green-Lagrange strains** reduce to

$$\varphi_1 = \frac{u_1}{r_{11}} - \frac{u_2}{r_{12}} - \frac{1}{A_1} \frac{\partial w}{\partial \xi_1} + \mathcal{O}(\varepsilon^3)$$

$$\varphi_2 = \frac{u_2}{r_{22}} - \frac{u_1}{r_{12}} - \frac{1}{A_2} \frac{\partial w}{\partial \xi_2} + \mathcal{O}(\varepsilon^3)$$

$$\varphi = \frac{1}{2} \left(\frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}} \right) + \mathcal{O}(\varepsilon^3)$$

$$\mathbf{e}_{11}^{\circ} = \frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} - \frac{u_2}{\rho_{11}} + \frac{w}{r_{11}} + \mathcal{O}(\varepsilon^3)$$

$$\mathbf{e}_{22}^{\circ} = \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{\rho_{22}} + \frac{w}{r_{22}} + \mathcal{O}(\varepsilon^3)$$

$$2\mathbf{e}_{12}^{\circ} = \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} + \frac{u_1}{\rho_{11}} - \frac{u_2}{\rho_{22}} - \frac{2w}{r_{12}} + \mathcal{O}(\varepsilon^3)$$

LINEARIZED REFERENCE-SURFACE STRAINS AND ROTATIONS - CONCLUDED

- Furthermore, the **derivatives of the displacement-vector field** reduce to

$$\frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} = \mathbf{e}_{11}^o \hat{\mathbf{a}}_1 + (\mathbf{e}_{12}^o + \varphi) \hat{\mathbf{a}}_2 - \varphi_1 \hat{\mathbf{n}} + \vec{\mathcal{O}}(\varepsilon^3)$$

$$\frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} = (\mathbf{e}_{12}^o - \varphi) \hat{\mathbf{a}}_1 + \mathbf{e}_{22}^o \hat{\mathbf{a}}_2 - \varphi_2 \hat{\mathbf{n}} + \vec{\mathcal{O}}(\varepsilon^3)$$

for **orthogonal** reference-surface Gaussian coordinates, where it is

noted that $\beta_1 = \mathbf{e}_{12}^o + \varphi + \mathcal{O}(\varepsilon^3)$ and $\beta_2 = \mathbf{e}_{12}^o - \varphi + \mathcal{O}(\varepsilon^3)$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

- The general expressions for the curvatures and torsions of the deformed reference surface presented previously herein are complicated nonlinear functions of the reference-surface displacement fields
- To aid in simplification of these expressions, it is convenient to have the corresponding linearized curvatures and torsions, expressed in terms of the linearized strain and rotation parameters
- Previously, it was shown herein that the **normal curvatures** of the deformed reference surface, along the convected Gaussian-coordinate curves, are defined by

$$\frac{1}{r_{11}} = \hat{a}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \quad \text{and} \quad \frac{1}{r_{22}} = \hat{a}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2}$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- Likewise, the **torsions** are defined by

$$\frac{1}{r_{12}} = -\hat{a}^2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \hat{u}}{\partial \xi_1} \quad \text{and} \quad \frac{1}{r_{21}} = \hat{a}^1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \hat{u}}{\partial \xi_2}$$

- Consider the first curvature expression given on the previous page

- Using $\hat{a}_1 = \left(\hat{a}_1 + \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \right) \left(1 - e_{11}^o + \mathcal{O}(\varepsilon^4) \right)$ and $\mathcal{A}_1 = A_1 (1 + e_{11}^o) + \mathcal{O}(\varepsilon^4)$ gives

$$\frac{1}{r_{11}} = \left(\hat{a}_1 + \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \right) \cdot \frac{1}{A_1} \frac{\partial \hat{u}}{\partial \xi_1} \left(1 - 2e_{11}^o + \mathcal{O}(\varepsilon^4) \right)$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- Next, using $\hat{\mathbf{u}} = \hat{\mathbf{n}} + \vec{\varphi}$, where $\vec{\varphi} = \varphi_1 \hat{\mathbf{a}}_1 + \varphi_2 \hat{\mathbf{a}}_2$ is the linear difference-vector field, gives

$$\frac{1}{A_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} = \frac{1}{A_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} + \frac{1}{A_1} \frac{\partial \vec{\varphi}}{\partial \xi_1}$$

- Then, using $\frac{1}{r_{11}} = \hat{\mathbf{a}}_1 \cdot \frac{1}{A_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1}$ and enforcing the fact that the components of the difference-vector field and the displacement derivatives have magnitudes of $\mathcal{O}(\varepsilon^2)$, gives

$$\frac{1}{r_{11}} = \frac{1}{r_{11}} (1 - 2\mathbf{e}_{11}^o) + \frac{1}{A_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{A_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} + \hat{\mathbf{a}}_1 \cdot \frac{1}{A_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} + \mathcal{O}(\varepsilon^4)$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- Now, using $\frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} = \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) \hat{a}_1 - \frac{\csc \theta_{12}}{r_{12}} \hat{a}_2$ and

$$\begin{aligned} \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} = & \left[\mathbf{e}_{11}^\circ \csc \theta_{12} - (\mathbf{e}_{12}^\circ + \varphi \sin \theta_{12}) \cot \theta_{12} \right] \csc \theta_{12} \hat{a}_1 \\ & + \left[(\mathbf{e}_{12}^\circ + \varphi \sin \theta_{12}) \csc \theta_{12} - \mathbf{e}_{11}^\circ \cot \theta_{12} \right] \csc \theta_{12} \hat{a}_2 - (\varphi_1 + \varphi_2 \cos \theta_{12}) \hat{n} + \vec{\mathcal{O}}(\varepsilon^3) \end{aligned}$$

gives $\frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} = \frac{\mathbf{e}_{11}^\circ}{r_{11}} + \frac{\mathbf{e}_{11}^\circ \cot \theta_{12} - \mathbf{e}_{12}^\circ \csc \theta_{12}}{r_{12}} - \frac{\varphi}{r_{12}} + \mathcal{O}(\varepsilon^3)$

- Thus,

$$\frac{1}{r_{11}} = \frac{1}{r_{11}} - \frac{\mathbf{e}_{11}^\circ}{r_{11}} + \frac{\mathbf{e}_{11}^\circ \cot \theta_{12} - \mathbf{e}_{12}^\circ \csc \theta_{12}}{r_{12}} - \frac{\varphi}{r_{12}} + \hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} + \mathcal{O}(\varepsilon^3)$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- Now consider the curvature expression $\frac{1}{r_{22}} = \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2}$

- Using $\hat{\mathbf{a}}_2 = \left(\hat{\mathbf{a}}_2 + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \left(1 - \mathbf{e}_{22}^o + \mathcal{O}(\varepsilon^4) \right)$ and $\mathcal{A}_2 = \mathbf{A}_2 \left(1 + \mathbf{e}_{22}^o \right) + \mathcal{O}(\varepsilon^4)$ gives

$$\frac{1}{r_{22}} = \left(\hat{\mathbf{a}}_2 + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \left(1 - 2\mathbf{e}_{22}^o + \mathcal{O}(\varepsilon^4) \right)$$

- Also, using $\hat{\mathbf{u}} = \hat{\mathbf{n}} + \vec{\boldsymbol{\phi}}$ gives $\frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} = \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\boldsymbol{\phi}}}{\partial \xi_2}$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- Then, using $\frac{1}{r_{22}} = \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2}$ and enforcing the fact that the components of the difference-vector field and the displacement derivatives have magnitudes of $\mathcal{O}(\varepsilon^2)$, gives

$$\frac{1}{r_{22}} = \frac{1}{r_{22}} (1 - 2\mathbf{e}_{22}^o) + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} + \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\boldsymbol{\varphi}}}{\partial \xi_2} + \mathcal{O}(\varepsilon^4)$$

- Now, using $\frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} = \frac{\csc \theta_{12}}{r_{21}} \hat{\mathbf{a}}_1 + \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right) \hat{\mathbf{a}}_2$ and

$$\begin{aligned} \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} = & \left[\left(\mathbf{e}_{12}^o - \varphi \sin \theta_{12} \right) \csc \theta_{12} - \mathbf{e}_{22}^o \cot \theta_{12} \right] \csc \theta_{12} \hat{\mathbf{a}}_1 \\ & + \left[\mathbf{e}_{22}^o \csc \theta_{12} - \left(\mathbf{e}_{12}^o - \varphi \sin \theta_{12} \right) \cot \theta_{12} \right] \csc \theta_{12} \hat{\mathbf{a}}_2 - \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) \hat{\mathbf{n}} + \vec{\mathcal{O}}(\varepsilon^3) \end{aligned}$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

gives
$$\frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} = \frac{e_{22}^o}{r_{22}} + \frac{e_{12}^o \csc \theta_{12} - e_{22}^o \cot \theta_{12}}{r_{21}} - \frac{\varphi}{r_{21}} + \mathcal{O}(\varepsilon^3)$$

- Thus,

$$\frac{1}{r_{22}} = \frac{1}{r_{22}} - \frac{e_{22}^o}{r_{22}} + \frac{e_{12}^o \csc \theta_{12} - e_{22}^o \cot \theta_{12}}{r_{21}} - \frac{\varphi}{r_{21}} + \hat{a}_2 \cdot \frac{1}{A_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} + \mathcal{O}(\varepsilon^3)$$

- Now consider the torsion expressions written as

$$\frac{1}{r_{12}} = -\frac{A_1}{\mathcal{A}_1} \hat{a}^2 \cdot \frac{1}{A_1} \frac{\partial \hat{u}}{\partial \xi_1} \quad \text{and} \quad \frac{1}{r_{21}} = \frac{A_2}{\mathcal{A}_2} \hat{a}^1 \cdot \frac{1}{A_2} \frac{\partial \hat{u}}{\partial \xi_2}$$

where $\hat{a}^1 = \hat{a}_1 \csc \theta_{12} - \hat{a}_2 \cot \theta_{12}$ and $\hat{a}^2 = \hat{a}_2 \csc \theta_{12} - \hat{a}_1 \cot \theta_{12}$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- For linearized strains,

$$\cot\theta_{12} = \cot\theta_{12} + \csc^3\theta_{12} \left[2\mathbf{e}_{12}^{\circ} - \cos\theta_{12}(\mathbf{e}_{11}^{\circ} + \mathbf{e}_{22}^{\circ}) \right] + \mathcal{O}(\varepsilon^4) \quad \text{and}$$

$$\csc\theta_{12} = \csc\theta_{12} + \csc^2\theta_{12}\cot\theta_{12} \left[2\mathbf{e}_{12}^{\circ} - \cos\theta_{12}(\mathbf{e}_{11}^{\circ} + \mathbf{e}_{22}^{\circ}) \right] + \mathcal{O}(\varepsilon^4)$$

- Using these expressions with

$$\mathcal{A}_1 = \mathbf{A}_1(1 + \mathbf{e}_{11}^{\circ}) + \mathcal{O}(\varepsilon^4) \quad \text{and} \quad \mathcal{A}_2 = \mathbf{A}_2(1 + \mathbf{e}_{22}^{\circ}) + \mathcal{O}(\varepsilon^4) \quad \text{gives}$$

$$\frac{\mathbf{A}_1}{\mathcal{A}_1} \hat{\mathbf{a}}^2 = \left(\csc\theta_{12} - \mathbf{e}_{11}^{\circ} \csc\theta_{12} \right) \hat{\mathbf{a}}_2 - \left(\cot\theta_{12} - \mathbf{e}_{11}^{\circ} \cot\theta_{12} \right) \hat{\mathbf{a}}_1 + \vec{\mathcal{O}}(\varepsilon^4)$$

$$\frac{\mathbf{A}_2}{\mathcal{A}_2} \hat{\mathbf{a}}^1 = \left(\csc\theta_{12} - \mathbf{e}_{22}^{\circ} \csc\theta_{12} \right) \hat{\mathbf{a}}_1 - \left(\cot\theta_{12} - \mathbf{e}_{22}^{\circ} \cot\theta_{12} \right) \hat{\mathbf{a}}_2 + \vec{\mathcal{O}}(\varepsilon^4)$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- Then, using

$$\hat{\mathbf{a}}_1 = \left(\hat{\mathbf{a}}_1 + \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \left(\mathbf{1} - \mathbf{e}_{11}^o + \mathcal{O}(\varepsilon^4) \right) \quad \text{and} \quad \hat{\mathbf{a}}_2 = \left(\hat{\mathbf{a}}_2 + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \left(\mathbf{1} - \mathbf{e}_{22}^o + \mathcal{O}(\varepsilon^4) \right)$$

gives

$$\begin{aligned} \frac{1}{\tau_{12}} = & \left(\cot \theta_{12} + \mathbf{F}_1 \right) \left(\hat{\mathbf{a}}_1 + \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \\ & + \left(\mathbf{F}_2 - \csc \theta_{12} \right) \left(\hat{\mathbf{a}}_2 + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\tau_{21}} = & \left(\csc \theta_{12} - \mathbf{F}_2 \right) \left(\hat{\mathbf{a}}_1 + \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \\ & - \left(\cot \theta_{12} + \mathbf{F}_3 \right) \left(\hat{\mathbf{a}}_2 + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

where
$$F_1 = \frac{1}{2} \csc^3 \theta_{12} \left[4e_{12}^o - (3e_{11}^o + 2e_{22}^o) \cos \theta_{12} + e_{11}^o \cos 3\theta_{12} \right]$$

$$F_2 = (e_{11}^o + e_{22}^o - 2e_{12}^o \cos \theta_{12}) \csc^3 \theta_{12}$$

$$F_3 = \frac{1}{2} \csc^3 \theta_{12} \left[4e_{12}^o - (2e_{11}^o + 3e_{22}^o) \cos \theta_{12} + e_{22}^o \cos 3\theta_{12} \right]$$

- Next, using $\hat{\mathbf{u}} = \hat{\mathbf{n}} + \vec{\varphi}$ and enforcing the fact that the components of the difference-vector field and the displacement derivatives have magnitudes of $\mathcal{O}(\varepsilon^2)$, gives

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

$$\begin{aligned} \frac{1}{r_{12}} = & \left(\cot\theta_{12} + F_1 \right) \left[\hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} + \hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} + \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \right] \\ & + \left(F_2 - \csc\theta_{12} \right) \left[\hat{a}_2 \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} + \hat{a}_2 \cdot \frac{1}{A_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} + \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\begin{aligned} \frac{1}{r_{21}} = & \left(\csc\theta_{12} - F_2 \right) \left[\hat{a}_1 \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} + \hat{a}_1 \cdot \frac{1}{A_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} \right] \\ & - \left(\cot\theta_{12} + F_3 \right) \left[\hat{a}_2 \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} + \hat{a}_2 \cdot \frac{1}{A_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} + \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

- Noting that products of F_1 , F_2 , and F_3 with the derivatives of $\vec{u}$ or $\vec{\varphi}$ are terms smaller than $\mathcal{O}(\varepsilon^2)$ gives

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

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$$\begin{aligned} \frac{1}{\mathbf{r}_{12}} = & \left(\cot\theta_{12} + F_1 \right) \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \right) + \cot\theta_{12} \left[\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} + \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \right] \\ & + \left(F_2 - \csc\theta_{12} \right) \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \right) - \csc\theta_{12} \left[\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\begin{aligned} \frac{1}{\mathbf{r}_{21}} = & \left(\csc\theta_{12} - F_2 \right) \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \right) + \csc\theta_{12} \left[\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} + \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \right] \\ & - \left(\cot\theta_{12} + F_3 \right) \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \right) - \cot\theta_{12} \left[\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

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- Now, using $\frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} = \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) \hat{a}_1 - \frac{\csc \theta_{12}}{r_{12}} \hat{a}_2$ and $\frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} = \frac{\csc \theta_{12}}{r_{21}} \hat{a}_1 + \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right) \hat{a}_2$ gives

$$\begin{aligned} \frac{1}{r_{12}} = & \frac{F_1 + F_2 \cos \theta_{12}}{r_{11}} + \frac{1 - F_2 \sin \theta_{12}}{r_{12}} + \cot \theta_{12} \left[\hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} + \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \right] \\ & - \csc \theta_{12} \left[\hat{a}_2 \cdot \frac{1}{A_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} + \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \right] + \mathcal{O}(\epsilon^4) \end{aligned}$$

where

$$F_1 + F_2 \cos \theta_{12} = 2e_{12}^{\circ} \csc \theta_{12} - 2e_{11}^{\circ} \cot \theta_{12}$$

$$1 - F_2 \sin \theta_{12} = 1 + 2e_{12}^{\circ} \cot \theta_{12} \csc \theta_{12} - (e_{11}^{\circ} + e_{22}^{\circ}) \csc^2 \theta_{12}$$

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- Likewise,

$$\frac{1}{r_{21}} = \frac{1 - F_2 \sin \theta_{12}}{r_{21}} - \frac{F_3 + F_2 \cos \theta_{12}}{r_{22}} + \csc \theta_{12} \left[\hat{\mathbf{a}}_1 \cdot \frac{1}{A_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{A_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \right] - \cot \theta_{12} \left[\hat{\mathbf{a}}_2 \cdot \frac{1}{A_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} + \frac{1}{A_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \cdot \frac{1}{A_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} \right] + \mathcal{O}(\varepsilon^4)$$

where

$$F_3 + F_2 \cos \theta_{12} = 2\mathbf{e}_{12}^\circ \csc \theta_{12} - 2\mathbf{e}_{22}^\circ \cot \theta_{12}$$

- Next, using

$$\frac{1}{A_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} = \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) \hat{\mathbf{a}}_1 - \frac{\csc \theta_{12}}{r_{12}} \hat{\mathbf{a}}_2$$

$$\frac{1}{A_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} = \frac{\csc \theta_{12}}{r_{21}} \hat{\mathbf{a}}_1 + \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right) \hat{\mathbf{a}}_2$$

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$$\begin{aligned} \frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} = & \left[\mathbf{e}_{11}^\circ \csc \theta_{12} - (\mathbf{e}_{12}^\circ + \varphi \sin \theta_{12}) \cot \theta_{12} \right] \csc \theta_{12} \hat{\mathbf{a}}_1 \\ & + \left[(\mathbf{e}_{12}^\circ + \varphi \sin \theta_{12}) \csc \theta_{12} - \mathbf{e}_{11}^\circ \cot \theta_{12} \right] \csc \theta_{12} \hat{\mathbf{a}}_2 - (\varphi_1 + \varphi_2 \cos \theta_{12}) \hat{\mathbf{n}} + \vec{\mathcal{O}}(\varepsilon^3) \end{aligned}$$

$$\begin{aligned} \frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} = & \left[(\mathbf{e}_{12}^\circ - \varphi \sin \theta_{12}) \csc \theta_{12} - \mathbf{e}_{22}^\circ \cot \theta_{12} \right] \csc \theta_{12} \hat{\mathbf{a}}_1 \\ & + \left[\mathbf{e}_{22}^\circ \csc \theta_{12} - (\mathbf{e}_{12}^\circ - \varphi \sin \theta_{12}) \cot \theta_{12} \right] \csc \theta_{12} \hat{\mathbf{a}}_2 - (\varphi_1 \cos \theta_{12} + \varphi_2) \hat{\mathbf{n}} + \vec{\mathcal{O}}(\varepsilon^3) \end{aligned}$$

gives

$$\frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \cdot \frac{1}{A_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} = \frac{\mathbf{e}_{11}^\circ}{r_{11}} + \frac{\mathbf{e}_{11}^\circ \cot \theta_{12} - \mathbf{e}_{12}^\circ \csc \theta_{12}}{r_{12}} - \frac{\varphi}{r_{12}} + \mathcal{O}(\varepsilon^3)$$

$$\frac{1}{A_1} \frac{\partial \vec{u}}{\partial \xi_1} \cdot \frac{1}{A_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2} = \frac{\mathbf{e}_{12}^\circ}{r_{22}} + \frac{\mathbf{e}_{11}^\circ \csc \theta_{12} - \mathbf{e}_{12}^\circ \cot \theta_{12}}{r_{21}} + \frac{\varphi}{r_{22}} \sin \theta_{12} - \frac{\varphi}{r_{21}} \cos \theta_{12} + \mathcal{O}(\varepsilon^3)$$

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$$\frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} = \frac{e_{12}^o}{r_{11}} + \frac{e_{12}^o \cot \theta_{12} - e_{22}^o \csc \theta_{12}}{r_{12}} - \frac{\varphi}{r_{11}} \sin \theta_{12} - \frac{\varphi}{r_{12}} \cos \theta_{12} + \mathcal{O}(\varepsilon^3)$$

$$\frac{1}{A_2} \frac{\partial \vec{u}}{\partial \xi_2} \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} = \frac{e_{22}^o}{r_{22}} + \frac{e_{12}^o \csc \theta_{12} - e_{22}^o \cot \theta_{12}}{r_{21}} - \frac{\varphi}{r_{21}} + \mathcal{O}(\varepsilon^3)$$

● Thus,

$$\begin{aligned} \frac{1}{r_{12}} - \frac{1}{r_{12}} &= \frac{e_{12}^o \csc \theta_{12} + \varphi - e_{11}^o \cot \theta_{12}}{r_{11}} - \frac{e_{11}^o}{r_{12}} \\ &+ \cot \theta_{12} \left[\hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right] - \csc \theta_{12} \left[\hat{a}_2 \cdot \frac{1}{A_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right] + \mathcal{O}(\varepsilon^3) \end{aligned} \quad \text{and}$$

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$$\frac{1}{\mathbf{r}_{21}} - \frac{1}{\mathbf{r}_{21}} = -\frac{\mathbf{e}_{22}^{\circ}}{\mathbf{r}_{21}} - \frac{\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \varphi - \mathbf{e}_{22}^{\circ} \cot \theta_{12}}{\mathbf{r}_{22}} + \csc \theta_{12} \left[\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right] - \cot \theta_{12} \left[\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right] + \mathcal{O}(\varepsilon^3)$$

- Recall that the **changes in reference-surface curvature**, $\mathbf{K}_{11}^{\circ}(\xi_1, \xi_2, t)$ and $\mathbf{K}_{22}^{\circ}(\xi_1, \xi_2, t)$, and the **change in reference-surface torsion**, $\mathbf{K}_{12}^{\circ}(\xi_1, \xi_2, t)$, caused by deformation are defined by

$$\mathbf{K}_{11}^{\circ} \equiv \frac{1}{\mathbf{r}_{11}} - \frac{1}{\mathbf{r}_{11}}$$

$$\mathbf{K}_{22}^{\circ} \equiv \frac{1}{\mathbf{r}_{22}} - \frac{1}{\mathbf{r}_{22}}$$

$$\mathbf{K}_{12}^{\circ} \equiv \frac{1}{2} \left[\left(\frac{1}{\mathbf{r}_{21}} - \frac{1}{\mathbf{r}_{21}} \right) - \left(\frac{1}{\mathbf{r}_{12}} - \frac{1}{\mathbf{r}_{12}} \right) \right]$$

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- These definitions give

$$\mathbf{K}_{11}^{\circ} = \boldsymbol{\chi}_{11}^{\circ} - \frac{\mathbf{e}_{11}^{\circ}}{\mathbf{r}_{11}} + \frac{\mathbf{e}_{11}^{\circ} \cot \theta_{12} - \mathbf{e}_{12}^{\circ} \csc \theta_{12}}{\mathbf{r}_{12}} + \mathcal{O}(\varepsilon^3)$$

$$\mathbf{K}_{22}^{\circ} = \boldsymbol{\chi}_{22}^{\circ} - \frac{\mathbf{e}_{22}^{\circ}}{\mathbf{r}_{22}} + \frac{\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \mathbf{e}_{22}^{\circ} \cot \theta_{12}}{\mathbf{r}_{21}} + \mathcal{O}(\varepsilon^3) \quad \text{and}$$

$$\begin{aligned} 2\mathbf{K}_{12}^{\circ} = & \left[2\boldsymbol{\chi}_{12}^{\circ} - \mathbf{e}_{12}^{\circ} \left(\frac{1}{\mathbf{r}_{11}} + \frac{1}{\mathbf{r}_{22}} \right) \right] \csc \theta_{12} \\ & - \cot \theta_{12} \left[\boldsymbol{\chi}_{11}^{\circ} - \frac{\mathbf{e}_{11}^{\circ}}{\mathbf{r}_{11}} + \boldsymbol{\chi}_{22}^{\circ} - \frac{\mathbf{e}_{22}^{\circ}}{\mathbf{r}_{22}} \right] + \frac{\mathbf{e}_{11}^{\circ}}{\mathbf{r}_{12}} - \frac{\mathbf{e}_{22}^{\circ}}{\mathbf{r}_{21}} + \mathcal{O}(\varepsilon^3) \end{aligned}$$

where the terms involving the linear rotation parameters are defined by

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

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$$\chi_{11}^{\circ} \equiv \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) - \frac{\varphi}{r_{12}}$$

$$\chi_{22}^{\circ} \equiv \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) - \frac{\varphi}{r_{21}}$$

$$\begin{aligned} 2\chi_{12}^{\circ} \equiv & \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) \\ & + \varphi \left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) \sin \theta_{12} - \varphi \left(\frac{1}{r_{21}} + \frac{1}{r_{12}} \right) \cos \theta_{12} \end{aligned}$$

- In these expressions, $\frac{1}{\mathbf{A}_{(\beta)}} \frac{\partial \vec{\varphi}}{\partial \xi_{\beta}} = \varphi^{(\alpha)} \Big|_{\langle \beta \rangle} \hat{\mathbf{a}}_{\alpha} + \varphi^{(3)} \Big|_{\langle \beta \rangle} \hat{\mathbf{n}}$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

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where, for **general Gaussian reference-surface coordinates**,

$$\varphi^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \varphi_1}{\partial \xi_1} - \frac{\varphi_2 \mathbf{csc} \theta_{12}}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\mathbf{csc} \theta_{12}}{\rho_{11}} \left(\varphi_1 \mathbf{cos} \theta_{12} + \varphi_2 \right)$$

$$\varphi^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} = \frac{1}{\mathbf{A}_1} \frac{\partial \varphi_2}{\partial \xi_1} + \frac{\mathbf{csc} \theta_{12}}{\rho_{11}} \left(\varphi_1 + \varphi_2 \mathbf{cos} \theta_{12} \right) + \frac{\varphi_2 \mathbf{cot} \theta_{12}}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1}$$

$$\varphi^{\langle 3 \rangle} \Big|_{\langle 1 \rangle} = -\frac{\varphi_1}{\mathbf{r}_{11}} + \varphi_2 \left(\frac{\mathbf{sin} \theta_{12}}{\mathbf{r}_{12}} - \frac{\mathbf{cos} \theta_{12}}{\mathbf{r}_{11}} \right)$$

$$\varphi^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \varphi}{\partial \xi_2} + \frac{\varphi_1 \mathbf{cot} \theta_{12}}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} - \frac{\mathbf{csc} \theta_{12}}{\rho_{22}} \left(\varphi_1 \mathbf{cos} \theta_{12} + \varphi_2 \right)$$

$$\varphi^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} = \frac{1}{\mathbf{A}_2} \frac{\partial \varphi_2}{\partial \xi_2} - \frac{\varphi_1 \mathbf{csc} \theta_{12}}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} + \frac{\mathbf{csc} \theta_{12}}{\rho_{22}} \left(\varphi_1 + \varphi_2 \mathbf{cos} \theta_{12} \right)$$

$$\varphi^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} = -\varphi_1 \left(\frac{\mathbf{sin} \theta_{12}}{\mathbf{r}_{21}} + \frac{\mathbf{cos} \theta_{12}}{\mathbf{r}_{22}} \right) - \frac{\varphi_2}{\mathbf{r}_{22}}$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- The expressions

$$\chi_{11}^{\circ} \equiv \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) - \frac{\varphi}{r_{12}} \quad \chi_{22}^{\circ} \equiv \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) - \frac{\varphi}{r_{21}}$$

$$2\chi_{12}^{\circ} \equiv \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) + \left(\frac{\varphi}{r_{22}} - \frac{\varphi}{r_{11}} \right) \sin \theta_{12} - \left(\frac{\varphi}{r_{21}} + \frac{\varphi}{r_{12}} \right) \cos \theta_{12}$$

become

$$\chi_{11}^{\circ} = \varphi^{(1)} \big|_{\langle 1 \rangle} + \varphi^{(2)} \big|_{\langle 1 \rangle} \cos \theta_{12} - \frac{\varphi}{r_{12}} \quad \chi_{22}^{\circ} = \varphi^{(1)} \big|_{\langle 2 \rangle} \cos \theta_{12} + \varphi^{(2)} \big|_{\langle 2 \rangle} - \frac{\varphi}{r_{21}}$$

$$2\chi_{12}^{\circ} = \varphi^{(1)} \big|_{\langle 2 \rangle} + \varphi^{(2)} \big|_{\langle 1 \rangle} + \left(\varphi^{(1)} \big|_{\langle 1 \rangle} + \varphi^{(2)} \big|_{\langle 2 \rangle} \right) \cos \theta_{12} + \left(\frac{\varphi}{r_{22}} - \frac{\varphi}{r_{11}} \right) \sin \theta_{12} - \left(\frac{\varphi}{r_{21}} + \frac{\varphi}{r_{12}} \right) \cos \theta_{12}$$

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- In contrast, using the previously derived general expressions for the components of the derivatives of a vector field given by

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{A_1} \frac{\partial \vec{V}}{\partial \xi_1} = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (V_1 + V_2 \cos \theta_{12}) - \frac{V_2 \sin \theta_{12}}{\rho_{11}} + \frac{V_3}{r_{11}}$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{A_1} \frac{\partial \vec{V}}{\partial \xi_1} = \left(\frac{1}{A_1} \frac{\partial V_1}{\partial \xi_1} + \frac{V_3}{r_{11}} \right) \cos \theta_{12} + \frac{1}{A_1} \frac{\partial V_2}{\partial \xi_1} + \left(\frac{V_1}{\rho_{11}} - \frac{V_3}{r_{12}} \right) \sin \theta_{12}$$

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{A_2} \frac{\partial \vec{V}}{\partial \xi_2} = \frac{1}{A_2} \frac{\partial V_1}{\partial \xi_2} + \left(\frac{1}{A_2} \frac{\partial V_2}{\partial \xi_2} + \frac{V_3}{r_{22}} \right) \cos \theta_{12} + \left(\frac{V_3}{r_{21}} - \frac{V_2}{\rho_{22}} \right) \sin \theta_{12}$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{A_2} \frac{\partial \vec{V}}{\partial \xi_2} = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (V_2 + V_1 \cos \theta_{12}) + \frac{V_1 \sin \theta_{12}}{\rho_{22}} + \frac{V_3}{r_{22}}$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

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yields the simpler forms

$$\chi_{11}^{\circ} = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (\varphi_1 + \varphi_2 \cos \theta_{12}) - \frac{\varphi_2 \sin \theta_{12}}{\rho_{11}} - \frac{\varphi}{r_{12}}$$

$$\chi_{22}^{\circ} = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\varphi_2 + \varphi_1 \cos \theta_{12}) + \frac{\varphi_1 \sin \theta_{12}}{\rho_{22}} - \frac{\varphi}{r_{21}}$$

$$\begin{aligned} 2\chi_{12}^{\circ} = & \frac{1}{A_2} \frac{\partial \varphi_1}{\partial \xi_2} + \frac{\cos \theta_{12}}{A_2} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\cos \theta_{12}}{A_1} \frac{\partial \varphi_1}{\partial \xi_1} + \frac{1}{A_1} \frac{\partial \varphi_2}{\partial \xi_1} \\ & + \left(\frac{\varphi_1}{\rho_{11}} - \frac{\varphi_2}{\rho_{22}} \right) \sin \theta_{12} + \left(\frac{\varphi}{r_{22}} - \frac{\varphi}{r_{11}} \right) \sin \theta_{12} - \left(\frac{\varphi}{r_{21}} + \frac{\varphi}{r_{12}} \right) \cos \theta_{12} \end{aligned}$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

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- These expressions for $\chi_{\alpha\beta}^{\circ}(\xi_1, \xi_2, t)$ are in complete agreement with the corresponding tensor equations given in:
 - Sanders, J. L.: Nonlinear Theories for Thin Shells. Quarterly Journal of Applied Mathematics, vol. 21, no. 1, 1963, pp. 21-36.
- For orthogonal Gaussian reference-surface coordinates, these expressions reduce to

$$\chi_{11}^{\circ} = \frac{1}{A_1} \frac{\partial \varphi_1}{\partial \xi_1} - \frac{\varphi_2}{\rho_{11}} - \frac{\varphi}{r_{12}}$$

$$\chi_{22}^{\circ} = \frac{1}{A_2} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\varphi_1}{\rho_{22}} + \frac{\varphi}{r_{12}}$$

$$2\chi_{12}^{\circ} = \frac{1}{A_1} \frac{\partial \varphi_2}{\partial \xi_1} + \frac{\varphi_1}{\rho_{11}} + \frac{1}{A_2} \frac{\partial \varphi_1}{\partial \xi_2} - \frac{\varphi_2}{\rho_{22}} - \varphi \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- The expressions

$$\chi_{11}^{\circ} \equiv \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) - \frac{\varphi}{r_{12}} \quad \chi_{22}^{\circ} \equiv \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) - \frac{\varphi}{r_{21}}$$

$$2\chi_{12}^{\circ} \equiv \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) + \left(\frac{\varphi}{r_{22}} - \frac{\varphi}{r_{11}} \right) \sin \theta_{12} - \left(\frac{\varphi}{r_{21}} + \frac{\varphi}{r_{12}} \right) \cos \theta_{12}$$

are expressed in terms of the rotation-vector field

$$\vec{\omega} = -\varphi_2 \hat{\mathbf{a}}^1 + \varphi_1 \hat{\mathbf{a}}^2 + \varphi \hat{\mathbf{n}} \quad \text{by using} \quad \vec{\varphi} = \vec{\omega} \times \hat{\mathbf{n}}$$

- In particular, differentiating the cross product yields

$$\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} = \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\omega}}{\partial \xi_1} \times \hat{\mathbf{n}} + \vec{\omega} \times \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1} \quad \text{and} \quad \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} = \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\omega}}{\partial \xi_2} \times \hat{\mathbf{n}} + \vec{\omega} \times \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2}$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- Next, from $\vec{\omega} = -\varphi_2 \hat{a}^1 + \varphi_1 \hat{a}^2 + \varphi \hat{n}$ it follows that

$$\vec{\omega} \times \hat{a}_1 = -(\varphi_1 + \varphi_2 \cos \theta_{12}) \hat{n} + \varphi \hat{a}^2 \quad \text{and} \quad \vec{\omega} \times \hat{a}_2 = -(\varphi_2 + \varphi_1 \cos \theta_{12}) \hat{n} - \varphi \hat{a}^1$$

- Thus, from

$$\frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} = \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) \hat{a}_1 - \frac{\csc \theta_{12}}{r_{12}} \hat{a}_2 \quad \text{and} \quad \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} = \frac{\csc \theta_{12}}{r_{21}} \hat{a}_1 + \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right) \hat{a}_2$$

it follows that

$$\begin{aligned} \vec{\omega} \times \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} &= \frac{\varphi \csc \theta_{12}}{r_{12}} \hat{a}^1 + \left(\frac{\varphi}{r_{11}} + \frac{\varphi \cot \theta_{12}}{r_{12}} \right) \hat{a}^2 \\ &\quad + \left[\frac{\varphi_2}{r_{12}} \sin \theta_{12} - \frac{1}{r_{11}} (\varphi_1 + \varphi_2 \cos \theta_{12}) \right] \hat{n} \end{aligned} \quad \text{and}$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

$$\vec{\omega} \times \frac{1}{\mathbf{A}_2} \frac{\partial \hat{n}}{\partial \xi_2} = - \left(\frac{\varphi}{r_{22}} - \frac{\varphi \cot \theta_{12}}{r_{21}} \right) \hat{a}^1 + \frac{\varphi \csc \theta_{12}}{r_{21}} \hat{a}^2 - \left[\frac{\varphi_1}{r_{21}} \sin \theta_{12} + \frac{1}{r_{22}} (\varphi_2 + \varphi_1 \cos \theta_{12}) \right] \hat{n}$$

- Next, the derivatives of the rotation-vector field are expressed as

$$\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\omega}}{\partial \xi_1} = \left(\hat{a}^1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\omega}}{\partial \xi_1} \right) \csc \theta_{12} \hat{a}_1 + \left(\hat{a}^2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\omega}}{\partial \xi_1} \right) \csc \theta_{12} \hat{a}_2 + \left(\hat{n} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\omega}}{\partial \xi_1} \right) \hat{n}$$

$$\frac{1}{\mathbf{A}_2} \frac{\partial \vec{\omega}}{\partial \xi_2} = \left(\hat{a}^1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\omega}}{\partial \xi_2} \right) \csc \theta_{12} \hat{a}_1 + \left(\hat{a}^2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\omega}}{\partial \xi_2} \right) \csc \theta_{12} \hat{a}_2 + \left(\hat{n} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\omega}}{\partial \xi_2} \right) \hat{n}$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- These expressions yield

$$\frac{1}{A_1} \frac{\partial \vec{\omega}}{\partial \xi_1} \times \hat{n} = \left(\hat{a}^2 \cdot \frac{1}{A_1} \frac{\partial \vec{\omega}}{\partial \xi_1} \right) \csc \theta_{12} \hat{a}^1 - \left(\hat{a}^1 \cdot \frac{1}{A_1} \frac{\partial \vec{\omega}}{\partial \xi_1} \right) \csc \theta_{12} \hat{a}^2 \quad \text{and}$$

$$\frac{1}{A_2} \frac{\partial \vec{\omega}}{\partial \xi_2} \times \hat{n} = \left(\hat{a}^2 \cdot \frac{1}{A_2} \frac{\partial \vec{\omega}}{\partial \xi_2} \right) \csc \theta_{12} \hat{a}^1 - \left(\hat{a}^1 \cdot \frac{1}{A_2} \frac{\partial \vec{\omega}}{\partial \xi_2} \right) \csc \theta_{12} \hat{a}^2$$

- The derivatives of the linear difference-vector field given by

$$\frac{1}{A_1} \frac{\partial \vec{\phi}}{\partial \xi_1} = \frac{1}{A_1} \frac{\partial \vec{\omega}}{\partial \xi_1} \times \hat{n} + \vec{\omega} \times \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \quad \text{and} \quad \frac{1}{A_2} \frac{\partial \vec{\phi}}{\partial \xi_2} = \frac{1}{A_2} \frac{\partial \vec{\omega}}{\partial \xi_2} \times \hat{n} + \vec{\omega} \times \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2}$$

become

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

$$\begin{aligned} \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} = & \left[\left(\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\omega}}{\partial \xi_1} \right) + \frac{\varphi}{r_{12}} \right] \mathbf{csc} \theta_{12} \hat{\mathbf{a}}^1 \\ & + \left[\left(\frac{\varphi}{r_{11}} + \frac{\varphi \cot \theta_{12}}{r_{12}} - \left(\hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\omega}}{\partial \xi_1} \right) \mathbf{csc} \theta_{12} \right) \right] \hat{\mathbf{a}}^2 \\ & + \left[\frac{\varphi_2}{r_{12}} \sin \theta_{12} - \frac{1}{r_{11}} (\varphi_1 + \varphi_2 \cos \theta_{12}) \right] \hat{\mathbf{n}} \end{aligned} \quad \text{and}$$

$$\begin{aligned} \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} = & \left[\left(\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\omega}}{\partial \xi_2} \right) \mathbf{csc} \theta_{12} - \left(\frac{\varphi}{r_{22}} - \frac{\varphi \cot \theta_{12}}{r_{21}} \right) \right] \hat{\mathbf{a}}^1 \\ & + \left[\frac{\varphi}{r_{21}} - \left(\hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\omega}}{\partial \xi_2} \right) \right] \mathbf{csc} \theta_{12} \hat{\mathbf{a}}^2 \\ & - \left[\frac{\varphi_1}{r_{21}} \sin \theta_{12} + \frac{1}{r_{22}} (\varphi_2 + \varphi_1 \cos \theta_{12}) \right] \hat{\mathbf{n}} \end{aligned}$$

LINEARIZED CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONCLUDED

- Using these expressions for the derivatives of the linear difference-vector field, the following expressions

$$\chi_{11}^{\circ} \equiv \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) - \frac{\varphi}{r_{12}} \quad \chi_{22}^{\circ} \equiv \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) - \frac{\varphi}{r_{21}}$$

$$2\chi_{12}^{\circ} \equiv \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) + \left(\frac{\varphi}{r_{22}} - \frac{\varphi}{r_{11}} \right) \sin \theta_{12} - \left(\frac{\varphi}{r_{21}} + \frac{\varphi}{r_{12}} \right) \cos \theta_{12}$$

become

$$\chi_{11}^{\circ} = \left(\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\omega}}{\partial \xi_1} \right) \quad \chi_{22}^{\circ} = - \left(\hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\omega}}{\partial \xi_2} \right)$$

$$2\chi_{12}^{\circ} = \left(\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\omega}}{\partial \xi_2} \right) - \left(\hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\omega}}{\partial \xi_1} \right)$$

“SMALL GREEN-LAGRANGE STRAINS IN TERMS OF LINEAR DEFORMATION MEASURES

- In *elasticity theory*, the Green-Lagrange strains are often expressed exactly by equations that are quadratic in the strains and rotations of linear theory, for deformations associated with “small” strains
- Nonlinear “small” strains in terms of **linear deformation measures** are obtained by using the following exact expressions

$$\hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} = \mathbf{e}_{11}^o \sin \theta_{12} - \beta_1 \cos \theta_{12}$$

$$\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} = \beta_1$$

$$\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} = - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)$$

$$\hat{\mathbf{a}}^1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} = \beta_2$$

$$\hat{\mathbf{a}}^2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} = \mathbf{e}_{22}^o \sin \theta_{12} - \beta_2 \cos \theta_{12}$$

$$\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} = - \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right)$$

“SMALL GREEN-LAGRANGE STRAINS IN TERMS OF LINEAR DEFORMATION MEASURES - CONTINUED

where (see also pp. 75 and 285-288)

$$\mathbf{e}_{11}^{\circ} \equiv \hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1}$$

$$\mathbf{e}_{22}^{\circ} \equiv \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2}$$

$$2\mathbf{e}_{12}^{\circ} \equiv \hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} + \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1}$$

$$\varphi_1 \equiv \mathbf{csc}\theta_{12} \left(\vec{\nabla} \times \vec{\mathbf{u}} \right) \cdot \hat{\mathbf{a}}_2$$

$$\varphi_2 \equiv -\mathbf{csc}\theta_{12} \left(\vec{\nabla} \times \vec{\mathbf{u}} \right) \cdot \hat{\mathbf{a}}_1$$

$$\varphi \equiv \frac{1}{2} \left(\vec{\nabla} \times \vec{\mathbf{u}} \right) \cdot \hat{\mathbf{n}}$$

$$\beta_1 \equiv \varphi + \mathbf{e}_{12}^{\circ} \mathbf{csc}\theta_{12} - \mathbf{e}_{11}^{\circ} \mathbf{cot}\theta_{12}$$

$$\beta_2 \equiv \mathbf{e}_{12}^{\circ} \mathbf{csc}\theta_{12} - \varphi - \mathbf{e}_{22}^{\circ} \mathbf{cot}\theta_{12}$$

- Likewise, the exact expressions for the **derivatives of the displacement vector** in terms of these quantities are

$$\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} = \left(\mathbf{e}_{11}^{\circ} - \beta_1 \mathbf{cot}\theta_{12} \right) \hat{\mathbf{a}}_1 + \beta_1 \mathbf{csc}\theta_{12} \hat{\mathbf{a}}_2 - \left(\varphi_1 + \varphi_2 \mathbf{cos}\theta_{12} \right) \hat{\mathbf{n}}$$

$$\frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} = \beta_2 \mathbf{csc}\theta_{12} \hat{\mathbf{a}}_1 + \left(\mathbf{e}_{22}^{\circ} - \beta_2 \mathbf{cot}\theta_{12} \right) \hat{\mathbf{a}}_2 - \left(\varphi_1 \mathbf{cos}\theta_{12} + \varphi_2 \right) \hat{\mathbf{n}}$$

“SMALL GREEN-LAGRANGE STRAINS IN TERMS OF LINEAR DEFORMATION MEASURES - CONTINUED

- By substituting these expressions into

$$\epsilon_{11}^o = \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) + \frac{1}{2} \left(\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right)$$

$$\epsilon_{22}^o = \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) + \frac{1}{2} \left(\frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right)$$

$$2\epsilon_{12}^o = \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) + \left(\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right)$$

the “*small*” nonlinear **Green-Lagrange strains** are expressed **exactly**, in terms of the linear strain and rotation measures as follows

$$\epsilon_{11}^o = \mathbf{e}_{11}^o + \frac{1}{2} \left[\left(\mathbf{e}_{11}^o \right)^2 + \beta_1^2 + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)^2 \right] + \mathcal{O}(\epsilon^4)$$

$$\epsilon_{22}^o = \mathbf{e}_{22}^o + \frac{1}{2} \left[\left(\mathbf{e}_{22}^o \right)^2 + \beta_2^2 + \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right)^2 \right] + \mathcal{O}(\epsilon^4)$$

“SMALL GREEN-LAGRANGE STRAINS IN TERMS OF LINEAR DEFORMATION MEASURES - CONTINUED

$$2\varepsilon_{12}^{\circ} = (\mathbf{e}_{11}^{\circ} + \mathbf{e}_{22}^{\circ})\cos\theta_{12} + (\beta_1 + \beta_2)\sin\theta_{12} + (\mathbf{e}_{11}^{\circ}\beta_2 + \mathbf{e}_{22}^{\circ}\beta_1)\sin\theta_{12} \\ + (\mathbf{e}_{11}^{\circ}\mathbf{e}_{22}^{\circ} - \beta_1\beta_2)\cos\theta_{12} + (\varphi_1 + \varphi_2\cos\theta_{12})(\varphi_1\cos\theta_{12} + \varphi_2) + \mathcal{O}(\varepsilon^4)$$

- Using the expressions for β_1 and β_2 gives

$$\varepsilon_{11}^{\circ} = \mathbf{e}_{11}^{\circ} + \frac{1}{2}\left[(\mathbf{e}_{11}^{\circ}\csc\theta_{12})^2 + (\mathbf{e}_{12}^{\circ}\csc\theta_{12} + \varphi)^2 \right. \\ \left. - 2\mathbf{e}_{11}^{\circ}(\mathbf{e}_{12}^{\circ}\csc\theta_{12} + \varphi)\cot\theta_{12} + (\varphi_1 + \varphi_2\cos\theta_{12})^2 \right] + \mathcal{O}(\varepsilon^4)$$

$$\varepsilon_{22}^{\circ} = \mathbf{e}_{22}^{\circ} + \frac{1}{2}\left[(\mathbf{e}_{22}^{\circ}\csc\theta_{12})^2 + (\mathbf{e}_{12}^{\circ}\csc\theta_{12} - \varphi)^2 \right. \\ \left. - 2\mathbf{e}_{22}^{\circ}(\mathbf{e}_{12}^{\circ}\csc\theta_{12} - \varphi)\cot\theta_{12} + (\varphi_1\cos\theta_{12} + \varphi_2)^2 \right] + \mathcal{O}(\varepsilon^4)$$

“SMALL GREEN-LAGRANGE STRAINS IN TERMS OF LINEAR DEFORMATION MEASURES - CONTINUED

$$\begin{aligned}
 2\varepsilon_{12}^{\circ} = & 2\mathbf{e}_{12}^{\circ} + \left[\mathbf{e}_{11}^{\circ}(\mathbf{e}_{12}^{\circ} - \varphi \sin \theta_{12}) + \mathbf{e}_{22}^{\circ}(\mathbf{e}_{12}^{\circ} + \varphi \sin \theta_{12}) \right] \csc^2 \theta_{12} \\
 & - \left[\mathbf{e}_{11}^{\circ} \mathbf{e}_{22}^{\circ} + (\mathbf{e}_{12}^{\circ} + \varphi \sin \theta_{12})(\mathbf{e}_{12}^{\circ} - \varphi \sin \theta_{12}) \right] \csc \theta_{12} \cot \theta_{12} \\
 & + (\varphi_1 + \varphi_2 \cos \theta_{12})(\varphi_1 \cos \theta_{12} + \varphi_2) + \mathcal{O}(\varepsilon^4)
 \end{aligned}$$

- These strain expressions are in complete agreement with the corresponding tensor equations given in:
 - Koiter, W. T.: On the Nonlinear Theory of Thin Elastic Shells. Proc. Koninklijke Nederlandse Akademie van Wetenschappen, B69, 1966, pp. 1-54.
 - Budiansky, B.: Notes on Nonlinear Shell Theory. Journal of Applied Mechanics, vol. 35, no. 2, 1968, pp. 393-401.

“SMALL GREEN-LAGRANGE STRAINS IN TERMS OF LINEAR DEFORMATION MEASURES - CONCLUDED

- For orthogonal reference-surface Gaussian coordinates, these expressions reduce to

$$\epsilon_{11}^o = e_{11}^o + \frac{1}{2} \left[(e_{11}^o)^2 + (e_{12}^o + \varphi)^2 + \varphi_1^2 \right] + \mathcal{O}(\epsilon^4)$$

$$\epsilon_{22}^o = e_{22}^o + \frac{1}{2} \left[(e_{12}^o - \varphi)^2 + (e_{22}^o)^2 + \varphi_2^2 \right] + \mathcal{O}(\epsilon^4)$$

$$2\epsilon_{12}^o = 2e_{12}^o + e_{11}^o(e_{12}^o - \varphi) + e_{22}^o(e_{12}^o + \varphi) + \varphi_1\varphi_2 + \mathcal{O}(\epsilon^4)$$

**GEOMETRY OF S_e FOR “SMALL” STRAINS
IN TERMS OF LINEAR DEFORMATION MEASURES**

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$

- To obtain the desired expressions for the metric coefficients and base vector fields of the deformed reference surface, it is convenient to write the exact expressions for the **derivatives of the displacement vector field** in terms of the linear deformation measures as

$$\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} = \Delta_{11} \hat{\mathbf{a}}_1 + \Delta_{12} \hat{\mathbf{a}}_2 + \Delta_{13} \hat{\mathbf{n}} \quad \text{and} \quad \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} = \Delta_{21} \hat{\mathbf{a}}_1 + \Delta_{22} \hat{\mathbf{a}}_2 + \Delta_{23} \hat{\mathbf{n}} \quad \text{where}$$

$$\Delta_{11} = \mathbf{e}_{11}^{\circ} \csc^2 \theta_{12} - \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} + \varphi \right) \cot \theta_{12}$$

$$\Delta_{12} = \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} + \varphi - \mathbf{e}_{11}^{\circ} \cot \theta_{12} \right) \csc \theta_{12}$$

$$\Delta_{13} = - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \quad \Delta_{23} = - \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right)$$

$$\Delta_{21} = \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \varphi - \mathbf{e}_{22}^{\circ} \cot \theta_{12} \right) \csc \theta_{12}$$

$$\Delta_{22} = \mathbf{e}_{22}^{\circ} \csc^2 \theta_{12} - \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \varphi \right) \cot \theta_{12}$$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- By substituting these expressions into

$$\epsilon_{11}^{\circ} = \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) + \frac{1}{2} \left(\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right)$$

$$\epsilon_{22}^{\circ} = \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) + \frac{1}{2} \left(\frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right)$$

$$2\epsilon_{12}^{\circ} = \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) + \left(\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right)$$

the “*small*” nonlinear **Green-Lagrange strains** are expressed exactly, in terms of linear deformation measures as follows

$$\epsilon_{11}^{\circ} = \Delta_{11} + \Delta_{12} \cos \theta_{12} + \frac{1}{2} \left(\Delta_{11}^2 + 2\Delta_{11}\Delta_{12} \cos \theta_{12} + \Delta_{12}^2 + \Delta_{13}^2 \right) + \mathcal{O}(\epsilon^4)$$

$$\epsilon_{22}^{\circ} = \Delta_{21} \cos \theta_{12} + \Delta_{22} + \frac{1}{2} \left(\Delta_{21}^2 + 2\Delta_{21}\Delta_{22} \cos \theta_{12} + \Delta_{22}^2 + \Delta_{23}^2 \right) + \mathcal{O}(\epsilon^4)$$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

$$2\epsilon_{12}^{\circ} = \Delta_{12} + \Delta_{21} + (\Delta_{11} + \Delta_{22})\cos\theta_{12} \\ + \Delta_{11}\Delta_{21} + \Delta_{12}\Delta_{22} + \Delta_{13}\Delta_{23} + (\Delta_{11}\Delta_{22} + \Delta_{12}\Delta_{21})\cos\theta_{12} + \mathcal{O}(\epsilon^4)$$

- These strain expressions are equivalent to the previous ones given:

$$\epsilon_{11}^{\circ} = e_{11}^{\circ} + \frac{1}{2} \left[\left(e_{11}^{\circ} \csc\theta_{12} \right)^2 + \left(e_{12}^{\circ} \csc\theta_{12} + \varphi \right)^2 \right. \\ \left. - 2e_{11}^{\circ} \left(e_{12}^{\circ} \csc\theta_{12} + \varphi \right) \cot\theta_{12} + \left(\varphi_1 + \varphi_2 \cos\theta_{12} \right)^2 \right] + \mathcal{O}(\epsilon^4)$$

$$\epsilon_{22}^{\circ} = e_{22}^{\circ} + \frac{1}{2} \left[\left(e_{22}^{\circ} \csc\theta_{12} \right)^2 + \left(e_{12}^{\circ} \csc\theta_{12} - \varphi \right)^2 \right. \\ \left. - 2e_{22}^{\circ} \left(e_{12}^{\circ} \csc\theta_{12} - \varphi \right) \cot\theta_{12} + \left(\varphi_1 \cos\theta_{12} + \varphi_2 \right)^2 \right] + \mathcal{O}(\epsilon^4)$$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

$$\begin{aligned}
 2\varepsilon_{12}^{\circ} = & 2\mathbf{e}_{12}^{\circ} + \left[\mathbf{e}_{11}^{\circ}(\mathbf{e}_{12}^{\circ} - \varphi \sin \theta_{12}) + \mathbf{e}_{22}^{\circ}(\mathbf{e}_{12}^{\circ} + \varphi \sin \theta_{12}) \right] \csc^2 \theta_{12} \\
 & - \left[\mathbf{e}_{11}^{\circ} \mathbf{e}_{22}^{\circ} + (\mathbf{e}_{12}^{\circ} + \varphi \sin \theta_{12})(\mathbf{e}_{12}^{\circ} - \varphi \sin \theta_{12}) \right] \csc \theta_{12} \cot \theta_{12} \\
 & + (\varphi_1 + \varphi_2 \cos \theta_{12})(\varphi_1 \cos \theta_{12} + \varphi_2) + \mathcal{O}(\varepsilon^4)
 \end{aligned}$$

- Using these expressions for the “small” **Green-Lagange strains**, the metric coefficients of the deformed reference surface, given by,

$$\mathcal{A}_1 = A_1 \left[1 + \varepsilon_{11}^{\circ} + \mathcal{O}(\varepsilon^4) \right] \quad \text{and} \quad \mathcal{A}_2 = A_2 \left[1 + \varepsilon_{22}^{\circ} + \mathcal{O}(\varepsilon^4) \right] \quad \text{become}$$

$$\mathcal{A}_1 = A_1 \left\{ 1 + \Delta_{11} + \Delta_{12} \cos \theta_{12} + \frac{1}{2} \left(\Delta_{11}^2 + 2\Delta_{11}\Delta_{12} \cos \theta_{12} + \Delta_{12}^2 + \Delta_{13}^2 \right) \right\} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{A}_2 = A_2 \left\{ 1 + \Delta_{21} \cos \theta_{12} + \Delta_{22} + \frac{1}{2} \left(\Delta_{21}^2 + 2\Delta_{21}\Delta_{22} \cos \theta_{12} + \Delta_{22}^2 + \Delta_{23}^2 \right) \right\} + \mathcal{O}(\varepsilon^4)$$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- Likewise, $\cos\theta_{12} = 2\varepsilon_{12}^{\circ} + \cos\theta_{12}\left(1 - \varepsilon_{11}^{\circ} - \varepsilon_{22}^{\circ}\right) + \mathcal{O}(\varepsilon^4)$ becomes

$$\begin{aligned}\cos\theta_{12} = & \Delta_{12} + \Delta_{21} + (\Delta_{11} + \Delta_{22})\cos\theta_{12} \\ & + \Delta_{11}\Delta_{21} + \Delta_{12}\Delta_{22} + \Delta_{13}\Delta_{23} + (\Delta_{11}\Delta_{22} + \Delta_{12}\Delta_{21})\cos\theta_{12} + \mathcal{O}(\varepsilon^4)\end{aligned}$$

- Then, by using the exact representations

$$\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} = \Delta_{11} \hat{\mathbf{a}}_1 + \Delta_{12} \hat{\mathbf{a}}_2 + \Delta_{13} \hat{\mathbf{n}} \quad \text{and} \quad \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} = \Delta_{21} \hat{\mathbf{a}}_1 + \Delta_{22} \hat{\mathbf{a}}_2 + \Delta_{23} \hat{\mathbf{n}}$$

it follows that

$$\vec{\mathbf{a}}_1(\xi_1, \xi_2, t) = \mathbf{A}_1 \left(\hat{\mathbf{a}}_1 + \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} \right) \quad \text{becomes}$$

$$\vec{\mathbf{a}}_1(\xi_1, \xi_2, t) = \mathbf{A}_1 \left[(1 + \Delta_{11}) \hat{\mathbf{a}}_1 + \Delta_{12} \hat{\mathbf{a}}_2 + \Delta_{13} \hat{\mathbf{n}} \right] \quad \text{and}$$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

$$\vec{\mathbf{a}}_2(\xi_1, \xi_2, t) = \mathbf{A}_2 \left(\hat{\mathbf{a}}_2 + \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} \right) \quad \text{becomes}$$

$$\vec{\mathbf{a}}_2(\xi_1, \xi_2, t) = \mathbf{A}_2 \left[\Delta_{21} \hat{\mathbf{a}}_1 + (1 + \Delta_{22}) \hat{\mathbf{a}}_2 + \Delta_{23} \hat{\mathbf{n}} \right]$$

- Next, using $\hat{\mathbf{a}}_\alpha = \frac{\vec{\mathbf{a}}_\alpha}{\mathcal{A}_{(\alpha)}}$ and $\frac{1}{\mathcal{A}_\alpha} = \frac{1}{\mathbf{A}_\alpha} \left[1 - \epsilon_{(\alpha\alpha)}^\circ + \mathcal{O}(\epsilon^4) \right]$ yields

$$\hat{\mathbf{a}}_1 = \left[(1 + \Delta_{11}) \hat{\mathbf{a}}_1 + \Delta_{12} \hat{\mathbf{a}}_2 + \Delta_{13} \hat{\mathbf{n}} \right] \left(1 - \epsilon_{11}^\circ + \mathcal{O}(\epsilon^4) \right) \quad \text{and}$$

$$\hat{\mathbf{a}}_2 = \left[\Delta_{21} \hat{\mathbf{a}}_1 + (1 + \Delta_{22}) \hat{\mathbf{a}}_2 + \Delta_{23} \hat{\mathbf{n}} \right] \left(1 - \epsilon_{22}^\circ + \mathcal{O}(\epsilon^4) \right)$$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- Recall, that the unit-magnitude vector field perpendicular to the tangent plane at $\mathcal{P} = \mathcal{D}(\mathbf{P}) \in \mathcal{S}_t$ of the deformed reference surface is given by

$$\hat{\mathbf{a}}(\xi_1, \xi_2, t) = \frac{\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2}{|\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2|} = \frac{\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2}{\sqrt{a}}$$

- Using $\vec{\mathbf{a}}_1(\xi_1, \xi_2, t) = \mathbf{A}_1 \left[(1 + \Delta_{11}) \hat{\mathbf{a}}_1 + \Delta_{12} \hat{\mathbf{a}}_2 + \Delta_{13} \hat{\mathbf{n}} \right]$ and

$$\vec{\mathbf{a}}_2(\xi_1, \xi_2, t) = \mathbf{A}_2 \left[\Delta_{21} \hat{\mathbf{a}}_1 + (1 + \Delta_{22}) \hat{\mathbf{a}}_2 + \Delta_{23} \hat{\mathbf{n}} \right] \quad \text{gives}$$

$$\begin{aligned} \frac{\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2}{\mathbf{A}_1 \mathbf{A}_2} = & \left[\Delta_{12} \Delta_{23} - \Delta_{13} (1 + \Delta_{22}) \right] \hat{\mathbf{a}}_1 + \left[\Delta_{13} \Delta_{21} - (1 + \Delta_{11}) \Delta_{23} \right] \hat{\mathbf{a}}_2 \\ & + \left[(1 + \Delta_{11})(1 + \Delta_{22}) - \Delta_{12} \Delta_{21} \right] \sin \theta_{12} \hat{\mathbf{n}} \end{aligned}$$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- Because $\sqrt{a} = |\vec{a}_1 \times \vec{a}_2|$, $\sqrt{a} = |\vec{a}_1| |\vec{a}_2| \sin \theta_{12} = \mathcal{A}_1 \mathcal{A}_2 \sin \theta_{12}$
- Using $\sin \theta_{12} = \sin \theta_{12} - \cot \theta_{12} \left[2\varepsilon_{12}^o - \cos \theta_{12} (\varepsilon_{11}^o + \varepsilon_{22}^o) \right] + \mathcal{O}(\varepsilon^4)$,
 $\mathcal{A}_1 = A_1 \left[1 + \varepsilon_{11}^o + \mathcal{O}(\varepsilon^4) \right]$, and $\mathcal{A}_2 = A_2 \left[1 + \varepsilon_{22}^o + \mathcal{O}(\varepsilon^4) \right]$ gives

$$\sqrt{a} = A_1 A_2 \sin \theta_{12} \left[1 - 2\varepsilon_{12}^o \cot \theta_{12} \csc \theta_{12} + (\varepsilon_{11}^o + \varepsilon_{22}^o) \csc^2 \theta_{12} \right] + \mathcal{O}(\varepsilon^4)$$

for “small strains”

- In addition,

$$\frac{\sqrt{a}}{\sqrt{a}} = 1 + 2\varepsilon_{12}^o \cot \theta_{12} \csc \theta_{12} - (\varepsilon_{11}^o + \varepsilon_{22}^o) \csc^2 \theta_{12} + \mathcal{O}(\varepsilon^4)$$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- Thus, $\hat{\mathbf{a}}(\xi_1, \xi_2, t)$ is now expressible as $\hat{\mathbf{a}} = \frac{\sqrt{a}}{\sqrt{a}} \vec{m}$, where

$$\vec{m} = m_1 \hat{\mathbf{a}}_1 + m_2 \hat{\mathbf{a}}_2 + m_3 \hat{\mathbf{n}} \quad \text{and}$$

$$m_1 = \left[\Delta_{12} \Delta_{23} - \Delta_{13} (1 + \Delta_{22}) \right] \csc^2 \theta_{12} - \left[\Delta_{13} \Delta_{21} - (1 + \Delta_{11}) \Delta_{23} \right] \csc \theta_{12} \cot \theta_{12}$$

$$m_2 = \left[\Delta_{13} \Delta_{21} - (1 + \Delta_{11}) \Delta_{23} \right] \csc^2 \theta_{12} - \left[\Delta_{12} \Delta_{23} - \Delta_{13} (1 + \Delta_{22}) \right] \csc \theta_{12} \cot \theta_{12}$$

$$m_3 = (1 + \Delta_{11})(1 + \Delta_{22}) - \Delta_{12} \Delta_{21}$$

and where $\hat{\mathbf{a}}^1 = \hat{\mathbf{a}}_1 \csc \theta_{12} - \hat{\mathbf{a}}_2 \cot \theta_{12}$ and $\hat{\mathbf{a}}^2 = \hat{\mathbf{a}}_2 \csc \theta_{12} - \hat{\mathbf{a}}_1 \cot \theta_{12}$ have been used

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- Substituting the expressions for Δ_{ij} in terms of the linear deformation measures into $\vec{m}$ and simplifying gives

$$m_1 = \varphi_1 - \left(e_{12}^o \csc \theta_{12} + \varphi \right) \left(\varphi_1 \cot \theta_{12} + \varphi_2 \csc \theta_{12} \right) + e_{22}^o \csc \theta_{12} \left(\varphi_1 \csc \theta_{12} + \varphi_2 \cot \theta_{12} \right)$$

$$m_2 = \varphi_2 - \left(e_{12}^o \csc \theta_{12} - \varphi \right) \left(\varphi_1 \csc \theta_{12} + \varphi_2 \cot \theta_{12} \right) + e_{11}^o \csc \theta_{12} \left(\varphi_1 \cot \theta_{12} + \varphi_2 \csc \theta_{12} \right)$$

$$m_3 = 1 + \varphi^2 + \left(e_{11}^o + e_{22}^o + e_{11}^o e_{22}^o - \left(e_{12}^o \right)^2 \right) \csc^2 \theta_{12} - 2e_{12}^o \cot \theta_{12} \csc \theta_{12}$$

- Note that $\vec{a}_1 \cdot \hat{n} = 0$ and $\vec{a}_2 \cdot \hat{n} = 0$ imply $\vec{a}_1 \cdot \vec{m} = 0$ and $\vec{a}_2 \cdot \vec{m} = 0$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- The **reciprocal base-vector fields** for the deformed surface are obtained

by using $\hat{\mathbf{a}}^1 \equiv \hat{\mathbf{a}}_2 \times \hat{\mathbf{n}}$ and $\hat{\mathbf{a}}^2 \equiv \hat{\mathbf{n}} \times \hat{\mathbf{a}}_1$ with $\hat{\mathbf{n}} = \frac{\sqrt{\mathbf{a}}}{\sqrt{a}} \vec{\mathbf{m}}$ to get

$$\hat{\mathbf{a}}^1 = \frac{\sqrt{a}}{\sqrt{a}} (\hat{\mathbf{a}}_2 \times \vec{\mathbf{m}}) \quad \text{and} \quad \hat{\mathbf{a}}^2 = \frac{\sqrt{a}}{\sqrt{a}} (\vec{\mathbf{m}} \times \hat{\mathbf{a}}_1)$$

- To obtain the desired expressions for the reciprocal base-vector fields, $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ are expressed in the form

$$\hat{\mathbf{a}}_1 = \frac{\mathbf{A}_1}{\mathcal{A}_1} \left[(1 + \Delta_{11}) \hat{\mathbf{a}}_1 + \Delta_{12} \hat{\mathbf{a}}_2 + \Delta_{13} \hat{\mathbf{n}} \right] \quad \text{and}$$

$$\hat{\mathbf{a}}_2 = \frac{\mathbf{A}_2}{\mathcal{A}_2} \left[\Delta_{21} \hat{\mathbf{a}}_1 + (1 + \Delta_{22}) \hat{\mathbf{a}}_2 + \Delta_{23} \hat{\mathbf{n}} \right]$$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- Then, it follows that

$$\hat{\mathbf{a}}^1 = \frac{\sqrt{a}}{\sqrt{a}} \frac{\mathbf{A}_2}{\mathcal{A}_2} \left[\Delta_{21} (\hat{\mathbf{a}}_1 \times \vec{m}) + (1 + \Delta_{22}) (\hat{\mathbf{a}}_2 \times \vec{m}) + \Delta_{23} (\hat{\mathbf{n}} \times \vec{m}) \right] \quad \text{and}$$

$$\hat{\mathbf{a}}^2 = \frac{\sqrt{a}}{\sqrt{a}} \frac{\mathbf{A}_1}{\mathcal{A}_1} \left[(1 + \Delta_{11}) (\vec{m} \times \hat{\mathbf{a}}_1) + \Delta_{12} (\vec{m} \times \hat{\mathbf{a}}_2) + \Delta_{13} (\vec{m} \times \hat{\mathbf{n}}) \right]$$

where

$$\frac{\sqrt{a}}{\sqrt{a}} \frac{\mathbf{A}_1}{\mathcal{A}_1} = 1 - \varepsilon_{11}^o + 2\varepsilon_{12}^o \cot\theta_{12} \csc\theta_{12} - \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \csc^2\theta_{12} + \mathcal{O}(\varepsilon^4) \quad \text{and}$$

$$\frac{\sqrt{a}}{\sqrt{a}} \frac{\mathbf{A}_2}{\mathcal{A}_2} = 1 - \varepsilon_{22}^o + 2\varepsilon_{12}^o \cot\theta_{12} \csc\theta_{12} - \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \csc^2\theta_{12} + \mathcal{O}(\varepsilon^4)$$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- Then, using $\hat{\mathbf{a}}_1 \times \vec{\mathbf{m}} = -m_3 \hat{\mathbf{a}}^2 + m_2 \sin\theta_{12} \hat{\mathbf{n}}$, $\hat{\mathbf{a}}_2 \times \vec{\mathbf{m}} = m_3 \hat{\mathbf{a}}^1 - m_1 \sin\theta_{12} \hat{\mathbf{n}}$,
and $\hat{\mathbf{n}} \times \vec{\mathbf{m}} = -m_2 \hat{\mathbf{a}}^1 + m_1 \hat{\mathbf{a}}^2$ gives the following desired equations

$$\hat{\mathbf{a}}^1 = \frac{\sqrt{a}}{\sqrt{a}} \frac{\mathbf{A}_2}{\mathcal{A}_2} \left[\left(m_3(1 + \Delta_{22}) - m_2 \Delta_{23} \right) \hat{\mathbf{a}}^1 + \left(m_1 \Delta_{23} - m_3 \Delta_{21} \right) \hat{\mathbf{a}}^2 \right. \\ \left. + \left(m_2 \Delta_{21} - m_1(1 + \Delta_{22}) \right) \sin\theta_{12} \hat{\mathbf{n}} \right]$$

$$\hat{\mathbf{a}}^2 = \frac{\sqrt{a}}{\sqrt{a}} \frac{\mathbf{A}_1}{\mathcal{A}_1} \left[\left(m_2 \Delta_{13} - m_3 \Delta_{12} \right) \hat{\mathbf{a}}^1 + \left(m_3(1 + \Delta_{11}) - m_1 \Delta_{13} \right) \hat{\mathbf{a}}^2 \right. \\ \left. + \left(m_1 \Delta_{12} - m_2(1 + \Delta_{11}) \right) \sin\theta_{12} \hat{\mathbf{n}} \right]$$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- For orthogonal Gaussian reference-surface coordinates,

$$\Delta_{11} = \mathbf{e}_{11}^{\circ}$$

$$\Delta_{12} = \mathbf{e}_{12}^{\circ} + \varphi$$

$$\Delta_{13} = -\varphi_1$$

$$\Delta_{21} = \mathbf{e}_{12}^{\circ} - \varphi$$

$$\Delta_{22} = \mathbf{e}_{22}^{\circ}$$

$$\Delta_{23} = -\varphi_2$$

$$\frac{1}{\mathbf{A}_1} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_1} = \mathbf{e}_{11}^{\circ} \hat{\mathbf{a}}_1 + (\mathbf{e}_{12}^{\circ} + \varphi) \hat{\mathbf{a}}_2 - \varphi_1 \hat{\mathbf{n}}$$

$$\frac{1}{\mathbf{A}_2} \frac{\partial \vec{\mathbf{u}}}{\partial \xi_2} = (\mathbf{e}_{12}^{\circ} - \varphi) \hat{\mathbf{a}}_1 + \mathbf{e}_{22}^{\circ} \hat{\mathbf{a}}_2 - \varphi_2 \hat{\mathbf{n}}$$

$$\boldsymbol{\varepsilon}_{11}^{\circ} = \mathbf{e}_{11}^{\circ} + \frac{1}{2} \left((\mathbf{e}_{11}^{\circ})^2 + (\mathbf{e}_{12}^{\circ} + \varphi)^2 + (\varphi_1)^2 \right) + \mathcal{O}(\varepsilon^4)$$

$$\boldsymbol{\varepsilon}_{22}^{\circ} = \mathbf{e}_{22}^{\circ} + \frac{1}{2} \left((\mathbf{e}_{12}^{\circ} - \varphi)^2 + (\mathbf{e}_{22}^{\circ})^2 + (\varphi_2)^2 \right) + \mathcal{O}(\varepsilon^4)$$

$$2\boldsymbol{\varepsilon}_{12}^{\circ} = 2\mathbf{e}_{12}^{\circ} + \mathbf{e}_{11}^{\circ}(\mathbf{e}_{12}^{\circ} - \varphi) + \mathbf{e}_{22}^{\circ}(\mathbf{e}_{12}^{\circ} + \varphi) + \varphi_1\varphi_2 + \mathcal{O}(\varepsilon^4)$$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONTINUED

- Likewise, for **orthogonal Gaussian reference-surface coordinates**,

$$\hat{\mathbf{a}}_1 = \frac{\mathbf{A}_1}{\mathcal{A}_1} \left[(1 + \mathbf{e}_{11}^\circ) \hat{\mathbf{a}}_1 + (\mathbf{e}_{12}^\circ + \varphi) \hat{\mathbf{a}}_2 - \varphi_1 \hat{\mathbf{n}} \right] \quad \frac{\mathbf{A}_1}{\mathcal{A}_1} = 1 - \varepsilon_{11}^\circ + \mathcal{O}(\varepsilon^4)$$

$$\hat{\mathbf{a}}_2 = \frac{\mathbf{A}_2}{\mathcal{A}_2} \left[(\mathbf{e}_{12}^\circ - \varphi) \hat{\mathbf{a}}_1 + (1 + \mathbf{e}_{22}^\circ) \hat{\mathbf{a}}_2 - \varphi_2 \hat{\mathbf{n}} \right] \quad \frac{\mathbf{A}_2}{\mathcal{A}_2} = 1 - \varepsilon_{22}^\circ + \mathcal{O}(\varepsilon^4)$$

$$\hat{\mathbf{u}} = \frac{\sqrt{\mathbf{a}}}{\sqrt{a}} \left(m_1 \hat{\mathbf{a}}_1 + m_2 \hat{\mathbf{a}}_2 + m_3 \hat{\mathbf{n}} \right) \quad \frac{\sqrt{\mathbf{a}}}{\sqrt{a}} = 1 - \varepsilon_{11}^\circ - \varepsilon_{22}^\circ + \mathcal{O}(\varepsilon^4)$$

$$m_1 = \varphi_1 (1 + \mathbf{e}_{22}^\circ) - \varphi_2 (\mathbf{e}_{12}^\circ + \varphi) \quad m_2 = \varphi_2 (1 + \mathbf{e}_{11}^\circ) - \varphi_1 (\mathbf{e}_{12}^\circ - \varphi)$$

$$m_3 = (1 + \mathbf{e}_{11}^\circ)(1 + \mathbf{e}_{22}^\circ) - (\mathbf{e}_{12}^\circ + \varphi)(\mathbf{e}_{12}^\circ - \varphi)$$

“SMALL-STRAIN” METRIC COEFFICIENTS AND BASE-VECTOR FIELDS OF $\mathcal{S}_t$ - CONCLUDED

- Moreover, for orthogonal Gaussian reference-surface coordinates,

$$\hat{\mathbf{a}}^1 = \frac{\sqrt{\mathbf{a}}}{\sqrt{a}} \frac{\mathbf{A}_2}{\mathcal{A}_2} \left[\left(m_3(1 + \mathbf{e}_{22}^o) + m_2\varphi_2 \right) \hat{\mathbf{a}}_1 - \left(m_1\varphi_2 + m_3(\mathbf{e}_{12}^o - \varphi) \right) \hat{\mathbf{a}}_2 + \left(m_2(\mathbf{e}_{12}^o - \varphi) - m_1(1 + \mathbf{e}_{22}^o) \right) \hat{\mathbf{n}} \right]$$

$$\hat{\mathbf{a}}^2 = \frac{\sqrt{\mathbf{a}}}{\sqrt{a}} \frac{\mathbf{A}_1}{\mathcal{A}_1} \left[- \left(m_2\varphi_1 + m_3(\mathbf{e}_{12}^o + \varphi) \right) \hat{\mathbf{a}}_1 + \left(m_3(1 + \mathbf{e}_{11}^o) + m_1\varphi_1 \right) \hat{\mathbf{a}}_2 + \left(m_1(\mathbf{e}_{12}^o + \varphi) - m_2(1 + \mathbf{e}_{11}^o) \right) \hat{\mathbf{n}} \right]$$

$$\frac{\sqrt{\mathbf{a}}}{\sqrt{a}} \frac{\mathbf{A}_1}{\mathcal{A}_1} = 1 - 2\epsilon_{11}^o - \epsilon_{22}^o + \mathcal{O}(\epsilon^4)$$

$$\frac{\sqrt{\mathbf{a}}}{\sqrt{a}} \frac{\mathbf{A}_2}{\mathcal{A}_2} = 1 - 2\epsilon_{22}^o - \epsilon_{11}^o + \mathcal{O}(\epsilon^4)$$

“SMALL-STRAIN” CURVATURES AND TORSIONS OF $\mathcal{S}_t$

- Previously, it was shown herein that the **curvatures of the deformed reference surface** are given by

$$\frac{1}{r_{11}} = \frac{\sqrt{a}}{\sqrt{a}} \left(\hat{a}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{m}}{\partial \xi_1} \right) \quad \text{and} \quad \frac{1}{r_{22}} = \frac{\sqrt{a}}{\sqrt{a}} \left(\hat{a}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{m}}{\partial \xi_2} \right)$$

and the **torsions of the deformed reference surface** are given by

$$\frac{1}{r_{12}} = - \frac{\sqrt{a}}{\sqrt{a}} \left(\hat{a}^2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{m}}{\partial \xi_1} \right) \quad \text{and} \quad \frac{1}{r_{21}} = \frac{\sqrt{a}}{\sqrt{a}} \left(\hat{a}^1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{m}}{\partial \xi_2} \right)$$

“SMALL-STRAIN” CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- Using

$$\hat{\mathbf{a}}_1 = \frac{\mathbf{A}_1}{\mathcal{A}_1} \left[(1 + \Delta_{11}) \hat{\mathbf{a}}_1 + \Delta_{12} \hat{\mathbf{a}}_2 + \Delta_{13} \hat{\mathbf{n}} \right] \quad \text{and}$$

$$\hat{\mathbf{a}}_2 = \frac{\mathbf{A}_2}{\mathcal{A}_2} \left[\Delta_{21} \hat{\mathbf{a}}_1 + (1 + \Delta_{22}) \hat{\mathbf{a}}_2 + \Delta_{23} \hat{\mathbf{n}} \right] \quad \text{gives}$$

$$\frac{1}{r_{11}} = \frac{\sqrt{\mathbf{a}}}{\sqrt{a}} \left(\frac{\mathbf{A}_1}{\mathcal{A}_1} \right)^2 \frac{1}{\bar{r}_{11}}$$

where

$$\frac{1}{\bar{r}_{11}} = \left[(1 + \Delta_{11}) \hat{\mathbf{a}}_1 + \Delta_{12} \hat{\mathbf{a}}_2 + \Delta_{13} \hat{\mathbf{n}} \right] \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{m}}{\partial \xi_1}$$

and

$$\frac{1}{r_{22}} = \frac{\sqrt{\mathbf{a}}}{\sqrt{a}} \left(\frac{\mathbf{A}_2}{\mathcal{A}_2} \right)^2 \frac{1}{\bar{r}_{22}}$$

where

$$\frac{1}{\bar{r}_{22}} = \left[\Delta_{21} \hat{\mathbf{a}}_1 + (1 + \Delta_{22}) \hat{\mathbf{a}}_2 + \Delta_{23} \hat{\mathbf{n}} \right] \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{m}}{\partial \xi_2}$$

“SMALL-STRAIN” CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- Similarly, using

$$\hat{\mathbf{a}}^2 = \frac{\sqrt{\mathbf{a}}}{\sqrt{\mathcal{A}_1}} \left[\left(m_2 \Delta_{13} - m_3 \Delta_{12} \right) \hat{\mathbf{a}}^1 + \left(m_3 (1 + \Delta_{11}) - m_1 \Delta_{13} \right) \hat{\mathbf{a}}^2 + \left(m_1 \Delta_{12} \sin \theta_{12} - m_2 (1 + \Delta_{11}) \sin \theta_{12} \right) \hat{\mathbf{n}} \right] \quad \text{gives}$$

$$\frac{1}{r_{12}} = \left(\frac{\sqrt{\mathbf{a}}}{\sqrt{\mathcal{A}_1}} \right)^2 \left(\frac{\mathbf{A}_1}{\mathcal{A}_1} \right)^2 \frac{1}{\bar{r}_{12}} \quad \text{where}$$

$$\frac{1}{\bar{r}_{12}} = \left\{ \left[m_2 \Delta_{13} - m_3 \Delta_{12} \right] \hat{\mathbf{a}}^1 + \left[m_3 (1 + \Delta_{11}) - m_1 \Delta_{13} \right] \hat{\mathbf{a}}^2 + \left[m_1 \Delta_{12} - m_2 (1 + \Delta_{11}) \right] \sin \theta_{12} \hat{\mathbf{n}} \right\} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{m}}{\partial \xi_1}$$

“SMALL-STRAIN” CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- And, using

$$\hat{\mathbf{a}}^1 = \frac{\sqrt{\mathbf{a}}}{\sqrt{\mathcal{a}}} \frac{\mathbf{A}_2}{\mathcal{A}_2} \left[\left(m_3(1 + \Delta_{22}) - m_2\Delta_{23} \right) \hat{\mathbf{a}}^1 + \left(m_1\Delta_{23} - m_3\Delta_{21} \right) \hat{\mathbf{a}}^2 + \left(m_2\Delta_{21} - m_1(1 + \Delta_{22}) \right) \sin\theta_{12} \hat{\mathbf{n}} \right] \quad \text{gives}$$

$$\frac{1}{r_{21}} = \left(\frac{\sqrt{\mathbf{a}}}{\sqrt{\mathcal{a}}} \right)^2 \left(\frac{\mathbf{A}_2}{\mathcal{A}_2} \right)^2 \frac{1}{\bar{r}_{21}} \quad \text{where}$$

$$\frac{1}{\bar{r}_{21}} = \left\{ \left[m_3(1 + \Delta_{22}) - m_2\Delta_{23} \right] \hat{\mathbf{a}}^1 + \left[m_1\Delta_{23} - m_3\Delta_{21} \right] \hat{\mathbf{a}}^2 + \left[m_2\Delta_{21} - m_1(1 + \Delta_{22}) \right] \sin\theta_{12} \hat{\mathbf{n}} \right\} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{m}}{\partial \xi_2}$$

“SMALL-STRAIN” CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- In these expressions,

$$\frac{\sqrt{a}}{\sqrt{a}} \left(\frac{\mathbf{A}_1}{\mathcal{A}_1} \right)^2 = 1 - 3\epsilon_{11}^o - \epsilon_{22}^o + \mathcal{O}(\epsilon^4)$$

$$\frac{\sqrt{a}}{\sqrt{a}} \left(\frac{\mathbf{A}_2}{\mathcal{A}_2} \right)^2 = 1 - 3\epsilon_{22}^o - \epsilon_{11}^o + \mathcal{O}(\epsilon^4)$$

$$\left(\frac{\sqrt{a}}{\sqrt{a}} \right)^2 \left(\frac{\mathbf{A}_1}{\mathcal{A}_1} \right)^2 = 1 - 4\epsilon_{11}^o - 2\epsilon_{22}^o + \mathcal{O}(\epsilon^4)$$

$$\left(\frac{\sqrt{a}}{\sqrt{a}} \right)^2 \left(\frac{\mathbf{A}_2}{\mathcal{A}_2} \right)^2 = 1 - 4\epsilon_{22}^o - 2\epsilon_{11}^o + \mathcal{O}(\epsilon^4)$$

- Recalling that the expressions for the **derivatives of** $\vec{m}$ are given by

$$\frac{1}{\mathbf{A}_1} \frac{\partial \vec{m}}{\partial \xi_1} = m^{(1)} \Big|_{\langle 1 \rangle} \hat{a}_1 + m^{(2)} \Big|_{\langle 1 \rangle} \hat{a}_2 + m^{(3)} \Big|_{\langle 1 \rangle} \hat{n}$$

$$\frac{1}{\mathbf{A}_2} \frac{\partial \vec{m}}{\partial \xi_2} = m^{(1)} \Big|_{\langle 2 \rangle} \hat{a}_1 + m^{(2)} \Big|_{\langle 2 \rangle} \hat{a}_2 + m^{(3)} \Big|_{\langle 2 \rangle} \hat{n}$$

“SMALL-STRAIN” CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

where, for **general** reference-surface Gaussian coordinates,

$$m^{(1)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_1}{\partial \xi_1} - \frac{m_2 \csc \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\csc \theta_{12}}{\rho_{11}} (m_1 \cos \theta_{12} + m_2) + m_3 \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right)$$

$$m^{(2)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_2}{\partial \xi_1} + \frac{\csc \theta_{12}}{\rho_{11}} (m_1 + m_2 \cos \theta_{12}) + \frac{m_2 \cot \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - m_3 \frac{\csc \theta_{12}}{r_{12}}$$

$$m^{(3)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_3}{\partial \xi_1} - \frac{m_1}{r_{11}} + m_2 \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right)$$

$$m^{(1)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_1}{\partial \xi_2} + \frac{m_1 \cot \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} - \frac{\csc \theta_{12}}{\rho_{22}} (m_1 \cos \theta_{12} + m_2) + m_3 \frac{\csc \theta_{12}}{r_{21}}$$

$$m^{(2)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_2}{\partial \xi_2} - \frac{m_1 \csc \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} + \frac{\csc \theta_{12}}{\rho_{22}} (m_1 + m_2 \cos \theta_{12}) + m_3 \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right)$$

“SMALL-STRAIN” CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

$$\boldsymbol{m}^{(3)} \Big|_{\langle 2 \rangle} = \frac{1}{A_2} \frac{\partial \boldsymbol{m}_3}{\partial \xi_2} - \boldsymbol{m}_1 \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) - \frac{\boldsymbol{m}_2}{r_{22}}$$

using

$$\hat{\boldsymbol{a}}_1 \cdot \frac{1}{A_1} \frac{\partial \vec{\boldsymbol{m}}}{\partial \xi_1} = \boldsymbol{m}^{(1)} \Big|_{\langle 1 \rangle} + \boldsymbol{m}^{(2)} \Big|_{\langle 1 \rangle} \cos \theta_{12}, \quad \hat{\boldsymbol{a}}_2 \cdot \frac{1}{A_1} \frac{\partial \vec{\boldsymbol{m}}}{\partial \xi_1} = \boldsymbol{m}^{(1)} \Big|_{\langle 1 \rangle} \cos \theta_{12} + \boldsymbol{m}^{(2)} \Big|_{\langle 1 \rangle},$$

and $\hat{\boldsymbol{n}} \cdot \frac{1}{A_1} \frac{\partial \vec{\boldsymbol{m}}}{\partial \xi_1} = \boldsymbol{m}^{(3)} \Big|_{\langle 1 \rangle}$ gives

$$\begin{aligned} \frac{1}{\bar{r}_{11}} = & (1 + \Delta_{11}) \left[\boldsymbol{m}^{(1)} \Big|_{\langle 1 \rangle} + \boldsymbol{m}^{(2)} \Big|_{\langle 1 \rangle} \cos \theta_{12} \right] \\ & + \Delta_{12} \left[\boldsymbol{m}^{(1)} \Big|_{\langle 1 \rangle} \cos \theta_{12} + \boldsymbol{m}^{(2)} \Big|_{\langle 1 \rangle} \right] + \Delta_{13} \left[\boldsymbol{m}^{(3)} \Big|_{\langle 1 \rangle} \right] \end{aligned}$$

“SMALL-STRAIN” CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- Similarly, using

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{A_2} \frac{\partial \vec{m}}{\partial \xi_2} = m^{(1)} \Big|_{\langle 2 \rangle} + m^{(2)} \Big|_{\langle 2 \rangle} \cos \theta_{12}, \quad \hat{\mathbf{a}}_2 \cdot \frac{1}{A_2} \frac{\partial \vec{m}}{\partial \xi_2} = m^{(1)} \Big|_{\langle 2 \rangle} \cos \theta_{12} + m^{(2)} \Big|_{\langle 2 \rangle},$$

and $\hat{\mathbf{n}} \cdot \frac{1}{A_2} \frac{\partial \vec{m}}{\partial \xi_2} = m^{(3)} \Big|_{\langle 2 \rangle}$ gives

$$\begin{aligned} \frac{1}{\bar{n}_{22}} = & (1 + \Delta_{22}) \left[m^{(1)} \Big|_{\langle 2 \rangle} \cos \theta_{12} + m^{(2)} \Big|_{\langle 2 \rangle} \right] \\ & + \Delta_{21} \left[m^{(1)} \Big|_{\langle 2 \rangle} + m^{(2)} \Big|_{\langle 2 \rangle} \cos \theta_{12} \right] + \Delta_{23} \left[m^{(3)} \Big|_{\langle 2 \rangle} \right] \end{aligned}$$

“SMALL-STRAIN” CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- Next, using

$$\hat{\mathbf{a}}^1 \cdot \frac{1}{A_1} \frac{\partial \vec{m}}{\partial \xi_1} = m^{(1)} \Big|_{\langle 1 \rangle} \sin \theta_{12}, \quad \hat{\mathbf{a}}^2 \cdot \frac{1}{A_1} \frac{\partial \vec{m}}{\partial \xi_1} = m^{(2)} \Big|_{\langle 1 \rangle} \sin \theta_{12}, \quad \text{and}$$

$$\hat{\mathbf{n}} \cdot \frac{1}{A_1} \frac{\partial \vec{m}}{\partial \xi_1} = m^{(3)} \Big|_{\langle 1 \rangle} \quad \text{gives}$$

$$\begin{aligned} \frac{1}{\bar{n}_{12}} = & \left[m_2 \Delta_{13} - m_3 \Delta_{12} \right] m^{(1)} \Big|_{\langle 1 \rangle} \sin \theta_{12} + \left[m_3 (1 + \Delta_{11}) - m_1 \Delta_{13} \right] m^{(2)} \Big|_{\langle 1 \rangle} \sin \theta_{12} \\ & + \left[m_1 \Delta_{12} - m_2 (1 + \Delta_{11}) \right] m^{(3)} \Big|_{\langle 1 \rangle} \sin \theta_{12} \end{aligned}$$

“SMALL-STRAIN” CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- And, using

$$\hat{\mathbf{a}}^1 \cdot \frac{1}{A_2} \frac{\partial \vec{m}}{\partial \xi_2} = m^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} \sin \theta_{12}, \quad \hat{\mathbf{a}}^2 \cdot \frac{1}{A_2} \frac{\partial \vec{m}}{\partial \xi_2} = m^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} \sin \theta_{12}, \quad \text{and}$$

$$\hat{\mathbf{n}} \cdot \frac{1}{A_2} \frac{\partial \vec{m}}{\partial \xi_2} = m^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} \quad \text{gives}$$

$$\begin{aligned} \frac{1}{\bar{n}_{21}} = & \left[m_3(1 + \Delta_{22}) - m_2 \Delta_{23} \right] m^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} \sin \theta_{12} + \left[m_1 \Delta_{23} - m_3 \Delta_{21} \right] m^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} \sin \theta_{12} \\ & + \left[m_2 \Delta_{21} - m_1(1 + \Delta_{22}) \right] m^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} \sin \theta_{12} \end{aligned}$$

“SMALL-STRAIN” CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONTINUED

- Substituting the expressions for Δ_{ij} in terms of the linear deformation measures into the expressions for $\bar{\tau}_{\alpha\beta}^{-1}$ and simplifying gives

$$\frac{1}{\bar{\tau}_{11}} = \left(1 + e_{11}^o\right) m^{(1)} \Big|_{\langle 1 \rangle} + \left(\cos\theta_{12} + e_{12}^o + \varphi \sin\theta_{12}\right) m^{(2)} \Big|_{\langle 1 \rangle} - \left(\varphi_1 + \varphi_2 \cos\theta_{12}\right) m^{(3)} \Big|_{\langle 1 \rangle}$$

$$\frac{1}{\bar{\tau}_{22}} = \left(\cos\theta_{12} + e_{12}^o - \varphi \sin\theta_{12}\right) m^{(1)} \Big|_{\langle 2 \rangle} + \left(1 + e_{22}^o\right) m^{(2)} \Big|_{\langle 2 \rangle} - \left(\varphi_1 \cos\theta_{12} + \varphi_2\right) m^{(3)} \Big|_{\langle 2 \rangle}$$

“SMALL-STRAIN” CURVATURES AND TORSIONS OF $\mathcal{S}_t$

CONCLUDED

$$\begin{aligned} \frac{1}{\bar{\kappa}_{12}} = & \left(1 + \mathbf{e}_{11}^{\circ} \csc^2 \theta_{12} - \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} + \varphi \right) \cot \theta_{12} \right) \left[\mathbf{m}_3 \mathbf{m}^{(2)} \Big|_{\langle 1 \rangle} - \mathbf{m}_2 \mathbf{m}^{(3)} \Big|_{\langle 1 \rangle} \right] \sin \theta_{12} \\ & + \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} + \varphi - \mathbf{e}_{11}^{\circ} \cot \theta_{12} \right) \left[\mathbf{m}_1 \mathbf{m}^{(3)} \Big|_{\langle 1 \rangle} - \mathbf{m}_3 \mathbf{m}^{(1)} \Big|_{\langle 1 \rangle} \right] \\ & + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \left[\mathbf{m}_1 \mathbf{m}^{(2)} \Big|_{\langle 1 \rangle} - \mathbf{m}_2 \mathbf{m}^{(1)} \Big|_{\langle 1 \rangle} \right] \sin \theta_{12} \end{aligned}$$

$$\begin{aligned} \frac{1}{\bar{\kappa}_{21}} = & \left(1 + \mathbf{e}_{22}^{\circ} \csc^2 \theta_{12} - \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \varphi \right) \cot \theta_{12} \right) \left[\mathbf{m}_3 \mathbf{m}^{(1)} \Big|_{\langle 2 \rangle} - \mathbf{m}_1 \mathbf{m}^{(3)} \Big|_{\langle 2 \rangle} \right] \sin \theta_{12} \\ & + \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \varphi - \mathbf{e}_{22}^{\circ} \cot \theta_{12} \right) \left[\mathbf{m}_2 \mathbf{m}^{(3)} \Big|_{\langle 2 \rangle} - \mathbf{m}_3 \mathbf{m}^{(2)} \Big|_{\langle 2 \rangle} \right] \\ & + \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) \left[\mathbf{m}_2 \mathbf{m}^{(1)} \Big|_{\langle 2 \rangle} - \mathbf{m}_1 \mathbf{m}^{(2)} \Big|_{\langle 2 \rangle} \right] \sin \theta_{12} \end{aligned}$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS

- The **changes in reference-surface curvatures**, $\overset{\circ}{K}_{11}(\xi_1, \xi_2, t)$ and $\overset{\circ}{K}_{22}(\xi_1, \xi_2, t)$, and the **change in reference-surface torsion**, $\overset{\circ}{K}_{12}(\xi_1, \xi_2, t)$, caused by deformation have been defined herein by

$$\overset{\circ}{K}_{11} \equiv \frac{1}{\kappa_{11}} - \frac{1}{r_{11}}$$

$$\overset{\circ}{K}_{22} \equiv \frac{1}{\kappa_{22}} - \frac{1}{r_{22}}$$

$$\overset{\circ}{K}_{12} \equiv - \left[\frac{1}{2} \left(\frac{1}{\kappa_{12}} - \frac{1}{\kappa_{21}} \right) - \frac{1}{2} \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) \right]$$

- Examination of the previously derived expressions for $\kappa_{\alpha\beta}^{-1}$ indicates that the changes in surface curvature and torsion are **complicated nonlinear functions** of the linear deformation parameters
 - Thus, these expressions appear to have very limited practical value
 - These expressions also have **no implicit limitations on the magnitude of the rotations** that may occur during deformation

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

- To obtain “small-strain” expressions for the changes in the reference-surface curvatures and torsions, it is convenient to express

$$K_{11}^o = \frac{1}{r_{11}} - \frac{1}{r_{11}} \quad \text{as} \quad \frac{\sqrt{a}}{\sqrt{a}} \left(\frac{\mathcal{A}_1}{\mathbf{A}_1} \right)^2 K_{11}^o = \frac{1}{\bar{r}_{11}} - \frac{\sqrt{a}}{\sqrt{a}} \left(\frac{\mathcal{A}_1}{\mathbf{A}_1} \right)^2 \frac{1}{r_{11}} \quad \text{and}$$

$$K_{22}^o = \frac{1}{r_{22}} - \frac{1}{r_{22}} \quad \text{as} \quad \frac{\sqrt{a}}{\sqrt{a}} \left(\frac{\mathcal{A}_2}{\mathbf{A}_2} \right)^2 K_{22}^o = \frac{1}{\bar{r}_{22}} - \frac{\sqrt{a}}{\sqrt{a}} \left(\frac{\mathcal{A}_2}{\mathbf{A}_2} \right)^2 \frac{1}{r_{22}}$$

where

$$\frac{\sqrt{a}}{\sqrt{a}} \left(\frac{\mathcal{A}_1}{\mathbf{A}_1} \right)^2 = 1 + 3\varepsilon_{11}^o + \varepsilon_{22}^o + \mathcal{O}(\varepsilon^4)$$

$$\frac{\sqrt{a}}{\sqrt{a}} \left(\frac{\mathcal{A}_2}{\mathbf{A}_2} \right)^2 = 1 + 3\varepsilon_{22}^o + \varepsilon_{11}^o + \mathcal{O}(\varepsilon^4)$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

- To obtain a similar expression for the change in the reference-surface torsion, the **geometric relation**

$$\frac{1}{\kappa_{12}} + \frac{1}{\kappa_{21}} = \cot\theta_{12} \left(\frac{1}{\kappa_{11}} - \frac{1}{\kappa_{22}} \right) \quad \text{is first added to} \quad 2\kappa_{12}^{\circ} = -\frac{1}{\kappa_{12}} + \frac{1}{\kappa_{21}} + \frac{1}{r_{12}} - \frac{1}{r_{21}}$$

to get

$$\frac{1}{\kappa_{21}} = \kappa_{12}^{\circ} + \frac{\cot\theta_{12}}{2} \left(\frac{1}{\kappa_{11}} - \frac{1}{\kappa_{22}} \right) + \frac{1}{2} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right)$$

and then the two equations are subtracted to get

$$-\frac{1}{\kappa_{12}} = \kappa_{12}^{\circ} - \frac{\cot\theta_{12}}{2} \left(\frac{1}{\kappa_{11}} - \frac{1}{\kappa_{22}} \right) + \frac{1}{2} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right)$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

- Then, using $\frac{1}{r_{11}} = \frac{1}{r_{11}} + K_{11}^{\circ}$ and $\frac{1}{r_{22}} = \frac{1}{r_{22}} + K_{22}^{\circ}$, it follows that

$$\frac{1}{r_{21}} = K_{12}^{\circ} + \frac{\cot\theta_{12}}{2} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} + K_{11}^{\circ} - K_{22}^{\circ} \right) + \frac{1}{2} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \quad \text{and}$$

$$-\frac{1}{r_{12}} = K_{12}^{\circ} - \frac{\cot\theta_{12}}{2} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} + K_{11}^{\circ} - K_{22}^{\circ} \right) + \frac{1}{2} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right)$$

- From $\frac{1}{r_{21}} = \left(\frac{\sqrt{a}}{\sqrt{a}} \right)^2 \left(\frac{A_2}{\mathcal{A}_2} \right)^2 \frac{1}{\bar{r}_{21}}$, it follows that

$$\frac{1}{\bar{r}_{21}} = \left(\frac{\sqrt{a} \mathcal{A}_2}{\sqrt{a} A_2} \right)^2 \left[K_{12}^{\circ} + \frac{\cot\theta_{12}}{2} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} + K_{11}^{\circ} - K_{22}^{\circ} \right) + \frac{1}{2} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \right]$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

- Likewise, from $\frac{1}{r_{12}} = \left(\frac{\sqrt{a}}{\sqrt{a}} \right)^2 \left(\frac{A_1}{\mathcal{A}_1} \right)^2 \frac{1}{\bar{r}_{12}}$, it follows that

$$-\frac{1}{\bar{r}_{12}} = \left(\frac{\sqrt{a} \mathcal{A}_1}{\sqrt{a} A_1} \right)^2 \left[K_{12}^{\circ} - \frac{\cot \theta_{12}}{2} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} + K_{11}^{\circ} - K_{22}^{\circ} \right) + \frac{1}{2} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \right]$$

- Adding these two equations for $\frac{1}{\bar{r}_{21}}$ and $\frac{1}{\bar{r}_{12}}$ yields

$$\frac{1}{\bar{r}_{21}} - \frac{1}{\bar{r}_{12}} = \left[2K_{12}^{\circ} + \frac{1}{r_{21}} - \frac{1}{r_{12}} \right] \mathbf{C}_1 + \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} + K_{11}^{\circ} - K_{22}^{\circ} \right) \mathbf{C}_2$$

where

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

$$C_1 = \frac{1}{2} \left(\frac{\sqrt{a} \mathcal{A}_2}{\sqrt{a} A_2} \right)^2 + \frac{1}{2} \left(\frac{\sqrt{a} \mathcal{A}_1}{\sqrt{a} A_1} \right)^2 = 1 + 3\varepsilon_{11}^o + 3\varepsilon_{22}^o + \mathcal{O}(\varepsilon^4)$$

$$C_2 = \left[\left(\frac{\sqrt{a} \mathcal{A}_2}{\sqrt{a} A_2} \right)^2 - \left(\frac{\sqrt{a} \mathcal{A}_1}{\sqrt{a} A_1} \right)^2 \right] \frac{\cot \theta_{12}}{2} = (\varepsilon_{22}^o - \varepsilon_{11}^o) \cot \theta_{12} + \mathcal{O}(\varepsilon^4)$$

● Now, $\cos \theta_{12} = 2\varepsilon_{12}^o + \cos \theta_{12} (1 - \varepsilon_{11}^o - \varepsilon_{22}^o) + \mathcal{O}(\varepsilon^4)$ and

$$\sin \theta_{12} = \sin \theta_{12} - \cot \theta_{12} \left[2\varepsilon_{12}^o - \cos \theta_{12} (\varepsilon_{11}^o + \varepsilon_{22}^o) \right] + \mathcal{O}(\varepsilon^4)$$

are used to get

$$\cot \theta_{12} = \cot \theta_{12} + 2\varepsilon_{12}^o \csc^3 \theta_{12} - (\varepsilon_{11}^o + \varepsilon_{22}^o) \csc^2 \theta_{12} \cot \theta_{12} + \mathcal{O}(\varepsilon^4)$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

- Thus, $\mathbf{C}_2 = (\epsilon_{22}^o - \epsilon_{11}^o) \cot \theta_{12} + \mathcal{O}(\epsilon^4)$
- At this point in the derivation, the equations of interest are:

$$\left(1 + 3\epsilon_{11}^o + \epsilon_{22}^o\right) \mathbf{K}_{11}^o = \frac{1}{\bar{\mathbf{r}}_{11}} - \frac{1 + 3\epsilon_{11}^o + \epsilon_{22}^o}{\mathbf{r}_{11}} + \mathcal{O}(\epsilon^4)$$

$$\left(1 + 3\epsilon_{22}^o + \epsilon_{11}^o\right) \mathbf{K}_{22}^o = \frac{1}{\bar{\mathbf{r}}_{22}} - \frac{1 + 3\epsilon_{22}^o + \epsilon_{11}^o}{\mathbf{r}_{22}} + \mathcal{O}(\epsilon^4)$$

$$\begin{aligned} \left(1 + 3\epsilon_{11}^o + 3\epsilon_{22}^o\right) 2\mathbf{K}_{12}^o + \left(\mathbf{K}_{11}^o - \mathbf{K}_{22}^o\right) \left(\epsilon_{22}^o - \epsilon_{11}^o\right) \cot \theta_{12} &= \frac{1}{\bar{\mathbf{r}}_{21}} - \frac{1}{\bar{\mathbf{r}}_{12}} \\ &- \left(1 + 3\epsilon_{11}^o + 3\epsilon_{22}^o\right) \left(\frac{1}{\mathbf{r}_{21}} - \frac{1}{\mathbf{r}_{12}}\right) - \left(\frac{1}{\mathbf{r}_{11}} - \frac{1}{\mathbf{r}_{22}}\right) \left(\epsilon_{22}^o - \epsilon_{11}^o\right) \cot \theta_{12} + \mathcal{O}(\epsilon^4) \end{aligned}$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

- **To get further simplification, it is presumed that the changes in surface curvatures and torsions are at most the same order of magnitude as the strains**
- Based on this presumption, the last three equations reduce to

$$\mathcal{K}_{11}^{\circ} = \left(\frac{1}{\bar{r}_{11}} - \frac{1}{r_{11}} \right) - \frac{3\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}}{r_{11}} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{K}_{22}^{\circ} = \left(\frac{1}{\bar{r}_{22}} - \frac{1}{r_{22}} \right) - \frac{3\varepsilon_{22}^{\circ} + \varepsilon_{11}^{\circ}}{r_{22}} + \mathcal{O}(\varepsilon^4)$$

$$2\mathcal{K}_{12}^{\circ} = - \left[\frac{1}{\bar{r}_{12}} - \frac{1}{\bar{r}_{21}} - \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) \right] + 3 \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) (\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}) \\ + \left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) (\varepsilon_{22}^{\circ} - \varepsilon_{11}^{\circ}) \cot \theta_{12} + \mathcal{O}(\varepsilon^4)$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

- These three equations give the **“small” changes in surface curvature and torsion associated with “small” strains**
- In these expressions, recall that

$$\frac{1}{\bar{\kappa}_{11}} = \left(1 + e_{11}^o\right) m^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} + \left(\cos\theta_{12} + e_{12}^o + \varphi \sin\theta_{12}\right) m^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} - \left(\varphi_1 + \varphi_2 \cos\theta_{12}\right) m^{\langle 3 \rangle} \Big|_{\langle 1 \rangle}$$

$$\frac{1}{\bar{\kappa}_{22}} = \left(\cos\theta_{12} + e_{12}^o - \varphi \sin\theta_{12}\right) m^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} + \left(1 + e_{22}^o\right) m^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} - \left(\varphi_1 \cos\theta_{12} + \varphi_2\right) m^{\langle 3 \rangle} \Big|_{\langle 2 \rangle}$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

$$\begin{aligned} \frac{1}{\bar{\kappa}_{12}} = & \left(1 + e_{11}^{\circ} \csc^2 \theta_{12} - (e_{12}^{\circ} \csc \theta_{12} + \varphi) \cot \theta_{12} \right) \left[m_3 m^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} - m_2 m^{\langle 3 \rangle} \Big|_{\langle 1 \rangle} \right] \sin \theta_{12} \\ & + \left(e_{12}^{\circ} \csc \theta_{12} + \varphi - e_{11}^{\circ} \cot \theta_{12} \right) \left[m_1 m^{\langle 3 \rangle} \Big|_{\langle 1 \rangle} - m_3 m^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} \right] \\ & + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \left[m_1 m^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} - m_2 m^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} \right] \sin \theta_{12} \end{aligned}$$

$$\begin{aligned} \frac{1}{\bar{\kappa}_{21}} = & \left(1 + e_{22}^{\circ} \csc^2 \theta_{12} - (e_{12}^{\circ} \csc \theta_{12} - \varphi) \cot \theta_{12} \right) \left[m_3 m^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} - m_1 m^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} \right] \sin \theta_{12} \\ & + \left(e_{12}^{\circ} \csc \theta_{12} - \varphi - e_{22}^{\circ} \cot \theta_{12} \right) \left[m_2 m^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} - m_3 m^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} \right] \\ & + \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) \left[m_2 m^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} - m_1 m^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} \right] \sin \theta_{12} \end{aligned}$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

$$\begin{aligned}\varepsilon_{11}^{\circ} = \mathbf{e}_{11}^{\circ} + \frac{1}{2} & \left[\left(\mathbf{e}_{11}^{\circ} \csc \theta_{12} \right)^2 + \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} + \varphi \right)^2 \right. \\ & \left. - 2\mathbf{e}_{11}^{\circ} \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} + \varphi \right) \cot \theta_{12} + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)^2 \right] + \mathcal{O}(\varepsilon^4)\end{aligned}$$

$$\begin{aligned}\varepsilon_{22}^{\circ} = \mathbf{e}_{22}^{\circ} + \frac{1}{2} & \left[\left(\mathbf{e}_{22}^{\circ} \csc \theta_{12} \right)^2 + \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \varphi \right)^2 \right. \\ & \left. - 2\mathbf{e}_{22}^{\circ} \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \varphi \right) \cot \theta_{12} + \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right)^2 \right] + \mathcal{O}(\varepsilon^4)\end{aligned}$$

$$\begin{aligned}2\varepsilon_{12}^{\circ} = 2\mathbf{e}_{12}^{\circ} + & \left[\mathbf{e}_{11}^{\circ} \left(\mathbf{e}_{12}^{\circ} - \varphi \sin \theta_{12} \right) + \mathbf{e}_{22}^{\circ} \left(\mathbf{e}_{12}^{\circ} + \varphi \sin \theta_{12} \right) \right] \csc^2 \theta_{12} \\ & - \left[\mathbf{e}_{11}^{\circ} \mathbf{e}_{22}^{\circ} + \left(\mathbf{e}_{12}^{\circ} + \varphi \sin \theta_{12} \right) \left(\mathbf{e}_{12}^{\circ} - \varphi \sin \theta_{12} \right) \right] \csc \theta_{12} \cot \theta_{12} \\ & + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) + \mathcal{O}(\varepsilon^4)\end{aligned}$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

$$m^{(1)} \Big|_{\langle 1 \rangle} = \frac{1}{A_1} \frac{\partial m_1}{\partial \xi_1} - \frac{m_2 \csc \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\csc \theta_{12}}{\rho_{11}} (m_1 \cos \theta_{12} + m_2) + m_3 \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right)$$

$$m^{(2)} \Big|_{\langle 1 \rangle} = \frac{1}{A_1} \frac{\partial m_2}{\partial \xi_1} + \frac{\csc \theta_{12}}{\rho_{11}} (m_1 + m_2 \cos \theta_{12}) + \frac{m_2 \cot \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - m_3 \frac{\csc \theta_{12}}{r_{12}}$$

$$m^{(3)} \Big|_{\langle 1 \rangle} = \frac{1}{A_1} \frac{\partial m_3}{\partial \xi_1} - \frac{m_1}{r_{11}} + m_2 \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right)$$

$$m^{(1)} \Big|_{\langle 2 \rangle} = \frac{1}{A_2} \frac{\partial m_1}{\partial \xi_2} + \frac{m_1 \cot \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} - \frac{\csc \theta_{12}}{\rho_{22}} (m_1 \cos \theta_{12} + m_2) + m_3 \frac{\csc \theta_{12}}{r_{21}}$$

$$m^{(2)} \Big|_{\langle 2 \rangle} = \frac{1}{A_2} \frac{\partial m_2}{\partial \xi_2} - \frac{m_1 \csc \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} + \frac{\csc \theta_{12}}{\rho_{22}} (m_1 + m_2 \cos \theta_{12}) + m_3 \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right)$$

$$m^{(3)} \Big|_{\langle 2 \rangle} = \frac{1}{A_2} \frac{\partial m_3}{\partial \xi_2} - m_1 \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) - \frac{m_2}{r_{22}}$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

$$\begin{aligned} m_1 = \varphi_1 - \left(e_{12}^o \csc \theta_{12} + \varphi \right) \left(\varphi_1 \cot \theta_{12} + \varphi_2 \csc \theta_{12} \right) \\ + e_{22}^o \csc \theta_{12} \left(\varphi_1 \csc \theta_{12} + \varphi_2 \cot \theta_{12} \right) \end{aligned}$$

$$\begin{aligned} m_2 = \varphi_2 - \left(e_{12}^o \csc \theta_{12} - \varphi \right) \left(\varphi_1 \csc \theta_{12} + \varphi_2 \cot \theta_{12} \right) \\ + e_{11}^o \csc \theta_{12} \left(\varphi_1 \cot \theta_{12} + \varphi_2 \csc \theta_{12} \right) \end{aligned}$$

$$\begin{aligned} m_3 = 1 + \varphi^2 + \left(e_{11}^o + e_{22}^o + e_{11}^o e_{22}^o - \left(e_{12}^o \right)^2 \right) \csc^2 \theta_{12} \\ - 2e_{12}^o \cot \theta_{12} \csc \theta_{12} \end{aligned}$$

$$\varphi_1 = \left(\frac{u_1}{r_{11}} - \frac{1}{A_1} \frac{\partial w}{\partial \xi_1} \right) \csc^2 \theta_{12} - u_1 \left(\frac{1}{r_{21}} + \frac{\cot \theta_{12}}{r_{22}} \right) \cot \theta_{12} + \left(\frac{\cot \theta_{12}}{A_2} \frac{\partial w}{\partial \xi_2} + \frac{u_2}{r_{21}} \right) \csc \theta_{12}$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

$$\varphi_2 = \left(\frac{u_2}{r_{22}} - \frac{1}{A_2} \frac{\partial w}{\partial \xi_2} \right) \csc^2 \theta_{12} + u_2 \left(\frac{1}{r_{12}} - \frac{\cot \theta_{12}}{r_{11}} \right) \cot \theta_{12} + \left(\frac{\cot \theta_{12}}{A_1} \frac{\partial w}{\partial \xi_1} - \frac{u_1}{r_{12}} \right) \csc \theta_{12}$$

$$2\varphi = \left(\frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} \right) \cot \theta_{12} + \left(\frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} \right) \csc \theta_{12} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}}$$

$$\mathbf{e}_{11}^o = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (u_1 + u_2 \cos \theta_{12}) - \frac{u_2 \sin \theta_{12}}{\rho_{11}} + \frac{w}{r_{11}}$$

$$\mathbf{e}_{22}^o = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (u_2 + u_1 \cos \theta_{12}) + \frac{u_1 \sin \theta_{12}}{\rho_{22}} + \frac{w}{r_{22}}$$

$$\begin{aligned} 2\mathbf{e}_{12}^o = & \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} + \left(\frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} + \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} \right) \cos \theta_{12} \\ & + \left(\frac{u_1}{\rho_{11}} - \frac{u_2}{\rho_{22}} \right) \sin \theta_{12} + w \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) \cos \theta_{12} + w \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \sin \theta_{12} \end{aligned}$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

- For orthogonal Gaussian reference-surface coordinates, the “small” changes in surface curvatures and surface torsion, for “small strains” and “finite rotations” are given by

$$K_{11}^{\circ} = \left(\frac{1}{\bar{\mathbf{r}}_{11}} - \frac{1}{\mathbf{r}_{11}} \right) - \frac{3\epsilon_{11}^{\circ} + \epsilon_{22}^{\circ}}{\mathbf{r}_{11}} + \mathcal{O}(\epsilon^4)$$

$$K_{22}^{\circ} = \left(\frac{1}{\bar{\mathbf{r}}_{22}} - \frac{1}{\mathbf{r}_{22}} \right) - \frac{3\epsilon_{22}^{\circ} + \epsilon_{11}^{\circ}}{\mathbf{r}_{22}} + \mathcal{O}(\epsilon^4)$$

$$2K_{12}^{\circ} = - \left[\frac{1}{\bar{\mathbf{r}}_{12}} - \frac{1}{\bar{\mathbf{r}}_{21}} - \left(\frac{1}{\mathbf{r}_{12}} - \frac{1}{\mathbf{r}_{21}} \right) \right] + 3 \left(\frac{1}{\mathbf{r}_{12}} - \frac{1}{\mathbf{r}_{21}} \right) (\epsilon_{11}^{\circ} + \epsilon_{22}^{\circ}) + \mathcal{O}(\epsilon^4)$$

where

$$\frac{1}{\bar{\mathbf{r}}_{11}} = \left(\mathbf{1} + \mathbf{e}_{11}^{\circ} \right) \mathbf{m}^{(1)} \Big|_{\langle 1 \rangle} + \left(\mathbf{e}_{12}^{\circ} + \varphi \right) \mathbf{m}^{(2)} \Big|_{\langle 1 \rangle} - \varphi_1 \mathbf{m}^{(3)} \Big|_{\langle 1 \rangle}$$

$$\frac{1}{\bar{\mathbf{r}}_{22}} = \left(\mathbf{e}_{12}^{\circ} - \varphi \right) \mathbf{m}^{(1)} \Big|_{\langle 2 \rangle} + \left(\mathbf{1} + \mathbf{e}_{22}^{\circ} \right) \mathbf{m}^{(2)} \Big|_{\langle 2 \rangle} - \varphi_2 \mathbf{m}^{(3)} \Big|_{\langle 2 \rangle}$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

$$\frac{1}{\bar{\kappa}_{12}} = (1 + e_{11}^o) \left[m_3 m^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} - m_2 m^{\langle 3 \rangle} \Big|_{\langle 1 \rangle} \right] \\ + (e_{12}^o + \varphi) \left[m_1 m^{\langle 3 \rangle} \Big|_{\langle 1 \rangle} - m_3 m^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} \right] + \varphi_1 \left[m_1 m^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} - m_2 m^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} \right]$$

$$\frac{1}{\bar{\kappa}_{21}} = (1 + e_{22}^o) \left[m_3 m^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} - m_1 m^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} \right] \\ + (e_{12}^o - \varphi) \left[m_2 m^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} - m_3 m^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} \right] + \varphi_2 \left[m_2 m^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} - m_1 m^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} \right]$$

$$\varepsilon_{11}^o = e_{11}^o + \frac{1}{2} \left[(e_{11}^o)^2 + (e_{12}^o + \varphi)^2 + \varphi_1^2 \right] + \mathcal{O}(\varepsilon^4)$$

$$\varepsilon_{22}^o = e_{22}^o + \frac{1}{2} \left[(e_{12}^o - \varphi)^2 + (e_{22}^o)^2 + \varphi_2^2 \right] + \mathcal{O}(\varepsilon^4)$$

$$2\varepsilon_{12}^o = 2e_{12}^o + e_{11}^o(e_{12}^o - \varphi) + e_{22}^o(e_{12}^o + \varphi) + \varphi_1\varphi_2 + \mathcal{O}(\varepsilon^4)$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONTINUED

$$m_1 = \varphi_1(1 + e_{22}^0) - \varphi_2(e_{12}^0 + \varphi)$$

$$m_2 = \varphi_2(1 + e_{11}^0) - \varphi_1(e_{12}^0 - \varphi)$$

$$m_3 = (1 + e_{11}^0)(1 + e_{22}^0) - (e_{12}^0 + \varphi)(e_{12}^0 - \varphi)$$

$$m^{(1)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_1}{\partial \xi_1} - \frac{m_2}{\rho_{11}} + \frac{m_3}{r_{11}}$$

$$m^{(2)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_2}{\partial \xi_1} + \frac{m_1}{\rho_{11}} - \frac{m_3}{r_{12}}$$

$$m^{(3)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_3}{\partial \xi_1} - \frac{m_1}{r_{11}} + \frac{m_2}{r_{12}}$$

$$m^{(1)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_1}{\partial \xi_2} - \frac{m_2}{\rho_{22}} - \frac{m_3}{r_{12}}$$

$$m^{(2)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_2}{\partial \xi_2} + \frac{m_1}{\rho_{22}} + \frac{m_3}{r_{22}}$$

$$m^{(3)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_3}{\partial \xi_2} + \frac{m_1}{r_{12}} - \frac{m_2}{r_{22}}$$

CHANGES IN THE CURVATURES AND TORSIONS OF $\mathcal{S}_0$ FOR “SMALL” STRAINS - CONCLUDED

$$\varphi_1 = \frac{u_1}{r_{11}} - \frac{u_2}{r_{12}} - \frac{1}{A_1} \frac{\partial w}{\partial \xi_1}$$

$$\varphi_2 = \frac{u_2}{r_{22}} - \frac{u_1}{r_{12}} - \frac{1}{A_2} \frac{\partial w}{\partial \xi_2}$$

$$\varphi = \frac{1}{2} \left(\frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}} \right)$$

$$e_{11}^o = \frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} - \frac{u_2}{\rho_{11}} + \frac{w}{r_{11}}$$

$$e_{22}^o = \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{\rho_{22}} + \frac{w}{r_{22}}$$

$$2e_{12}^o = \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} + \frac{u_1}{\rho_{11}} - \frac{u_2}{\rho_{22}} - 2 \frac{w}{r_{12}}$$

$$\frac{1}{\rho_{11}} = - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2}$$

$$\frac{1}{\rho_{22}} = \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \xi_1}$$

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

- Previously, the following **three independent compatibility equations** for the undeformed reference surface were obtained:

$$\frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\rho_{11}} \right) - \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\rho_{22}} \right) + \frac{\partial}{\partial \xi_1} \left(\frac{\partial \theta_{12}}{\partial \xi_2} \right) = \mathbf{A}_1 \mathbf{A}_2 \sin \theta_{12} \mathbf{K}_G$$

$$\begin{aligned} \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{r_{22}} \right) \cos \theta_{12} + \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{r_{21}} \right) \sin \theta_{12} - \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{r_{11}} \right) + \frac{\mathbf{A}_1 \mathbf{A}_2}{\rho_{11}} \left(\frac{\cos \theta_{12}}{r_{21}} - \frac{\sin \theta_{12}}{r_{22}} \right) \\ - \frac{\mathbf{A}_1 \mathbf{A}_2}{\rho_{22} r_{12}} + \left[\frac{\mathbf{A}_2}{r_{21}} \cos \theta_{12} - \frac{\mathbf{A}_2}{r_{22}} \sin \theta_{12} \right] \frac{\partial \theta_{12}}{\partial \xi_1} + \left(\frac{\mathbf{A}_1}{r_{12}} \right) \frac{\partial \theta_{12}}{\partial \xi_2} = 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{r_{22}} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{r_{12}} \right) \sin \theta_{12} - \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{r_{11}} \right) \cos \theta_{12} - \frac{\mathbf{A}_1 \mathbf{A}_2}{\rho_{22}} \left(\frac{\cos \theta_{12}}{r_{12}} + \frac{\sin \theta_{12}}{r_{11}} \right) \\ + \frac{\mathbf{A}_1 \mathbf{A}_2}{\rho_{11} r_{21}} + \left(\frac{\mathbf{A}_2}{r_{21}} \right) \frac{\partial \theta_{12}}{\partial \xi_1} + \left[\frac{\mathbf{A}_1}{r_{12}} \cos \theta_{12} + \frac{\mathbf{A}_1}{r_{11}} \sin \theta_{12} \right] \frac{\partial \theta_{12}}{\partial \xi_2} = 0 \end{aligned}$$

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

CONTINUED

where $K_G \equiv \frac{1}{r_{11}r_{22}} + \frac{1}{r_{12}r_{21}} + \cot\theta_{12}\left(\frac{1}{r_{12}r_{22}} - \frac{1}{r_{11}r_{21}}\right)$ is the **Gaussian curvature** of the undeformed reference surface

- By direct analogy, the compatibility equations for the deformed reference surface are:

$$\frac{\partial}{\partial \xi_2} \left(\frac{\mathcal{A}_1}{\rho_{11}} \right) - \frac{\partial}{\partial \xi_1} \left(\frac{\mathcal{A}_2}{\rho_{22}} \right) + \frac{\partial}{\partial \xi_1} \left(\frac{\partial \theta_{12}}{\partial \xi_2} \right) = \mathcal{A}_1 \mathcal{A}_2 \sin \theta_{12} K_G$$

$$\begin{aligned} \frac{\partial}{\partial \xi_1} \left(\frac{\mathcal{A}_2}{r_{22}} \right) \cos \theta_{12} + \frac{\partial}{\partial \xi_1} \left(\frac{\mathcal{A}_2}{r_{21}} \right) \sin \theta_{12} - \frac{\partial}{\partial \xi_2} \left(\frac{\mathcal{A}_1}{r_{11}} \right) + \frac{\mathcal{A}_1 \mathcal{A}_2}{\rho_{11}} \left(\frac{\cos \theta_{12}}{r_{21}} - \frac{\sin \theta_{12}}{r_{22}} \right) \\ - \frac{\mathcal{A}_1 \mathcal{A}_2}{\rho_{22} r_{12}} + \left[\frac{\mathcal{A}_2}{r_{21}} \cos \theta_{12} - \frac{\mathcal{A}_2}{r_{22}} \sin \theta_{12} \right] \frac{\partial \theta_{12}}{\partial \xi_1} + \left(\frac{\mathcal{A}_1}{r_{12}} \right) \frac{\partial \theta_{12}}{\partial \xi_2} = 0 \end{aligned}$$

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

CONTINUED

$$\begin{aligned} \frac{\partial}{\partial \xi_1} \left(\frac{\mathcal{A}_2}{r_{22}} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{\mathcal{A}_1}{r_{12}} \right) \sin \theta_{12} - \frac{\partial}{\partial \xi_2} \left(\frac{\mathcal{A}_1}{r_{11}} \right) \cos \theta_{12} - \frac{\mathcal{A}_1 \mathcal{A}_2}{\rho_{22}} \left(\frac{\cos \theta_{12}}{r_{12}} + \frac{\sin \theta_{12}}{r_{11}} \right) \\ + \frac{\mathcal{A}_1 \mathcal{A}_2}{\rho_{11} r_{21}} + \left(\frac{\mathcal{A}_2}{r_{21}} \right) \frac{\partial \theta_{12}}{\partial \xi_1} + \left[\frac{\mathcal{A}_1}{r_{12}} \cos \theta_{12} + \frac{\mathcal{A}_1}{r_{11}} \sin \theta_{12} \right] \frac{\partial \theta_{12}}{\partial \xi_2} = 0 \end{aligned}$$

where $K_G = \frac{1}{r_{11} r_{22}} + \frac{1}{r_{12} r_{21}} + \cot \theta_{12} \left(\frac{1}{r_{12} r_{22}} - \frac{1}{r_{11} r_{21}} \right)$ is the **Gaussian**

curvature of the deformed reference surface

- Next, by direct analogy, it also follows that the derivatives of the metric coefficients for the deformed reference surface are given by

$$\frac{\partial \mathcal{A}_1}{\partial \xi_2} = \mathcal{A}_1 \mathcal{A}_2 \left[\frac{\cot \theta_{12}}{\rho_{22}} - \frac{\csc \theta_{12}}{\rho_{11}} - \frac{\csc \theta_{12}}{\mathcal{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\cot \theta_{12}}{\mathcal{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right] \text{ and}$$

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

CONTINUED

$$\frac{\partial \mathcal{A}_2}{\partial \xi_1} = \mathcal{A}_1 \mathcal{A}_2 \left[\frac{\csc \theta_{12}}{\rho_{22}} - \frac{\cot \theta_{12}}{\rho_{11}} - \frac{\cot \theta_{12}}{\mathcal{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\csc \theta_{12}}{\mathcal{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right]$$

- By using these derivative expressions, the compatibility equations are also expressed as

$$\begin{aligned} & \frac{1}{\mathcal{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{\rho_{11}} \right) - \frac{1}{\mathcal{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{\rho_{22}} \right) + \left(\frac{\cot \theta_{12}}{\rho_{22}} - \frac{\csc \theta_{12}}{\rho_{11}} \right) \left[\frac{1}{\rho_{11}} + \frac{1}{\mathcal{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right] \\ & + \left(\frac{\csc \theta_{12}}{\rho_{22}} - \frac{\cot \theta_{12}}{\rho_{11}} \right) \left[-\frac{1}{\rho_{22}} + \frac{1}{\mathcal{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right] + \frac{1}{\mathcal{A}_1 \mathcal{A}_2} \frac{\partial}{\partial \xi_1} \left(\frac{\partial \theta_{12}}{\partial \xi_2} \right) = \sin \theta_{12} K_G \end{aligned}$$

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

CONTINUED

$$\begin{aligned} & \cos\theta_{12} \frac{1}{\mathcal{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{22}} \right) + \sin\theta_{12} \frac{1}{\mathcal{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{21}} \right) - \frac{1}{\mathcal{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{11}} \right) \\ & + \csc\theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \left(\frac{1}{\rho_{11}} + \frac{1}{\mathcal{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) + \frac{2}{r_{12}} \left(-\frac{1}{\rho_{22}} + \frac{1}{\mathcal{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) = 0 \end{aligned}$$

$$\begin{aligned} & \frac{1}{\mathcal{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{22}} \right) + \sin\theta_{12} \frac{1}{\mathcal{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{12}} \right) - \cos\theta_{12} \frac{1}{\mathcal{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{11}} \right) \\ & + \frac{2}{r_{21}} \left(\frac{1}{\rho_{11}} + \frac{1}{\mathcal{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) + \csc\theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \left(-\frac{1}{\rho_{22}} + \frac{1}{\mathcal{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) = 0 \end{aligned}$$

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

CONTINUED

- For “small” strains, the following approximations are valid:

$$\cos \theta_{12} = 2\varepsilon_{12}^{\circ} + \cos \theta_{12} \left(1 - \varepsilon_{11}^{\circ} - \varepsilon_{22}^{\circ} \right) + \mathcal{O}(\varepsilon^4)$$

$$\sin \theta_{12} = \sin \theta_{12} - \cot \theta_{12} \left[2\varepsilon_{12}^{\circ} - \cos \theta_{12} \left(\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ} \right) \right] + \mathcal{O}(\varepsilon^4)$$

$$\cot \theta_{12} = \cot \theta_{12} + 2\varepsilon_{12}^{\circ} \csc^3 \theta_{12} - \left(\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ} \right) \csc^2 \theta_{12} \cot \theta_{12} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{A}_1 = A_1 \left(1 + \varepsilon_{11}^{\circ} + \mathcal{O}(\varepsilon^4) \right)$$

$$\mathcal{A}_2 = A_2 \left(1 + \varepsilon_{22}^{\circ} + \mathcal{O}(\varepsilon^4) \right)$$

$$\frac{1}{r_{11}} = \frac{1}{r_{11}} + K_{11}^{\circ}$$

$$\frac{1}{r_{22}} = \frac{1}{r_{22}} + K_{22}^{\circ}$$

- In addition, $\theta_{12} = \cos^{-1} \left[2\varepsilon_{12}^{\circ} + \cos \theta_{12} \left(1 - \varepsilon_{11}^{\circ} - \varepsilon_{22}^{\circ} \right) + \mathcal{O}(\varepsilon^4) \right]$ becomes

$$\theta_{12} = \theta_{12} - 2\varepsilon_{12}^{\circ} \csc \theta_{12} + \left(\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ} \right) \cot \theta_{12} + \mathcal{O}(\varepsilon^4)$$

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

CONTINUED

- Moreover, the torsions of the deformed reference surface are

$$\begin{aligned} \frac{1}{\mathbf{r}_{12}} &= \frac{1}{\mathbf{r}_{12}} - \mathbf{K}_{12}^{\circ} + \frac{1}{2} \cot \theta_{12} (\mathbf{K}_{11}^{\circ} - \mathbf{K}_{22}^{\circ}) \\ &+ \frac{1}{2} \left[2\boldsymbol{\varepsilon}_{12}^{\circ} \left(\frac{1}{\mathbf{r}_{11}} - \frac{1}{\mathbf{r}_{22}} \right) \csc \theta_{12} - (\boldsymbol{\varepsilon}_{11}^{\circ} + \boldsymbol{\varepsilon}_{22}^{\circ}) \left(\frac{1}{\mathbf{r}_{12}} + \frac{1}{\mathbf{r}_{21}} \right) \right] \csc^2 \theta_{12} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\mathbf{r}_{21}} &= \frac{1}{\mathbf{r}_{21}} + \mathbf{K}_{12}^{\circ} + \frac{1}{2} \cot \theta_{12} (\mathbf{K}_{11}^{\circ} - \mathbf{K}_{22}^{\circ}) \\ &+ \frac{1}{2} \left[2\boldsymbol{\varepsilon}_{12}^{\circ} \left(\frac{1}{\mathbf{r}_{11}} - \frac{1}{\mathbf{r}_{22}} \right) \csc \theta_{12} - (\boldsymbol{\varepsilon}_{11}^{\circ} + \boldsymbol{\varepsilon}_{22}^{\circ}) \left(\frac{1}{\mathbf{r}_{12}} + \frac{1}{\mathbf{r}_{21}} \right) \right] \csc^2 \theta_{12} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

and the corresponding geodesic curvatures are

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

CONTINUED

$$\frac{1}{\rho_{11}} = \frac{1}{\rho_{11}} \left[1 + 2\varepsilon_{12}^{\circ} \cot\theta_{12} \csc\theta_{12} - \csc^2\theta_{12} (\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}) \right] + \frac{\csc\theta_{12}}{\mathbf{A}_1 \mathbf{A}_2} \left[\frac{\partial}{\partial \xi_1} (\mathbf{A}_2 [2\varepsilon_{12}^{\circ} - \varepsilon_{11}^{\circ} \cos\theta_{12}]) - \frac{\partial}{\partial \xi_2} [\mathbf{A}_1 \varepsilon_{11}^{\circ}] \right] + \mathcal{O}(\varepsilon^4)$$

and

$$\frac{1}{\rho_{22}} = \frac{1}{\rho_{22}} \left[1 + 2\varepsilon_{12}^{\circ} \cot\theta_{12} \csc\theta_{12} - \csc^2\theta_{12} (\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}) \right] - \frac{\csc\theta_{12}}{\mathbf{A}_1 \mathbf{A}_2} \left[\frac{\partial}{\partial \xi_2} (\mathbf{A}_1 [2\varepsilon_{12}^{\circ} - \cos\theta_{12} \varepsilon_{22}^{\circ}]) - \frac{\partial}{\partial \xi_1} [\mathbf{A}_2 \varepsilon_{22}^{\circ}] \right] + \mathcal{O}(\varepsilon^4)$$

- First, substituting these “small” strain approximations into the Gaussian curvature K_G of the deformed reference surface and using the identity $\frac{1}{r_{12}} + \frac{1}{r_{21}} = \cot\theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$ yields $K_G = K_G + K_G$, where the change in Gaussian curvature caused by deformation is given by

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

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$$\begin{aligned} K_G = & \frac{K_{22}^o}{r_{11}} + \frac{K_{11}^o}{r_{22}} + K_{12}^o \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) + \left[\frac{K_{22}^o}{r_{12}} - \frac{K_{11}^o}{r_{21}} - K_{12}^o \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) \right] \cot \theta_{12} \\ & + \left[2\varepsilon_{12}^o \cot^2 \theta_{12} - \csc^2 \theta_{12} (\varepsilon_{11}^o + \varepsilon_{22}^o) \right] \left(\frac{1}{r_{12}r_{22}} - \frac{1}{r_{21}r_{11}} \right) \cot \theta_{12} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

- In addition,

$$\begin{aligned} \mathcal{A}_1 \mathcal{A}_2 \sin \theta_{12} K_G = & \mathbf{A}_1 \mathbf{A}_2 \sin \theta_{12} (K_G + K_G) \\ & + \mathbf{A}_1 \mathbf{A}_2 \left[(\varepsilon_{11}^o + \varepsilon_{22}^o) \csc \theta_{12} - 2\varepsilon_{12}^o \cot \theta_{12} \right] K_G + \mathcal{O}(\varepsilon^4) \end{aligned}$$

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- Similarly,

$$\frac{\mathcal{A}_1}{\rho_{11}} = \frac{\mathbf{A}_1}{\rho_{11}} \left[1 + \epsilon_{11}^o + 2\epsilon_{12}^o \cot\theta_{12} \csc\theta_{12} - \csc^2\theta_{12} (\epsilon_{11}^o + \epsilon_{22}^o) \right] + \frac{\csc\theta_{12}}{\mathbf{A}_1 \mathbf{A}_2} \left[\frac{\partial}{\partial \xi_1} \left(\mathbf{A}_2 [2\epsilon_{12}^o - \epsilon_{11}^o \cos\theta_{12}] \right) - \frac{\partial}{\partial \xi_2} [\mathbf{A}_1 \epsilon_{11}^o] \right] + \mathcal{O}(\epsilon^4)$$

$$\frac{\mathcal{A}_2}{\rho_{22}} = \frac{\mathbf{A}_2}{\rho_{22}} \left[1 + \epsilon_{22}^o + 2\epsilon_{12}^o \cot\theta_{12} \csc\theta_{12} - \csc^2\theta_{12} (\epsilon_{11}^o + \epsilon_{22}^o) \right] - \frac{\csc\theta_{12}}{\mathbf{A}_1 \mathbf{A}_2} \left[\frac{\partial}{\partial \xi_2} \left(\mathbf{A}_1 [2\epsilon_{12}^o - \cos\theta_{12} \epsilon_{22}^o] \right) - \frac{\partial}{\partial \xi_1} [\mathbf{A}_2 \epsilon_{22}^o] \right] + \mathcal{O}(\epsilon^4)$$

- Thus, $\frac{\partial}{\partial \xi_2} \left(\frac{\mathcal{A}_1}{\rho_{11}} \right) - \frac{\partial}{\partial \xi_1} \left(\frac{\mathcal{A}_2}{\rho_{22}} \right) + \frac{\partial}{\partial \xi_1} \left(\frac{\partial \theta_{12}}{\partial \xi_2} \right) = \mathcal{A}_1 \mathcal{A}_2 \sin\theta_{12} K_G$ becomes

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$$\begin{aligned}
 & \frac{\partial}{\partial \xi_2} \left\{ \frac{\mathbf{A}_1}{\rho_{11}} \left[\epsilon_{11}^{\circ} + 2\epsilon_{12}^{\circ} \cot \theta_{12} \csc \theta_{12} - \csc^2 \theta_{12} (\epsilon_{11}^{\circ} + \epsilon_{22}^{\circ}) \right] \right\} \\
 & + \frac{\partial}{\partial \xi_2} \left\{ \frac{\csc \theta_{12}}{\mathbf{A}_2} \left[\frac{\partial}{\partial \xi_1} \left(\mathbf{A}_2 \left[2\epsilon_{12}^{\circ} - \epsilon_{11}^{\circ} \cos \theta_{12} \right] \right) - \frac{\partial}{\partial \xi_2} \left[\mathbf{A}_1 \epsilon_{11}^{\circ} \right] \right] \right\} \\
 & - \frac{\partial}{\partial \xi_1} \left\{ \frac{\mathbf{A}_2}{\rho_{22}} \left[\epsilon_{22}^{\circ} + 2\epsilon_{12}^{\circ} \cot \theta_{12} \csc \theta_{12} - \csc^2 \theta_{12} (\epsilon_{11}^{\circ} + \epsilon_{22}^{\circ}) \right] \right\} \\
 & + \frac{\partial}{\partial \xi_1} \left\{ \frac{\csc \theta_{12}}{\mathbf{A}_1} \left[\frac{\partial}{\partial \xi_2} \left(\mathbf{A}_1 \left[2\epsilon_{12}^{\circ} - \cos \theta_{12} \epsilon_{22}^{\circ} \right] \right) - \frac{\partial}{\partial \xi_1} \left[\mathbf{A}_2 \epsilon_{22}^{\circ} \right] \right] \right\} \\
 & + \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} \left[\left(\epsilon_{11}^{\circ} + \epsilon_{22}^{\circ} \right) \cot \theta_{12} - 2\epsilon_{12}^{\circ} \csc \theta_{12} \right] = \mathbf{A}_1 \mathbf{A}_2 \sin \theta_{12} K_G \\
 & \quad + \mathbf{A}_1 \mathbf{A}_2 \left[\left(\epsilon_{11}^{\circ} + \epsilon_{22}^{\circ} \right) \csc \theta_{12} - 2\epsilon_{12}^{\circ} \cot \theta_{12} \right] K_G + \mathcal{O}(\epsilon^4)
 \end{aligned}$$

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- Because of the complexity of this expression and, particularly, the other two corresponding compatibility equations that follow, it is convenient

to express
$$\frac{\partial}{\partial \xi_2} \left(\frac{\mathcal{A}_1}{\rho_{11}} \right) - \frac{\partial}{\partial \xi_1} \left(\frac{\mathcal{A}_2}{\rho_{22}} \right) + \frac{\partial}{\partial \xi_1} \left(\frac{\partial \theta_{12}}{\partial \xi_2} \right) = \mathcal{A}_1 \mathcal{A}_2 \sin \theta_{12} K_G \quad \text{as}$$

$$\mathcal{C}_{11}(\boldsymbol{\varepsilon}_{11}^\circ) + \mathcal{C}_{12}(\boldsymbol{\varepsilon}_{22}^\circ) + \mathcal{C}_{13}(\boldsymbol{\varepsilon}_{12}^\circ) + \mathcal{C}_{14}(\mathbf{K}_{11}^\circ) + \mathcal{C}_{15}(\mathbf{K}_{22}^\circ) + \mathcal{C}_{16}(\mathbf{K}_{12}^\circ) = 0$$

where

$$\begin{aligned} \mathcal{C}_{11}(\boldsymbol{\varepsilon}_{11}^\circ) = & \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\rho_{22}} \csc^2 \theta_{12} \boldsymbol{\varepsilon}_{11}^\circ \right) - \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\rho_{11}} \cot^2 \theta_{12} \boldsymbol{\varepsilon}_{11}^\circ \right) \\ & - \frac{\partial}{\partial \xi_2} \left(\frac{\csc \theta_{12}}{\mathbf{A}_2} \left[\frac{\partial}{\partial \xi_1} (\mathbf{A}_2 \cos \theta_{12} \boldsymbol{\varepsilon}_{11}^\circ) + \frac{\partial}{\partial \xi_2} (\mathbf{A}_1 \boldsymbol{\varepsilon}_{11}^\circ) \right] \right) + \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} (\cot \theta_{12} \boldsymbol{\varepsilon}_{11}^\circ) \\ & - \left(\frac{1}{\mathbf{r}_{11} \mathbf{r}_{22}} + \frac{1}{\mathbf{r}_{12} \mathbf{r}_{21}} \right) \mathbf{A}_1 \mathbf{A}_2 \csc \theta_{12} \boldsymbol{\varepsilon}_{11}^\circ + \mathcal{O}(\varepsilon^4) \end{aligned}$$

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$$\begin{aligned} \mathcal{C}_{12}(\boldsymbol{\varepsilon}_{22}^{\circ}) &= \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\rho_{22}} \cot^2 \theta_{12} \boldsymbol{\varepsilon}_{22}^{\circ} \right) - \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\rho_{11}} \csc^2 \theta_{12} \boldsymbol{\varepsilon}_{22}^{\circ} \right) \\ &\quad - \frac{\partial}{\partial \xi_1} \left(\frac{\csc \theta_{12}}{\mathbf{A}_1} \left[\frac{\partial}{\partial \xi_2} (\mathbf{A}_1 \cos \theta_{12} \boldsymbol{\varepsilon}_{22}^{\circ}) + \frac{\partial}{\partial \xi_1} (\mathbf{A}_2 \boldsymbol{\varepsilon}_{22}^{\circ}) \right] \right) + \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} (\cot \theta_{12} \boldsymbol{\varepsilon}_{22}^{\circ}) \\ &\quad - \left(\frac{1}{\mathbf{r}_{11} \mathbf{r}_{22}} + \frac{1}{\mathbf{r}_{12} \mathbf{r}_{21}} \right) \mathbf{A}_1 \mathbf{A}_2 \csc \theta_{12} \boldsymbol{\varepsilon}_{22}^{\circ} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\mathcal{C}_{14}(\mathbf{K}_{11}^{\circ}) = -\mathbf{A}_1 \mathbf{A}_2 \left(\frac{\sin \theta_{12}}{\mathbf{r}_{22}} - \frac{\cos \theta_{12}}{\mathbf{r}_{21}} \right) \mathbf{K}_{11}^{\circ} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{15}(\mathbf{K}_{22}^{\circ}) = -\mathbf{A}_1 \mathbf{A}_2 \left(\frac{\sin \theta_{12}}{\mathbf{r}_{11}} + \frac{\cos \theta_{12}}{\mathbf{r}_{12}} \right) \mathbf{K}_{22}^{\circ} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{16}(\mathbf{K}_{12}^{\circ}) = -\mathbf{A}_1 \mathbf{A}_2 \left[\left(\frac{1}{\mathbf{r}_{12}} - \frac{1}{\mathbf{r}_{21}} \right) \sin \theta_{12} - \left(\frac{1}{\mathbf{r}_{11}} + \frac{1}{\mathbf{r}_{22}} \right) \cos \theta_{12} \right] \mathbf{K}_{12}^{\circ} + \mathcal{O}(\varepsilon^4)$$

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$$\begin{aligned}
 \mathcal{C}_{13}(\boldsymbol{\varepsilon}_{12}^{\circ}) &= \frac{\partial}{\partial \xi_1} \left(\frac{\csc \theta_{12}}{\mathbf{A}_1} \frac{\partial}{\partial \xi_2} (\mathbf{A}_1 \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ}) \right) - \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\rho_{22}} \cot \theta_{12} \csc \theta_{12} \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ} \right) \\
 &\quad + \frac{\partial}{\partial \xi_2} \left(\frac{\csc \theta_{12}}{\mathbf{A}_2} \frac{\partial}{\partial \xi_1} (\mathbf{A}_2 \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ}) \right) + \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\rho_{11}} \cot \theta_{12} \csc \theta_{12} \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ} \right) \\
 &\quad - \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} (\csc \theta_{12} \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ}) + \mathbf{A}_1 \mathbf{A}_2 (1 - \cos \theta_{12}) \cot^2 \theta_{12} \left(\frac{1}{\mathbf{r}_{12} \mathbf{r}_{22}} - \frac{1}{\mathbf{r}_{21} \mathbf{r}_{11}} \right) \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ} \\
 &\quad + \mathbf{A}_1 \mathbf{A}_2 \cot \theta_{12} \left(\frac{1}{\mathbf{r}_{11} \mathbf{r}_{22}} + \frac{1}{\mathbf{r}_{12} \mathbf{r}_{21}} \right) \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ} + \mathcal{O}(\varepsilon^4)
 \end{aligned}$$

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- Similarly, the compatibility equation

$$\begin{aligned} & \cos\theta_{12} \frac{1}{\mathcal{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{22}} \right) + \sin\theta_{12} \frac{1}{\mathcal{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{21}} \right) - \frac{1}{\mathcal{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{11}} \right) \\ & + \csc\theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \left(\frac{1}{\rho_{11}} + \frac{1}{\mathcal{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) + \frac{2}{r_{12}} \left(-\frac{1}{\rho_{22}} + \frac{1}{\mathcal{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) = 0 \end{aligned}$$

becomes

$$\mathcal{C}_{21}(\epsilon_{11}^\circ) + \mathcal{C}_{22}(\epsilon_{22}^\circ) + \mathcal{C}_{23}(\epsilon_{12}^\circ) + \mathcal{C}_{24}(\kappa_{11}^\circ) + \mathcal{C}_{25}(\kappa_{22}^\circ) + \mathcal{C}_{26}(\kappa_{12}^\circ) = 0$$

and the compatibility equation

$$\begin{aligned} & \frac{1}{\mathcal{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{22}} \right) + \sin\theta_{12} \frac{1}{\mathcal{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{12}} \right) - \cos\theta_{12} \frac{1}{\mathcal{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{11}} \right) \\ & + \frac{2}{r_{21}} \left(\frac{1}{\rho_{11}} + \frac{1}{\mathcal{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) + \csc\theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \left(-\frac{1}{\rho_{22}} + \frac{1}{\mathcal{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) = 0 \end{aligned}$$

becomes

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$$\mathcal{C}_{31}(\boldsymbol{\varepsilon}_{11}^{\circ}) + \mathcal{C}_{32}(\boldsymbol{\varepsilon}_{22}^{\circ}) + \mathcal{C}_{33}(\boldsymbol{\varepsilon}_{12}^{\circ}) + \mathcal{C}_{34}(\mathbf{K}_{11}^{\circ}) + \mathcal{C}_{35}(\mathbf{K}_{22}^{\circ}) + \mathcal{C}_{36}(\mathbf{K}_{12}^{\circ}) = 0$$

where

$$\begin{aligned} \mathcal{C}_{21}(\boldsymbol{\varepsilon}_{11}^{\circ}) = & - \left[\frac{\csc\theta_{12}}{2} \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \right] \frac{1}{\mathbf{A}_1} \frac{\partial \boldsymbol{\varepsilon}_{11}^{\circ}}{\partial \xi_1} - \left[\left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc^2\theta_{12} - \frac{2\cot\theta_{12}}{r_{12}} \right] \frac{1}{\mathbf{A}_2} \frac{\partial \boldsymbol{\varepsilon}_{11}^{\circ}}{\partial \xi_2} \\ & - \csc\theta_{12} \left[\frac{1}{2\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) + \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{22}} \right) \sin 2\theta_{12} - \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{21}} \right) \cos 2\theta_{12} \right] \boldsymbol{\varepsilon}_{11}^{\circ} \\ & + \csc\theta_{12} \left[\left(-\frac{1}{\rho_{22}} + \frac{1}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \csc\theta_{12} + \frac{1}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \cot\theta_{12} \right] \boldsymbol{\varepsilon}_{11}^{\circ} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

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$$\begin{aligned}
 \mathcal{C}_{22}(\epsilon_{22}^{\circ}) = & \frac{\csc\theta_{12}}{2} \left[\frac{1}{r_{21}} - \frac{3}{r_{12}} \right] \frac{1}{A_1} \frac{\partial \epsilon_{22}^{\circ}}{\partial \xi_1} \\
 & + \frac{\csc\theta_{12}}{2} \left[\frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{21}} \right) \cos 2\theta_{12} - \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{12}} \right) \right] \epsilon_{22}^{\circ} \\
 & + \left[\left(\frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{11}} \right) - \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{22}} \right) \cos \theta_{12} \right) - \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \frac{\cot \theta_{12} \csc \theta_{12}}{\rho_{11}} \right] \epsilon_{22}^{\circ} \\
 & - \csc \theta_{12} \left(\frac{1}{\rho_{11}} + \frac{1}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) \left[\left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc^2 \theta_{12} - \frac{4 \cot \theta_{12}}{r_{12}} \right] \epsilon_{22}^{\circ} \\
 & - \cot \theta_{12} \left(-\frac{1}{\rho_{22}} + \frac{1}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \left[\left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc^2 \theta_{12} - \frac{2 \cot \theta_{12}}{r_{12}} \right] \epsilon_{22}^{\circ} + \mathcal{O}(\epsilon^4)
 \end{aligned}$$

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$$\begin{aligned} \mathcal{C}_{23}(\boldsymbol{\varepsilon}_{12}^{\circ}) = & \left[\frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc^2 \theta_{12} + \frac{2}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{22}} \right) - 2 \cot \theta_{12} \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{21}} \right) \right] \boldsymbol{\varepsilon}_{12}^{\circ} \\ & + \csc^2 \theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \frac{1}{\mathbf{A}_1} \frac{\partial \boldsymbol{\varepsilon}_{12}^{\circ}}{\partial \xi_1} \\ & - \csc^2 \theta_{12} \left[\left(\frac{1}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) \left(\frac{1}{r_{21}} + \frac{5}{r_{12}} \right) + \frac{2}{\rho_{11}} \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) \right] \boldsymbol{\varepsilon}_{12}^{\circ} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\begin{aligned} \mathcal{C}_{24}(\mathbf{K}_{11}^{\circ}) = & \frac{\cos \theta_{12}}{2} \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{K}_{11}^{\circ}}{\partial \xi_1} - \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{K}_{11}^{\circ}}{\partial \xi_2} \\ & + \left[\csc \theta_{12} \left(\frac{1}{\rho_{11}} + \frac{1}{2} \frac{1}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) + \cot \theta_{12} \left(-\frac{1}{\rho_{22}} + \frac{1}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \right] \mathbf{K}_{11}^{\circ} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

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$$\mathcal{C}_{25}(\mathbf{K}_{22}^{\circ}) = \frac{\cos\theta_{12}}{2} \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{K}_{22}^{\circ}}{\partial \xi_1} - \left[\csc\theta_{12} \left(\frac{1}{\rho_{11}} + \frac{1}{2} \frac{1}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) + \cot\theta_{12} \left(-\frac{1}{\rho_{22}} + \frac{1}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \right] \mathbf{K}_{22}^{\circ} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{26}(\mathbf{K}_{12}^{\circ}) = \sin\theta_{12} \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{K}_{12}^{\circ}}{\partial \xi_1} - 2 \left(-\frac{1}{\rho_{22}} + \frac{1}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \mathbf{K}_{12}^{\circ} + \mathcal{O}(\varepsilon^4)$$

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$$\begin{aligned}
 \mathcal{C}_{31}(\boldsymbol{\varepsilon}_{11}^{\circ}) &= \frac{\csc\theta_{12}}{2} \left[\frac{3}{r_{21}} - \frac{1}{r_{12}} \right] \frac{1}{\mathbf{A}_2} \frac{\partial \boldsymbol{\varepsilon}_{11}^{\circ}}{\partial \xi_2} \\
 &+ \frac{\csc\theta_{12}}{2} \left[\frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{21}} \right) - \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{12}} \right) \cos 2\theta_{12} \right] \boldsymbol{\varepsilon}_{11}^{\circ} \\
 &+ \left[\frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{22}} \right) - \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{11}} \right) \cos \theta_{12} - \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \frac{\cot \theta_{12} \csc \theta_{12}}{\rho_{22}} \right] \boldsymbol{\varepsilon}_{11}^{\circ} \\
 &- \cot \theta_{12} \left(\frac{1}{\rho_{11}} + \frac{1}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) \left[\left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) \csc^2 \theta_{12} + \frac{2 \cot \theta_{12}}{r_{21}} \right] \boldsymbol{\varepsilon}_{11}^{\circ} \\
 &- \csc \theta_{12} \left(-\frac{1}{\rho_{22}} + \frac{1}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \left[\left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) \csc^2 \theta_{12} + \frac{4 \cot \theta_{12}}{r_{21}} \right] \boldsymbol{\varepsilon}_{11}^{\circ} + \mathcal{O}(\varepsilon^4)
 \end{aligned}$$

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$$\begin{aligned} \mathcal{C}_{32}(\boldsymbol{\varepsilon}_{22}^{\circ}) = & - \left[\left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) \csc^2 \theta_{12} + \frac{2 \cot \theta_{12}}{r_{21}} \right] \frac{1}{A_1} \frac{\partial \boldsymbol{\varepsilon}_{22}^{\circ}}{\partial \xi_1} + \left[\frac{\csc \theta_{12}}{2} \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \right] \frac{1}{A_2} \frac{\partial \boldsymbol{\varepsilon}_{22}^{\circ}}{\partial \xi_2} \\ & - \csc \theta_{12} \left[\frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{11}} \right) \sin 2\theta_{12} + \frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{12}} \right) \cos 2\theta_{12} - \frac{1}{2A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \right] \boldsymbol{\varepsilon}_{22}^{\circ} \\ & + \csc \theta_{12} \left[\left(\frac{1}{\rho_{11}} + \frac{1}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \csc \theta_{12} - \frac{1}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \cot \theta_{12} \right] \boldsymbol{\varepsilon}_{22}^{\circ} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\begin{aligned} \mathcal{C}_{34}(\boldsymbol{\kappa}_{11}^{\circ}) = & \frac{\cos \theta_{12}}{2} \frac{1}{A_2} \frac{\partial \boldsymbol{\kappa}_{11}^{\circ}}{\partial \xi_2} \\ & - \left[\cot \theta_{12} \left(\frac{1}{\rho_{11}} + \frac{1}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) + \csc \theta_{12} \left(-\frac{1}{\rho_{22}} + \frac{1}{2} \frac{1}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \right] \boldsymbol{\kappa}_{11}^{\circ} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

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CONTINUED

$$\begin{aligned} \mathcal{C}_{33}(\boldsymbol{\varepsilon}_{12}^{\circ}) = & \left[\frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{\mathbf{r}_{22}} - \frac{1}{\mathbf{r}_{11}} \right) \csc^2 \theta_{12} + \frac{2}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{\mathbf{r}_{11}} \right) + 2 \cot \theta_{12} \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{\mathbf{r}_{12}} \right) \right] \boldsymbol{\varepsilon}_{12}^{\circ} \\ & + \csc^2 \theta_{12} \left(\frac{1}{\mathbf{r}_{22}} - \frac{1}{\mathbf{r}_{11}} \right) \frac{1}{\mathbf{A}_2} \frac{\partial \boldsymbol{\varepsilon}_{12}^{\circ}}{\partial \xi_2} \\ & + \csc^2 \theta_{12} \left[\left(\frac{1}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \left(\frac{1}{\mathbf{r}_{12}} + \frac{5}{\mathbf{r}_{12}} \right) + \frac{2}{\rho_{22}} \left(\frac{1}{\mathbf{r}_{12}} - \frac{1}{\mathbf{r}_{21}} \right) \right] \boldsymbol{\varepsilon}_{12}^{\circ} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\begin{aligned} \mathcal{C}_{35}(\mathbf{K}_{22}^{\circ}) = & - \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{K}_{22}^{\circ}}{\partial \xi_1} + \frac{\cos \theta_{12}}{2} \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{K}_{22}^{\circ}}{\partial \xi_2} \\ & + \left[\cot \theta_{12} \left(\frac{1}{\rho_{11}} + \frac{1}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) + \csc \theta_{12} \left(- \frac{1}{\rho_{22}} + \frac{1}{2} \frac{1}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \right] \mathbf{K}_{22}^{\circ} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

CONTINUED

$$\mathcal{C}_{36}(\mathbf{K}_{12}^{\circ}) = \sin\theta_{12} \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{K}_{12}^{\circ}}{\partial \xi_2} - 2 \left(\frac{1}{\rho_{11}} + \frac{1}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) \mathbf{K}_{12}^{\circ} + \mathcal{O}(\varepsilon^4)$$

- In obtaining these expressions for the **differential compatibility operators**, $\mathcal{C}_{ij}(\)$, the following expressions were used

$$\frac{\partial \mathbf{A}_2}{\partial \xi_1} = \mathbf{A}_1 \mathbf{A}_2 \left(\frac{\csc\theta_{12}}{\rho_{22}} - \frac{\cot\theta_{12}}{\rho_{11}} \right) - \left(\frac{\cot\theta_{12}}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} + \frac{\csc\theta_{12}}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \csc\theta_{12}$$

$$\frac{\partial \mathbf{A}_1}{\partial \xi_2} = \mathbf{A}_1 \mathbf{A}_2 \left(\frac{\cot\theta_{12}}{\rho_{22}} - \frac{\csc\theta_{12}}{\rho_{11}} \right) - \left(\frac{\csc\theta_{12}}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} + \frac{\cot\theta_{12}}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \csc\theta_{12}$$

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

CONTINUED

- For orthogonal reference-surface Gaussian coordinates, the **Gaussian curvature reduces to** $K_G = \frac{1}{r_{11}r_{22}} - \frac{1}{r_{12}^2}$ **and the corresponding change**

due to deformation becomes
$$K_G = \frac{K_{22}^o}{r_{11}} + \frac{K_{11}^o}{r_{22}} + \frac{2K_{12}^o}{r_{12}} + \mathcal{O}(\varepsilon^4)$$

- Likewise, for orthogonal reference-surface Gaussian coordinates, the differential operators reduce to:

$$\begin{aligned} \mathcal{C}_{11}(\boldsymbol{\varepsilon}_{11}^o) = & \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\rho_{22}} \boldsymbol{\varepsilon}_{11}^o \right) - \frac{\partial}{\partial \xi_2} \left(\frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} (\mathbf{A}_1 \boldsymbol{\varepsilon}_{11}^o) \right) \\ & - \left(\frac{1}{r_{11}r_{22}} - \frac{1}{(r_{12})^2} \right) \mathbf{A}_1 \mathbf{A}_2 \boldsymbol{\varepsilon}_{11}^o + \mathcal{O}(\varepsilon^4) \end{aligned}$$

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

CONTINUED

$$\begin{aligned} \mathcal{C}_{12}(\boldsymbol{\varepsilon}_{22}^{\circ}) = & -\frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\rho_{11}} \boldsymbol{\varepsilon}_{22}^{\circ} \right) - \frac{\partial}{\partial \xi_1} \left(\frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} (\mathbf{A}_2 \boldsymbol{\varepsilon}_{22}^{\circ}) \right) \\ & - \left(\frac{1}{\mathbf{r}_{11} \mathbf{r}_{22}} - \frac{1}{(\mathbf{r}_{12})^2} \right) \mathbf{A}_1 \mathbf{A}_2 \boldsymbol{\varepsilon}_{22}^{\circ} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\begin{aligned} \mathcal{C}_{13}(\boldsymbol{\varepsilon}_{12}^{\circ}) = & \frac{\partial}{\partial \xi_1} \left(\frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_2} (\mathbf{A}_1 2\boldsymbol{\varepsilon}_{12}^{\circ}) \right) + \frac{\partial}{\partial \xi_2} \left(\frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_1} (\mathbf{A}_2 2\boldsymbol{\varepsilon}_{12}^{\circ}) \right) \\ & - \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} (2\boldsymbol{\varepsilon}_{12}^{\circ}) + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\mathcal{C}_{14}(\mathbf{K}_{11}^{\circ}) = -\frac{\mathbf{A}_1 \mathbf{A}_2}{\mathbf{r}_{22}} \mathbf{K}_{11}^{\circ} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{15}(\mathbf{K}_{22}^{\circ}) = -\frac{\mathbf{A}_1 \mathbf{A}_2}{\mathbf{r}_{11}} \mathbf{K}_{22}^{\circ} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{16}(\mathbf{K}_{12}^{\circ}) = -\frac{2\mathbf{A}_1 \mathbf{A}_2}{\mathbf{r}_{12}} \mathbf{K}_{12}^{\circ} + \mathcal{O}(\varepsilon^4)$$

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

CONTINUED

$$\mathcal{C}_{21}(\boldsymbol{\varepsilon}_{11}^{\circ}) = - \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \frac{1}{\mathbf{A}_2} \frac{\partial \boldsymbol{\varepsilon}_{11}^{\circ}}{\partial \xi_2} + \frac{\boldsymbol{\varepsilon}_{11}^{\circ}}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{12}} \right) + \frac{1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1} \left(\frac{2}{r_{12}} \right) \boldsymbol{\varepsilon}_{11}^{\circ} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{22}(\boldsymbol{\varepsilon}_{22}^{\circ}) = \frac{\boldsymbol{\varepsilon}_{22}^{\circ}}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{11}} \right) - \frac{2}{r_{12}} \frac{1}{\mathbf{A}_1} \frac{\partial \boldsymbol{\varepsilon}_{22}^{\circ}}{\partial \xi_1} + \frac{\boldsymbol{\varepsilon}_{22}^{\circ}}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{23}(\boldsymbol{\varepsilon}_{12}^{\circ}) = \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) \boldsymbol{\varepsilon}_{12}^{\circ} + \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \frac{1}{\mathbf{A}_1} \frac{\partial \boldsymbol{\varepsilon}_{12}^{\circ}}{\partial \xi_1} + \frac{1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} \frac{4 \boldsymbol{\varepsilon}_{12}^{\circ}}{r_{12}} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{24}(\mathbf{K}_{11}^{\circ}) = - \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{K}_{11}^{\circ}}{\partial \xi_2} - \frac{1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} \mathbf{K}_{11}^{\circ} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{25}(\mathbf{K}_{22}^{\circ}) = \frac{1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} \mathbf{K}_{22}^{\circ} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{26}(\mathbf{K}_{12}^{\circ}) = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{K}_{12}^{\circ}}{\partial \xi_1} + \frac{2}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1} \mathbf{K}_{12}^{\circ} + \mathcal{O}(\varepsilon^4)$$

“SMALL-STRAIN” COMPATIBILITY CONDITIONS FOR $\mathcal{S}_t$

CONCLUDED

$$\mathcal{C}_{31}(\boldsymbol{\varepsilon}_{11}^{\circ}) = \left[\frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{\mathbf{r}_{22}} \right) \right] \boldsymbol{\varepsilon}_{11}^{\circ} - \frac{2}{\mathbf{r}_{12}} \frac{1}{\mathbf{A}_2} \frac{\partial \boldsymbol{\varepsilon}_{11}^{\circ}}{\partial \xi_2} + \frac{1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1} \left[\left(\frac{1}{\mathbf{r}_{22}} - \frac{1}{\mathbf{r}_{11}} \right) \right] \boldsymbol{\varepsilon}_{11}^{\circ} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{32}(\boldsymbol{\varepsilon}_{22}^{\circ}) = - \left(\frac{1}{\mathbf{r}_{22}} - \frac{1}{\mathbf{r}_{11}} \right) \frac{1}{\mathbf{A}_1} \frac{\partial \boldsymbol{\varepsilon}_{22}^{\circ}}{\partial \xi_1} + \frac{\boldsymbol{\varepsilon}_{22}^{\circ}}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{\mathbf{r}_{12}} \right) + \frac{1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} \frac{2 \boldsymbol{\varepsilon}_{22}^{\circ}}{\mathbf{r}_{12}} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{33}(\boldsymbol{\varepsilon}_{12}^{\circ}) = \left[\frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{\mathbf{r}_{22}} + \frac{1}{\mathbf{r}_{11}} \right) \right] \boldsymbol{\varepsilon}_{12}^{\circ} + \left(\frac{1}{\mathbf{r}_{22}} - \frac{1}{\mathbf{r}_{11}} \right) \frac{1}{\mathbf{A}_2} \frac{\partial \boldsymbol{\varepsilon}_{12}^{\circ}}{\partial \xi_2} + \frac{1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1} \frac{4 \boldsymbol{\varepsilon}_{12}^{\circ}}{\mathbf{r}_{12}} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{34}(\mathbf{K}_{11}^{\circ}) = \frac{1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1} \mathbf{K}_{11}^{\circ} + \mathcal{O}(\varepsilon^4)$$

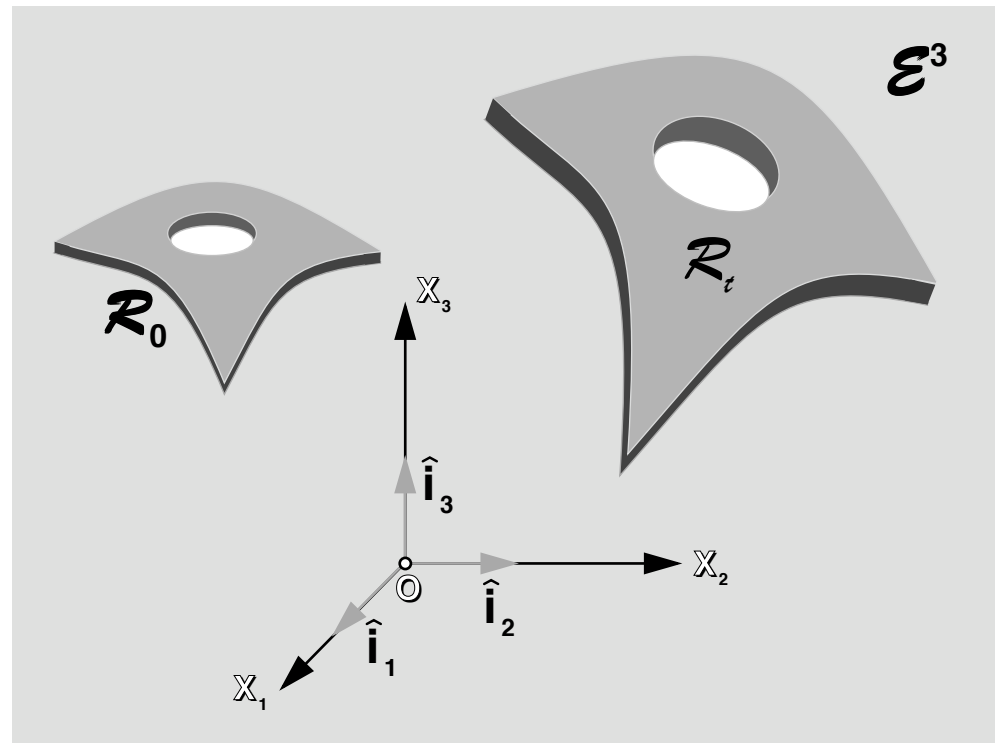
$$\mathcal{C}_{35}(\mathbf{K}_{22}^{\circ}) = - \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{K}_{22}^{\circ}}{\partial \xi_1} - \frac{1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1} \mathbf{K}_{22}^{\circ} + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{C}_{36}(\mathbf{K}_{12}^{\circ}) = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{K}_{12}^{\circ}}{\partial \xi_2} + \frac{2}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} \mathbf{K}_{12}^{\circ} + \mathcal{O}(\varepsilon^4)$$

MATHEMATICAL DESCRIPTION OF DEFORMED-SHELL GEOMETRY

MATHEMATICAL DESCRIPTION OF THE DEFORMED SHELL, $\mathcal{R}_t$

- As discussed previously, a shell structure is described mathematically as a set of points (material particles), $\mathbf{B}$, that occupy a region, $\mathcal{R}_t$, of **three-dimensional Euclidean space** $\mathcal{E}^3$ at time t
- The time $t = 0$ is defined as the **reference configuration or reference state** of the shell, which occupies $\mathcal{R}_0 \subset \mathcal{E}^3$
- In the **reference configuration**, or state, the shell is presumed to be **undeformed and unstressed**



MATHEMATICAL DESCRIPTION OF THE DEFORMED SHELL, $\mathcal{R}_t$ - CONTINUED

- In addition, the undeformed shell is modelled as a set of **contiguous parallel surfaces** that fill $\mathcal{R}_0 \subset \mathcal{E}^3$
- Recall that **points of the shell in the reference state** are described parametrically by the Cartesian coordinates $\mathbf{X}_k = \mathbf{X}_k(\xi_1, \xi_2, \xi_3)$, where the parameters ξ_1 , ξ_2 , and ξ_3 define a system of curvilinear coordinates for the region $\mathcal{R}_0 \subset \mathcal{E}^3$
- The coordinates $\mathbf{x}_k = \mathbf{X}_k(\xi_1, \xi_2, 0)$ define points of the corresponding reference surface at time $t = 0$
- Values of $\mathbf{c}_3^-(\xi_1, \xi_2) \leq \xi_3 \leq \mathbf{c}_3^+(\xi_1, \xi_2)$ define the set of **contiguous parallel surfaces**

MATHEMATICAL DESCRIPTION OF THE DEFORMED SHELL, $\mathcal{R}_t$ - CONTINUED

- During deformation, the set of material points initially in $\mathcal{R}_0$ occupy a region, $\mathcal{R}_t \subset \mathcal{E}^3$, at time t
- Material points initially in the reference surface $\mathcal{S}_0 \subset \mathcal{R}_0$ occupy a region, $\mathcal{S}_t \subset \mathcal{R}_t$, at time t
- Material points initially in the parallel surface $\mathcal{S}_0(\xi_3) \subset \mathcal{R}_0$ occupy a region, $\mathcal{S}_t(\xi_3) \subset \mathcal{R}_t$, at time t
- Let points of $\mathcal{R}_t$ have the Cartesian coordinates (x_1, x_2, x_3) , with respect to the $\mathbf{O} - \mathbf{X}_1 - \mathbf{X}_2 - \mathbf{X}_3$ coordinate frame shown in the previous figure
- Points of $\mathcal{S}_t$ have the Cartesian coordinates (x_1, x_2, x_3)

MATHEMATICAL DESCRIPTION OF THE DEFORMED SHELL, $\mathcal{R}_t$ - CONTINUED

- By using the **Lagrangian description** of motion and deformation, the coordinates $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3)$ of a generic point $\mathcal{P} \in \mathcal{R}_t$ are expressed in terms of the coordinates (X_1, X_2, X_3) in the undeformed configuration of the corresponding point $P \in \mathcal{R}_0$; that is, $\mathcal{X}_k = \mathcal{X}_k(X_1, X_2, X_3, t)$
- Thus, the parametric representation of $\mathcal{R}_0$ given by $X_k = X_k(\xi_1, \xi_2, \xi_3)$ induces the parametric representation for $\mathcal{R}_t$ given by

$$\mathcal{X}_k = \mathcal{X}_k(\xi_1, \xi_2, \xi_3, t) \quad \text{such that} \quad \mathcal{X}_k(\xi_1, \xi_2, \xi_3, 0) = X_k(\xi_1, \xi_2, \xi_3)$$

- These equations constitute a parametric representation in which the undeformed, unstressed material region $\mathcal{R}_0$ is **transformed into the deformed image $\mathcal{R}_t$**

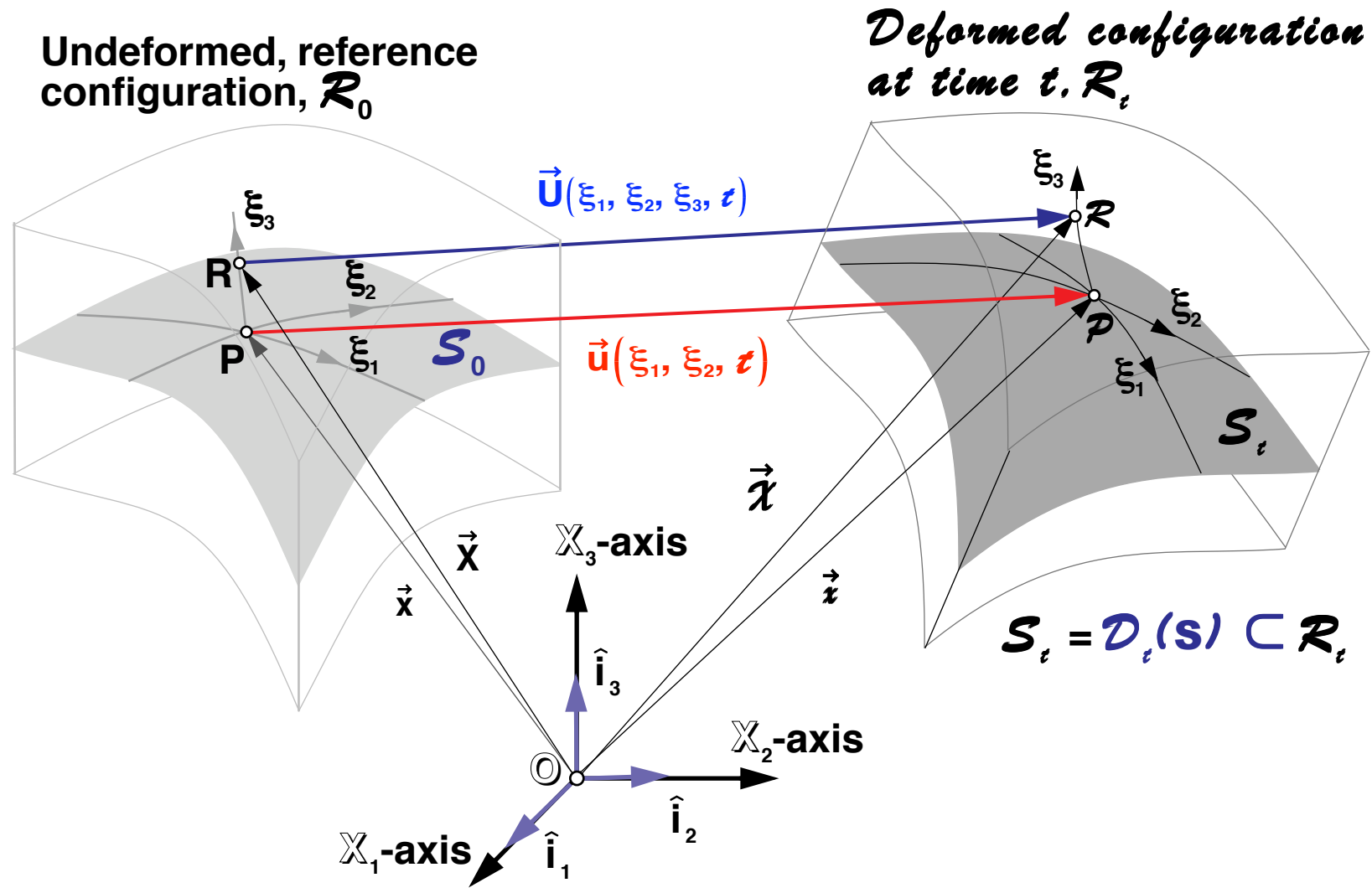
MATHEMATICAL DESCRIPTION OF THE DEFORMED SHELL, $\mathcal{R}_t$ - CONTINUED

- Thus, the curvilinear-coordinate net that defines points of $\mathcal{R}_0$ deforms into a curvilinear-coordinate net that defines points of $\mathcal{R}_t$
- The convection of the set of points $\mathcal{R}_0$ through $\mathcal{E}^3$ is represented mathematically by a continuous family of **deformation mappings** $\mathcal{D}_t()$, in which the set $\mathcal{R}_0$ is mapped onto the set $\mathcal{R}_t$ in a one-to-one manner at every given instant of time
- The **convected nature** of the curvilinear coordinates enables the definition of vector fields associated with points of the deformed configuration $\mathcal{R}_t$ that facilitate characterization of its deformation

MATHEMATICAL DESCRIPTION OF THE DEFORMED SHELL, $\mathcal{R}_t$ - CONTINUED

- To characterize the geometry of the deformed region $\mathcal{R}_t$, it is convenient to define the position vector $\vec{\mathcal{X}} = \mathcal{X}_k(\xi_1, \xi_2, \xi_3, t) \hat{i}_k$, with respect to the same fixed coordinate frame used to define $\vec{X} = X_k(\xi_1, \xi_2, \xi_3) \hat{i}_k$
- The position vector $\vec{\mathcal{X}}$ locates the deformed image, $\mathcal{P} = \mathcal{D}_t(P) \in \mathcal{R}_t$, of a generic point, $P \in \mathcal{R}_0$, at time t
- The **displacement** of points of the reference configuration are described by the displacement-vector field $\vec{U}(\xi_1, \xi_2, \xi_3, t)$ such that $\vec{\mathcal{X}} = \vec{X} + \vec{U}$, $\vec{U}(\xi_1, \xi_2, \xi_3, 0) = \vec{0}$, and $\vec{U}(\xi_1, \xi_2, 0, t) = \vec{u}(\xi_1, \xi_2, t)$
- Here, $\vec{u}(\xi_1, \xi_2, t)$ is the displacement-vector field associated with points of the undeformed reference surface $\mathcal{S}_0 \subset \mathcal{R}_0$

MATHEMATICAL DESCRIPTION OF THE DEFORMED SHELL, $\mathcal{R}_t$ - CONCLUDED



CONVECTED BASE-VECTOR FIELDS FOR $\mathcal{R}_t$

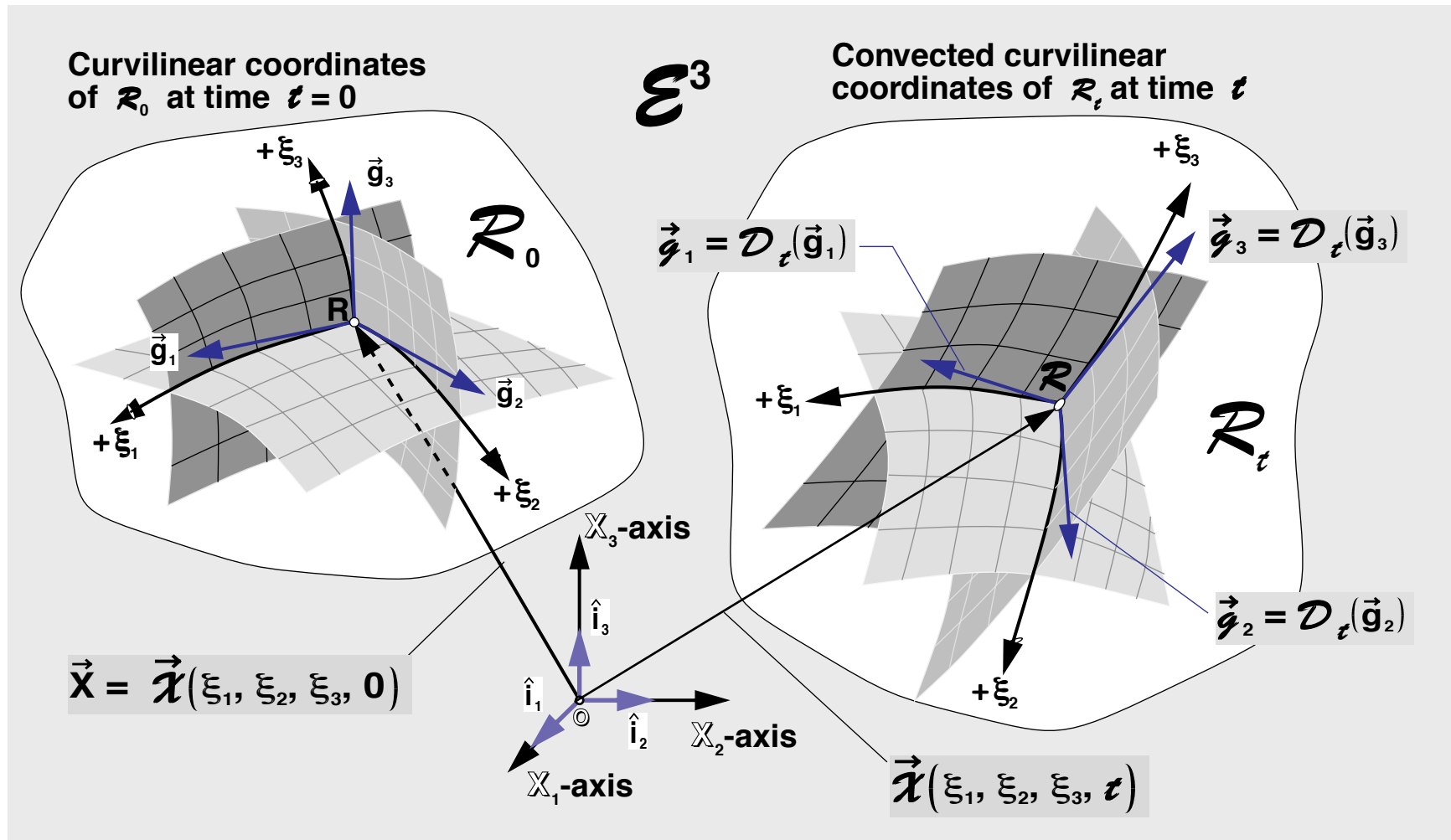
- To characterize the deformation of a shell, vector fields associated with points of the deformed configuration $\mathcal{R}_t$ are needed
- Moreover, to describe and manipulate these vector fields, a corresponding set of base-vector fields is also needed
- By **direct analogy** with the curvilinear coordinates of the undeformed configuration $\mathcal{R}_0$, the **natural base-vector fields** of the convected curvilinear coordinate system (ξ_1, ξ_2, ξ_3) are given by

$$\vec{\mathcal{g}}_k(\xi_1, \xi_2, \xi_3, t) \equiv \frac{\partial \vec{\mathcal{X}}}{\partial \xi_k}, \text{ where } \vec{\mathcal{g}}_k(\xi_1, \xi_2, \xi_3, 0) = \vec{\mathbf{g}}_k(\xi_1, \xi_2, \xi_3)$$

- Each of the base vectors $\{\vec{\mathcal{g}}_1, \vec{\mathcal{g}}_2, \vec{\mathcal{g}}_3\}$ is tangent to one of the convected curvilinear-coordinate curves at a given point of the deformed shell, as depicted in the following figure

CONVECTED BASE-VECTOR FIELDS FOR $\mathcal{R}_t$

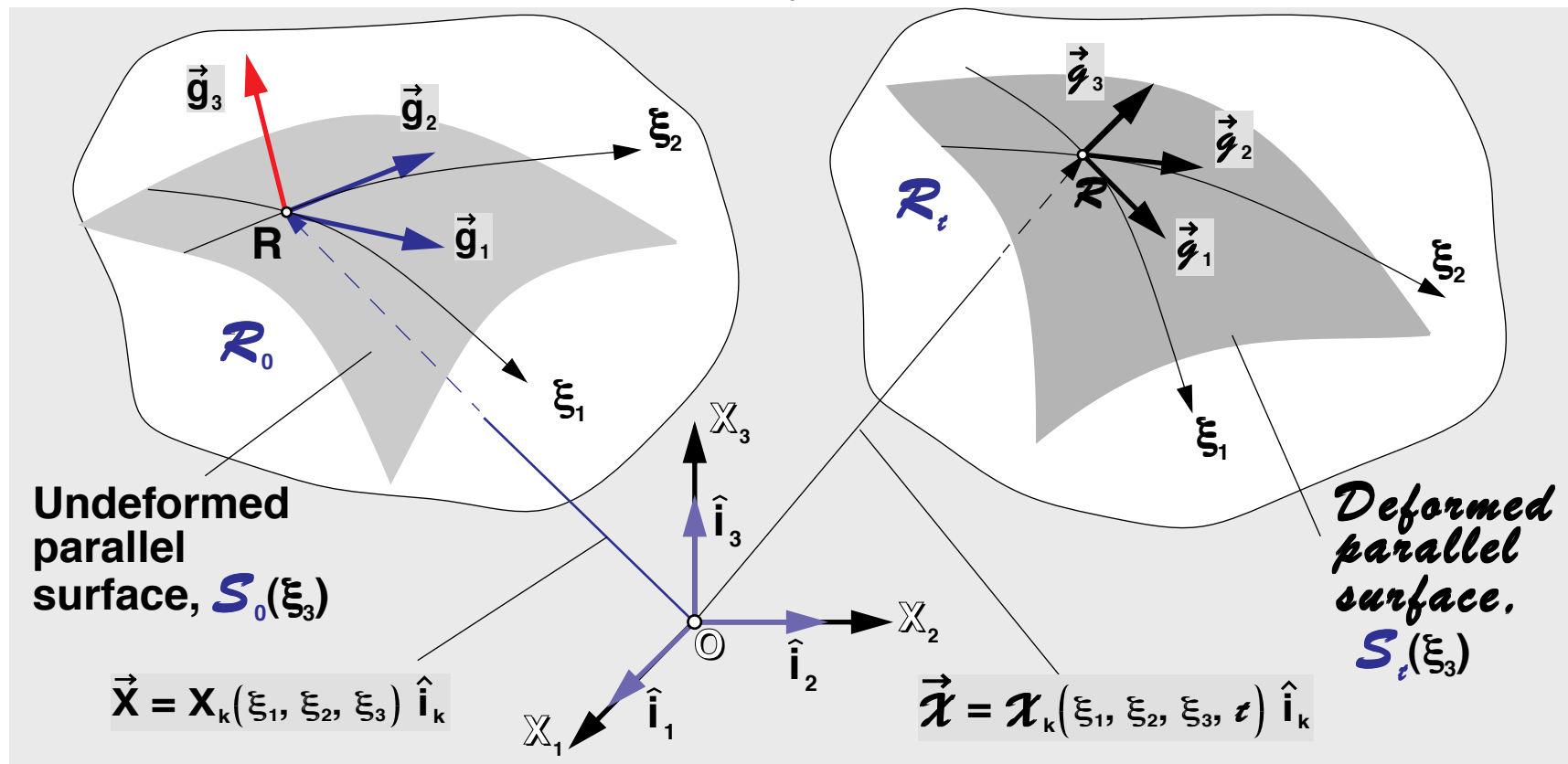
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CONVECTED BASE-VECTOR FIELDS FOR $\mathcal{R}_t$

CONTINUED

- Thus, it follows that $\vec{g}_1$ and $\vec{g}_2$ are tangent to the convected Gaussian-coordinate curves of a given **deformed parallel surface** defined by a specific value of the coordinate ξ_3



CONVECTED BASE-VECTOR FIELDS FOR $\mathcal{R}_t$

CONTINUED

- Recall, that the vectors $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$, defined for the undeformed configuration $\mathcal{R}_0$, are **noncolinear and noncoplanar**, for a meaningful coordinate system formed by three distinct coordinate surfaces
- Thus, $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ are **inherently linearly independent** at each point of $\mathcal{R}_0$, and, as a result, constitute a basis for representing vector field associated with the points of $\mathcal{R}_0$
- Therefore, any vector field $\vec{V}(\xi_1, \xi_2, \xi_3)$ associated with points of $\mathcal{R}_0$ can be expressed uniquely by the linear combination $\vec{V} = V^k \vec{g}_k$, where $V^k = V^k(\xi_1, \xi_2, \xi_3)$

CONVECTED BASE-VECTOR FIELDS FOR $\mathcal{R}_t$

CONTINUED

- For physically meaningful deformations, the vector fields $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ remain noncolinear and noncoplanar and, as a result, **remain linearly independent during deformation**
- Thus, they form a basis for points of the deformed configuration $\mathcal{R}_t$ such that any vector field $\vec{v}(\xi_1, \xi_2, \xi_3, t)$, associated with points of $\mathcal{R}_t$ can be expressed uniquely as the linear combination
$$\vec{v} = v^k \vec{g}_k, \text{ where } v^k = v^k(\xi_1, \xi_2, \xi_3, t)$$
- In addition, $v^k(\xi_1, \xi_2, \xi_3, 0) = V^k(\xi_1, \xi_2, \xi_3)$ because of the convective nature of the curvilinear coordinates

CONVECTED BASE-VECTOR FIELDS FOR $\mathcal{R}_t$

CONTINUED

- Another convenient form of the convected natural basis that is used herein is the set of **unit-magnitude vector fields** $\{\hat{\vec{g}}_1, \hat{\vec{g}}_2, \hat{\vec{g}}_3\}$ defined by

$$\hat{\vec{g}}_k \equiv \frac{\vec{\vec{g}}_k}{\mathcal{H}_{(k)}} = \frac{1}{\mathcal{H}_{(k)}} \frac{\partial \vec{\mathcal{X}}}{\partial \xi_k} \quad \text{with} \quad \mathcal{H}_k(\xi_1, \xi_2, \xi_3, t) \equiv |\vec{\vec{g}}_k| = \sqrt{\vec{\vec{g}}_k \cdot \vec{\vec{g}}_{(k)}}$$

and where the parentheses enclosing the subscript indicates suspension of summation of repeated indices

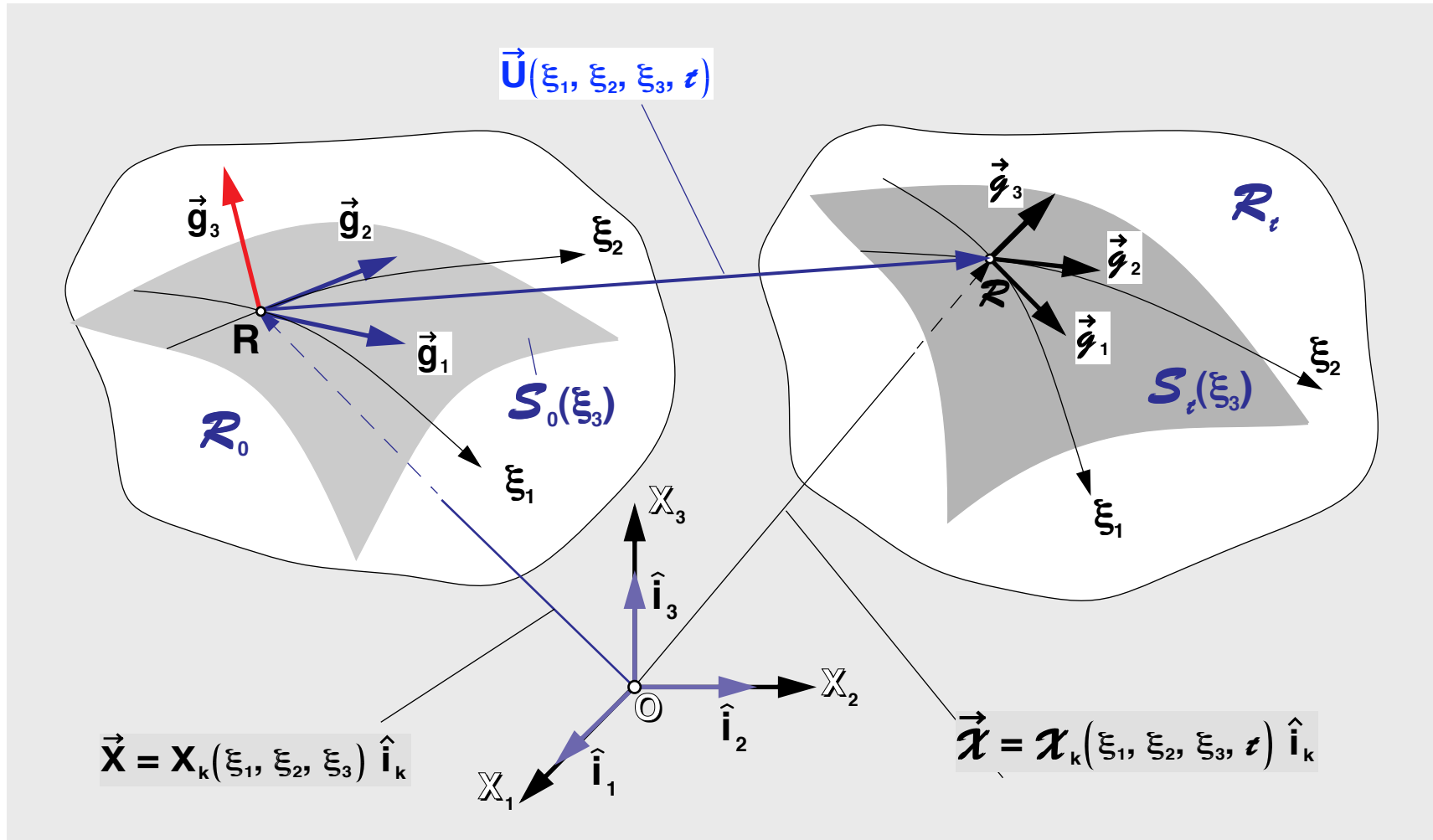
- The relationships between $\{\vec{\mathbf{g}}_1, \vec{\mathbf{g}}_2, \vec{\mathbf{g}}_3\}$ and $\{\vec{\vec{g}}_1, \vec{\vec{g}}_2, \vec{\vec{g}}_3\}$ are found by

substituting $\vec{\mathcal{X}} = \vec{\mathbf{X}} + \vec{\mathbf{U}}$ into $\vec{\vec{g}}_k \equiv \frac{\partial \vec{\mathcal{X}}}{\partial \xi_k}$ and then using $\vec{\mathbf{g}}_k = \frac{\partial \vec{\mathbf{X}}}{\partial \xi_k}$ to get

$$\vec{\vec{g}}_k = \vec{\mathbf{g}}_k + \frac{\partial \vec{\mathbf{U}}}{\partial \xi_k} \quad \text{or} \quad \vec{\vec{g}}_k = \mathbf{H}_{(k)} \left(\hat{\vec{g}}_k + \frac{1}{\mathbf{H}_{(k)}} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_k} \right)$$

CONVECTED BASE-VECTOR FIELDS FOR $\mathcal{R}_t$

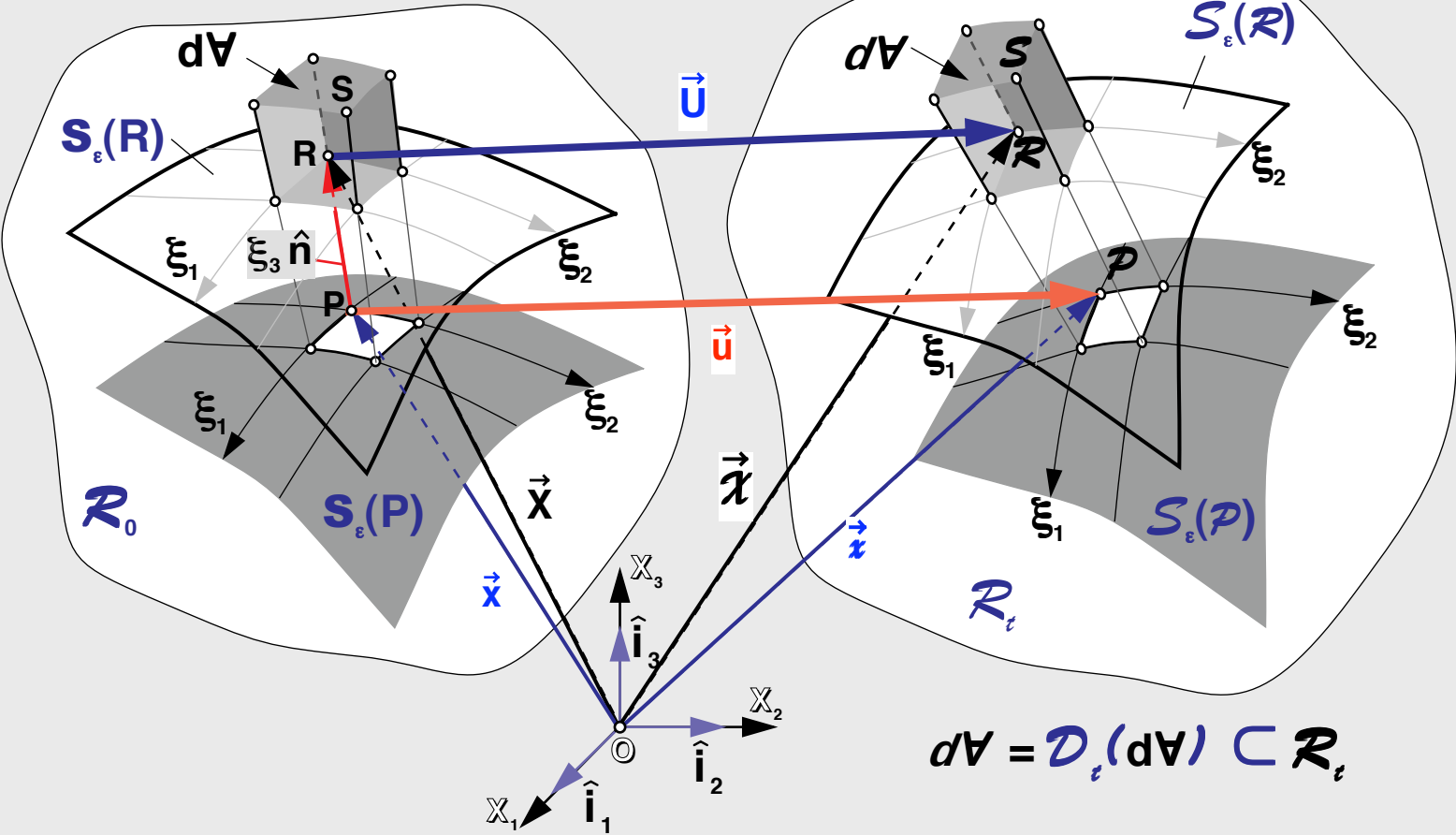
CONCLUDED



METRIC COEFFICIENTS FOR $\mathcal{R}_t$

- The **metric coefficients of the deformed shell**, $\mathcal{R}_t$, are obtained following the same process used for the undeformed shell, $\mathcal{R}_0$
- First, consider the deformation of the differential volume element $dV \subset \mathcal{R}_0$ into another differential volume element $dV \subset \mathcal{R}_t$, as depicted in the following figure
 - The point $R \in \mathcal{R}_0$ has coordinates (X_1, X_2, X_3)
 - The base of $dV \subset \mathcal{R}_0$ is located on the parallel surface $S_\varepsilon(R)$
 - $\mathcal{R} = \mathcal{D}_t(R) \in \mathcal{R}_t$, is the image of $R \in \mathcal{R}_0$, at time t and has coordinates (χ_1, χ_2, χ_3)
 - The base of $dV \subset \mathcal{R}_t$ is located on the parallel surface $S_\varepsilon(\mathcal{R})$

METRIC COEFFICIENTS FOR $\mathcal{R}_t$ - CONTINUED



METRIC COEFFICIENTS FOR $\mathcal{R}_t$ - CONTINUED

- Recall that the position vectors to points P and R in the undeformed shell are $\vec{\mathbf{x}} = \mathbf{x}_k(\xi_1, \xi_2) \hat{\mathbf{i}}_k$ and $\vec{\mathbf{X}}(\xi_1, \xi_2, \xi_3) = \vec{\mathbf{x}}(\xi_1, \xi_2) + \xi_3 \hat{\mathbf{n}}(\xi_1, \xi_2)$, respectively

- Likewise, the position vector to point S is

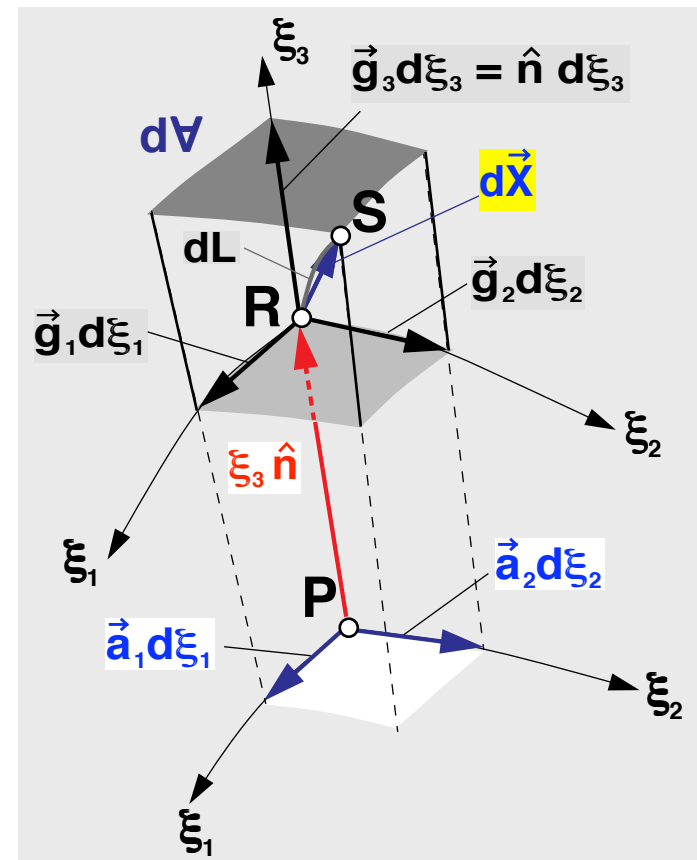
$$\vec{\mathbf{X}}(\xi_1 + \mathbf{d}\xi_1, \xi_2 + \mathbf{d}\xi_2, \xi_3 + \mathbf{d}\xi_3) = \vec{\mathbf{X}}(\xi_1, \xi_2, \xi_3) + \mathbf{d}\vec{\mathbf{X}} \quad \text{and}$$

the vector from point R to point S is $\vec{dX}$

- The length of arc of the material curve between points R and S is given by

$$dL = \widehat{RS} = \left| d\vec{X} \right|,$$

to a first approximation in differentials



METRIC COEFFICIENTS FOR $\mathcal{R}_t$ - CONTINUED

- For the deformed shell, the position vectors to points $\mathcal{P}$ and $\mathcal{R}$ are

$$\vec{\mathbf{x}} = \mathbf{x}_k(\xi_1, \xi_2, \xi_3, t) \hat{\mathbf{i}}_k \quad \text{and} \quad \vec{\mathbf{x}} = \mathbf{x}_k(\xi_1, \xi_2, \xi_3, t) \hat{\mathbf{i}}_k, \quad \text{respectively}$$

- Likewise, the position vector to point $\mathcal{S}$ is

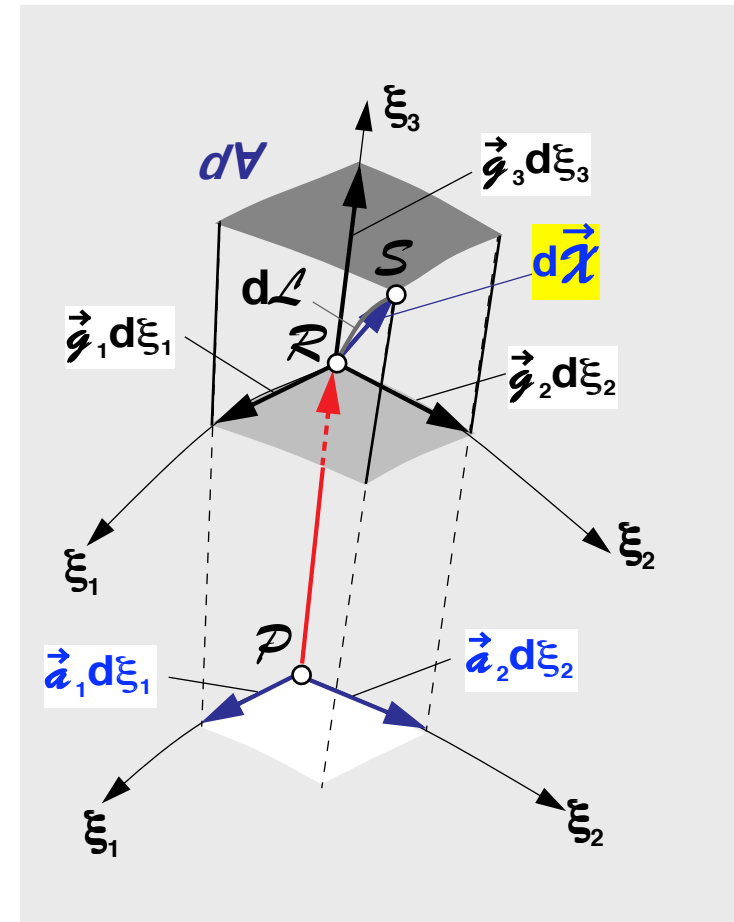
$$\vec{\mathbf{x}}(\xi_1 + d\xi_1, \xi_2 + d\xi_2, \xi_3 + d\xi_3) = \vec{\mathbf{x}}(\xi_1, \xi_2, \xi_3) + d\vec{\mathbf{x}} \quad \text{and the}$$

vector from point $\mathcal{R}$ to point $\mathcal{S}$ is $d\vec{\mathbf{x}}$

- The length of arc of the material curve between points $\mathcal{R}$ and $\mathcal{S}$ is

$$d\mathcal{L} = \mathcal{D}_t(dL) \quad \text{and} \quad d\mathcal{L} = \widehat{\mathcal{RS}} = |d\vec{\mathbf{x}}|,$$

to a first approximation in differentials



METRIC COEFFICIENTS FOR $\mathcal{R}_t$ - CONTINUED

- The chain rule of differentiation gives

$$d\vec{\mathcal{X}}(\xi_1, \xi_2, \xi_3, t) = \frac{\partial \vec{\mathcal{X}}}{\partial \xi_k} d\xi_k = \vec{\mathcal{g}}_k(\xi_1, \xi_2, \xi_3, t) d\xi_k$$

- From $d\mathcal{L} = |d\vec{\mathcal{X}}|$, it follows that $d\mathcal{L}^2 = d\vec{\mathcal{X}} \cdot d\vec{\mathcal{X}} = (\vec{\mathcal{g}}_j \cdot \vec{\mathcal{g}}_k) d\xi_j d\xi_k$

- Using $\mathcal{H}_k(\xi_1, \xi_2, \xi_3, t) \equiv |\vec{\mathcal{g}}_k| = \sqrt{\vec{\mathcal{g}}_k \cdot \vec{\mathcal{g}}_{(k)}}$ and the presumption of a nonorthogonal convected-coordinate mesh for $\mathcal{R}_t$ gives

$$d\mathcal{L}^2 = (\mathcal{H}_1 d\xi_1)^2 + 2\mathcal{H}_1 \mathcal{H}_2 \cos \Theta_{12} d\xi_1 d\xi_2 + (\mathcal{H}_2 d\xi_2)^2 + 2\mathcal{H}_1 \mathcal{H}_3 \cos \Theta_{13} d\xi_1 d\xi_3 + 2\mathcal{H}_2 \mathcal{H}_3 \cos \Theta_{23} d\xi_2 d\xi_3 + (\mathcal{H}_3 d\xi_3)^2$$

where $\Theta_{jk}(\xi_1, \xi_2, \xi_3, t)$ are the angles between the generally nonorthogonal intersecting coordinate curves of the deformed shell that are defined by $\mathcal{H}_j \mathcal{H}_k \cos \Theta_{(jk)} = \vec{\mathcal{g}}_j \cdot \vec{\mathcal{g}}_k$

METRIC COEFFICIENTS FOR $\mathcal{R}_t$ - CONTINUED

- By substituting $\vec{\mathcal{G}}_k = \vec{g}_k + \frac{\partial \vec{U}}{\partial \xi_k} = H_{(k)} \left(\hat{g}_k + \frac{1}{H_{(k)}} \frac{\partial \vec{U}}{\partial \xi_k} \right)$ into $\mathcal{H}_k = \sqrt{\vec{\mathcal{G}}_k \cdot \vec{\mathcal{G}}_{(k)}}$, the metric coefficients of the deformed shell are found, in terms of the displacement vector field, to be

$$\mathcal{H}_1 = H_1 \sqrt{1 + 2 \left(\hat{g}_1 \cdot \frac{1}{H_1} \frac{\partial \vec{U}}{\partial \xi_1} \right) + \left(\frac{1}{H_1} \frac{\partial \vec{U}}{\partial \xi_1} \cdot \frac{1}{H_1} \frac{\partial \vec{U}}{\partial \xi_1} \right)}$$

$$\mathcal{H}_2 = H_2 \sqrt{1 + 2 \left(\hat{g}_2 \cdot \frac{1}{H_2} \frac{\partial \vec{U}}{\partial \xi_2} \right) + \left(\frac{1}{H_2} \frac{\partial \vec{U}}{\partial \xi_2} \cdot \frac{1}{H_2} \frac{\partial \vec{U}}{\partial \xi_2} \right)}$$

$$\mathcal{H}_3 = H_3 \sqrt{1 + 2 \left(\hat{g}_3 \cdot \frac{1}{H_3} \frac{\partial \vec{U}}{\partial \xi_3} \right) + \left(\frac{1}{H_3} \frac{\partial \vec{U}}{\partial \xi_3} \cdot \frac{1}{H_3} \frac{\partial \vec{U}}{\partial \xi_3} \right)}$$

or

$$\mathcal{H}_k = H_k \sqrt{1 + 2 \left(\hat{g}_{(k)} \cdot \frac{1}{H_{(k)}} \frac{\partial \vec{U}}{\partial \xi_{(k)}} \right) + \left(\frac{1}{H_{(k)}} \frac{\partial \vec{U}}{\partial \xi_{(k)}} \cdot \frac{1}{H_{(k)}} \frac{\partial \vec{U}}{\partial \xi_{(k)}} \right)}$$

METRIC COEFFICIENTS FOR $\mathcal{R}_t$ - CONTINUED

- Similarly, $\mathcal{H}_j \mathcal{H}_k \cos \Theta_{(jk)} = \vec{g}_j \cdot \vec{g}_k$ gives

$$\frac{\mathcal{H}_1 \mathcal{H}_2}{H_1 H_2} \cos \Theta_{12} - \cos \Theta_{12} = \left(\hat{g}_1 \cdot \frac{1}{H_2} \frac{\partial \vec{U}}{\partial \xi_2} \right) + \left(\hat{g}_2 \cdot \frac{1}{H_1} \frac{\partial \vec{U}}{\partial \xi_1} \right) + \left(\frac{1}{H_1} \frac{\partial \vec{U}}{\partial \xi_1} \cdot \frac{1}{H_2} \frac{\partial \vec{U}}{\partial \xi_2} \right)$$

$$\frac{\mathcal{H}_1 \mathcal{H}_3}{H_1 H_3} \cos \Theta_{13} - \cos \Theta_{13} = \left(\hat{g}_1 \cdot \frac{1}{H_3} \frac{\partial \vec{U}}{\partial \xi_3} \right) + \left(\hat{g}_3 \cdot \frac{1}{H_1} \frac{\partial \vec{U}}{\partial \xi_1} \right) + \left(\frac{1}{H_1} \frac{\partial \vec{U}}{\partial \xi_1} \cdot \frac{1}{H_3} \frac{\partial \vec{U}}{\partial \xi_3} \right)$$

$$\frac{\mathcal{H}_2 \mathcal{H}_3}{H_2 H_3} \cos \Theta_{23} - \cos \Theta_{23} = \left(\hat{g}_2 \cdot \frac{1}{H_3} \frac{\partial \vec{U}}{\partial \xi_3} \right) + \left(\hat{g}_3 \cdot \frac{1}{H_2} \frac{\partial \vec{U}}{\partial \xi_2} \right) + \left(\frac{1}{H_2} \frac{\partial \vec{U}}{\partial \xi_2} \cdot \frac{1}{H_3} \frac{\partial \vec{U}}{\partial \xi_3} \right)$$

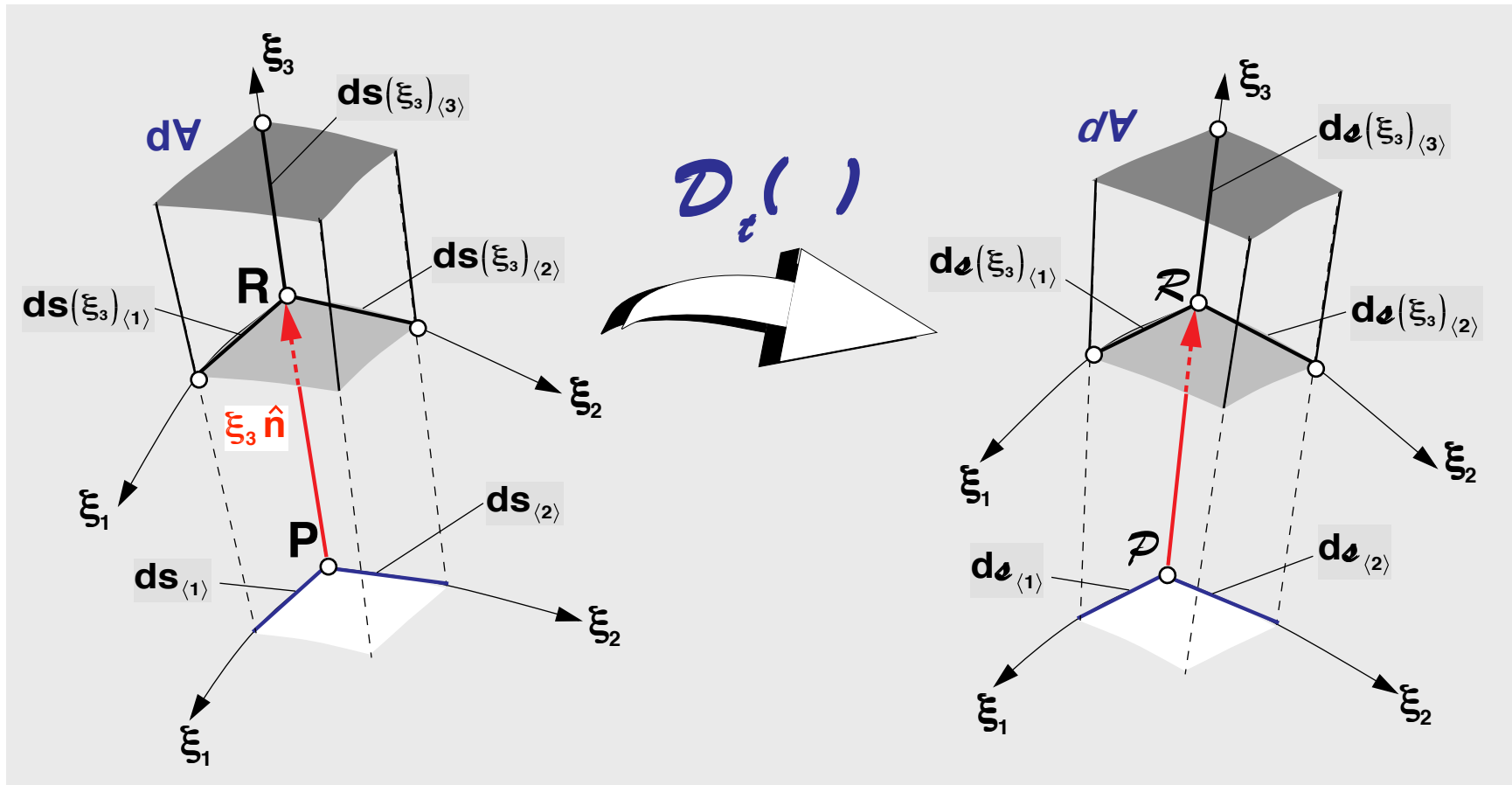
METRIC COEFFICIENTS FOR $\mathcal{R}_t$ - CONTINUED

- The lengths of the differential arcs along the convected coordinate curves are obtained directly from the expression

$$d\mathcal{L}^2 = \left(\mathcal{H}_1 d\xi_1\right)^2 + 2\mathcal{H}_1\mathcal{H}_2\cos\Theta_{12} d\xi_1 d\xi_2 + \left(\mathcal{H}_2 d\xi_2\right)^2 + \\ 2\mathcal{H}_1\mathcal{H}_3\cos\Theta_{13} d\xi_1 d\xi_3 + 2\mathcal{H}_2\mathcal{H}_3\cos\Theta_{23} d\xi_2 d\xi_3 + \left(\mathcal{H}_3 d\xi_3\right)^2$$

- Along the convected ξ_1 -coordinate curve, $d\xi_2 = d\xi_3 = 0$ and $d\mathcal{L} \rightarrow d\mathcal{L}(\xi_3)_{\langle 1 \rangle}$, where $d\mathcal{L}(\xi_3)_{\langle 1 \rangle} = \mathcal{H}_1 d\xi_1$
- Along the convected ξ_2 -coordinate curve, $d\xi_1 = d\xi_3 = 0$ and $d\mathcal{L} \rightarrow d\mathcal{L}(\xi_3)_{\langle 2 \rangle}$, where $d\mathcal{L}(\xi_3)_{\langle 2 \rangle} = \mathcal{H}_2 d\xi_2$
- Along the convected ξ_3 -coordinate curve, $d\xi_1 = d\xi_2 = 0$ and $d\mathcal{L} \rightarrow d\mathcal{L}(\xi_3)_{\langle 3 \rangle}$, where $d\mathcal{L}(\xi_3)_{\langle 3 \rangle} = \mathcal{H}_3 d\xi_3$

METRIC COEFFICIENTS FOR $\mathcal{R}_t$ - CONTINUED



METRIC COEFFICIENTS FOR $\mathcal{R}_t$ - CONCLUDED

- In terms of the displacement vector-field,

$$d\ell(\xi_3)_{\langle 1 \rangle} = \sqrt{1 + 2\left(\hat{\mathbf{g}}_1 \cdot \frac{1}{H_1} \frac{\partial \vec{U}}{\partial \xi_1}\right) + \left(\frac{1}{H_1} \frac{\partial \vec{U}}{\partial \xi_1} \cdot \frac{1}{H_1} \frac{\partial \vec{U}}{\partial \xi_1}\right)} H_1 d\xi_1$$

$$d\ell(\xi_3)_{\langle 2 \rangle} = \sqrt{1 + 2\left(\hat{\mathbf{g}}_2 \cdot \frac{1}{H_2} \frac{\partial \vec{U}}{\partial \xi_2}\right) + \left(\frac{1}{H_2} \frac{\partial \vec{U}}{\partial \xi_2} \cdot \frac{1}{H_2} \frac{\partial \vec{U}}{\partial \xi_2}\right)} H_2 d\xi_2$$

$$d\ell(\xi_3)_{\langle 3 \rangle} = \sqrt{1 + 2\left(\hat{\mathbf{g}}_3 \cdot \frac{1}{H_3} \frac{\partial \vec{U}}{\partial \xi_3}\right) + \left(\frac{1}{H_3} \frac{\partial \vec{U}}{\partial \xi_3} \cdot \frac{1}{H_3} \frac{\partial \vec{U}}{\partial \xi_3}\right)} H_3 d\xi_3$$

RECIPROCAL BASIS FOR $\mathcal{R}_t$

- Previously, the vector fields $\{\vec{g}^1, \vec{g}^2, \vec{g}^3\}$ were introduced for the undeformed shell, $\mathcal{R}_0$, such that

$$\vec{g}_1 \times \vec{g}_2 = H \vec{g}^3 \quad \vec{g}_3 \times \vec{g}_1 = H \vec{g}^2 \quad \vec{g}_2 \times \vec{g}_3 = H \vec{g}^1$$

and $\vec{g}^m \cdot \vec{g}_n = \delta_n^m$, where δ_n^m is the **Kronecker delta symbol** and

$$H^2 = \begin{vmatrix} (H_1)^2 & H_1 H_2 \cos \Theta_{12} & H_1 H_3 \cos \Theta_{13} \\ H_1 H_2 \cos \Theta_{12} & (H_2)^2 & H_2 H_3 \cos \Theta_{23} \\ H_1 H_3 \cos \Theta_{13} & H_2 H_3 \cos \Theta_{23} & (H_3)^2 \end{vmatrix}$$

- A set of expressions $\vec{g}^k = g^{kp} \vec{g}_p$ that relate the basis $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ to the reciprocal basis $\{\vec{g}^1, \vec{g}^2, \vec{g}^3\}$ were derived for general nonorthogonal coordinates

RECIPROCAL BASIS FOR $\mathcal{R}_t$ - CONTINUED

where

$$\begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{12} & g^{22} & g^{23} \\ g^{13} & g^{23} & g^{33} \end{bmatrix} = \begin{bmatrix} (H_1)^2 & H_1 H_2 \cos \Theta_{12} & H_1 H_3 \cos \Theta_{13} \\ H_1 H_2 \cos \Theta_{12} & (H_2)^2 & H_2 H_3 \cos \Theta_{23} \\ H_1 H_3 \cos \Theta_{13} & H_2 H_3 \cos \Theta_{23} & (H_3)^2 \end{bmatrix}^{-1}$$

- Therefore, it follows that reciprocal base vector fields for the deformed shell $\mathcal{R}_t$ are found by direct analogy
- Specifically, a convected basis $\{\vec{g}^1, \vec{g}^2, \vec{g}^3\}$ that is reciprocal to the convected basis $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ is defined for $\mathcal{R}_t$ by

$$\vec{g}_1 \times \vec{g}_2 = \mathcal{H} \vec{g}^3 \quad \vec{g}_3 \times \vec{g}_1 = \mathcal{H} \vec{g}^2 \quad \vec{g}_2 \times \vec{g}_3 = \mathcal{H} \vec{g}^1$$

and $\vec{g}^m \cdot \vec{g}_n = \delta_n^m$, where δ_n^m is the Kronecker delta symbol and

RECIPROCAL BASIS FOR $\mathcal{R}_t$ - CONTINUED

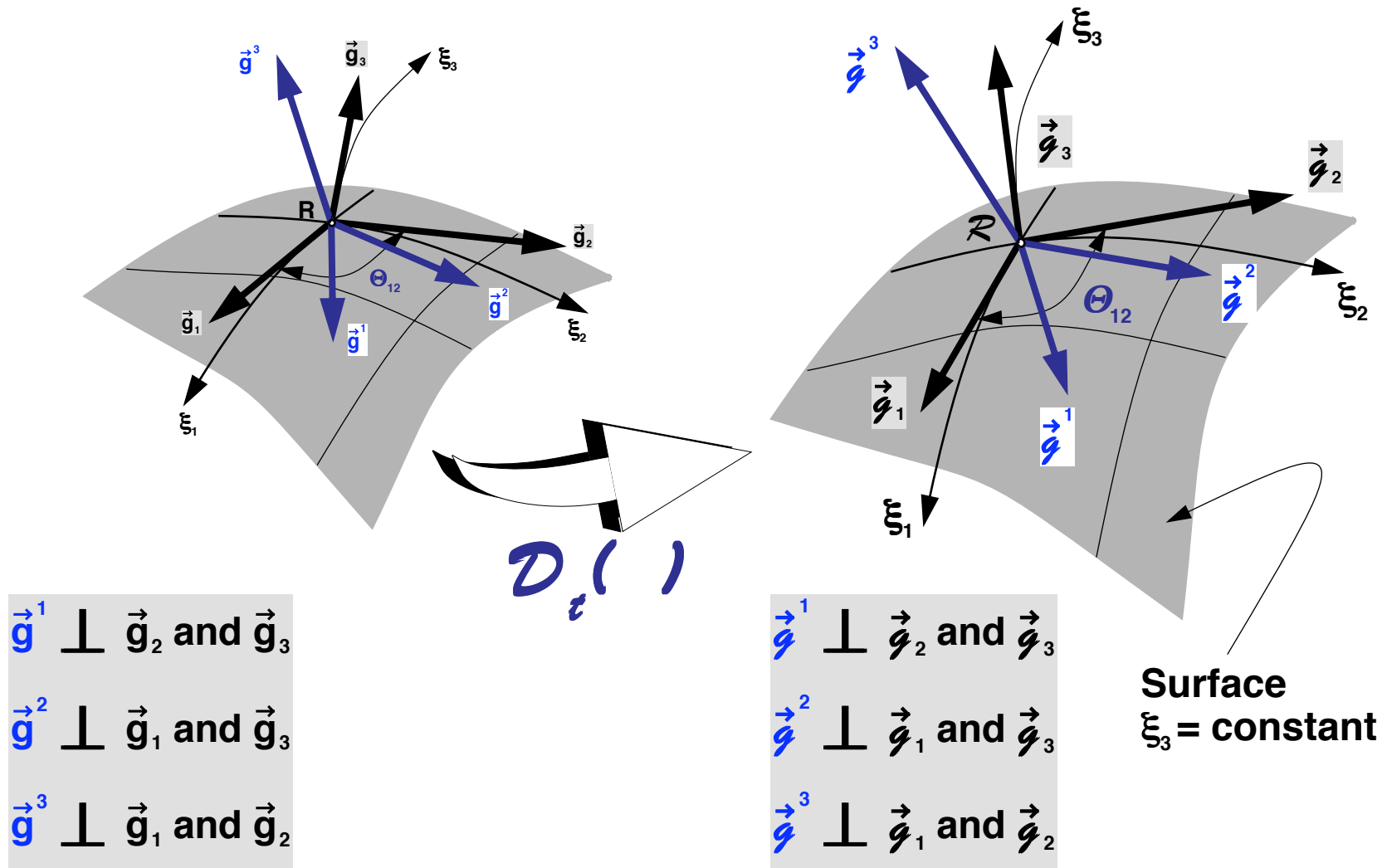
$$\mathcal{H}^2 = \begin{vmatrix} (\mathcal{H}_1)^2 & \mathcal{H}_1 \mathcal{H}_2 \cos \Theta_{12} & \mathcal{H}_1 \mathcal{H}_3 \cos \Theta_{13} \\ \mathcal{H}_1 \mathcal{H}_2 \cos \Theta_{12} & (\mathcal{H}_2)^2 & \mathcal{H}_2 \mathcal{H}_3 \cos \Theta_{23} \\ \mathcal{H}_1 \mathcal{H}_3 \cos \Theta_{13} & \mathcal{H}_2 \mathcal{H}_3 \cos \Theta_{23} & (\mathcal{H}_3)^2 \end{vmatrix}$$

- The relationship between the two bases is given by $\vec{q}^k = q^{kp} \vec{q}_p$, where

$$\begin{bmatrix} q^{11} & q^{12} & q^{13} \\ q^{12} & q^{22} & q^{23} \\ q^{13} & q^{23} & q^{33} \end{bmatrix} = \begin{bmatrix} (\mathcal{H}_1)^2 & \mathcal{H}_1 \mathcal{H}_2 \cos \Theta_{12} & \mathcal{H}_1 \mathcal{H}_3 \cos \Theta_{13} \\ \mathcal{H}_1 \mathcal{H}_2 \cos \Theta_{12} & (\mathcal{H}_2)^2 & \mathcal{H}_2 \mathcal{H}_3 \cos \Theta_{23} \\ \mathcal{H}_1 \mathcal{H}_3 \cos \Theta_{13} & \mathcal{H}_2 \mathcal{H}_3 \cos \Theta_{23} & (\mathcal{H}_3)^2 \end{bmatrix}^{-1}$$

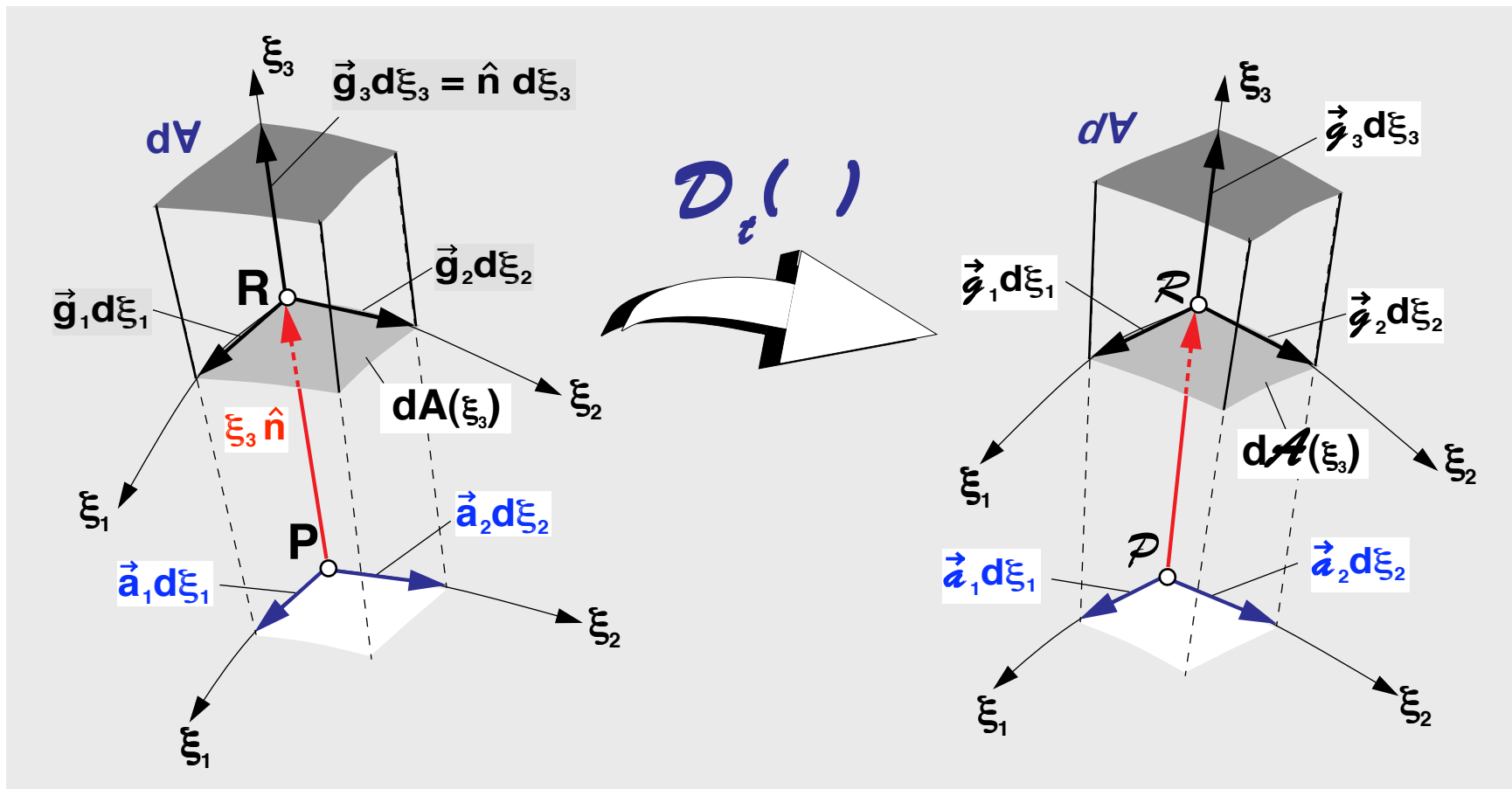
- Also, $(\vec{q}_1 \times \vec{q}_2) \cdot \vec{q}_3 = (\vec{q}_2 \times \vec{q}_3) \cdot \vec{q}_1 = (\vec{q}_3 \times \vec{q}_1) \cdot \vec{q}_2 = \mathcal{H}$

RECIPROCAL BASIS FOR $\mathcal{R}_t$ - CONCLUDED



DIFFERENTIAL AREAS AND VOLUMES FOR $\mathcal{R}_t$

- Consider the deformation of the differential volume dV into $d\mathcal{V}$, as shown in the figure



DIFFERENTIAL AREAS AND VOLUMES FOR $\mathcal{R}_t$

CONTINUED

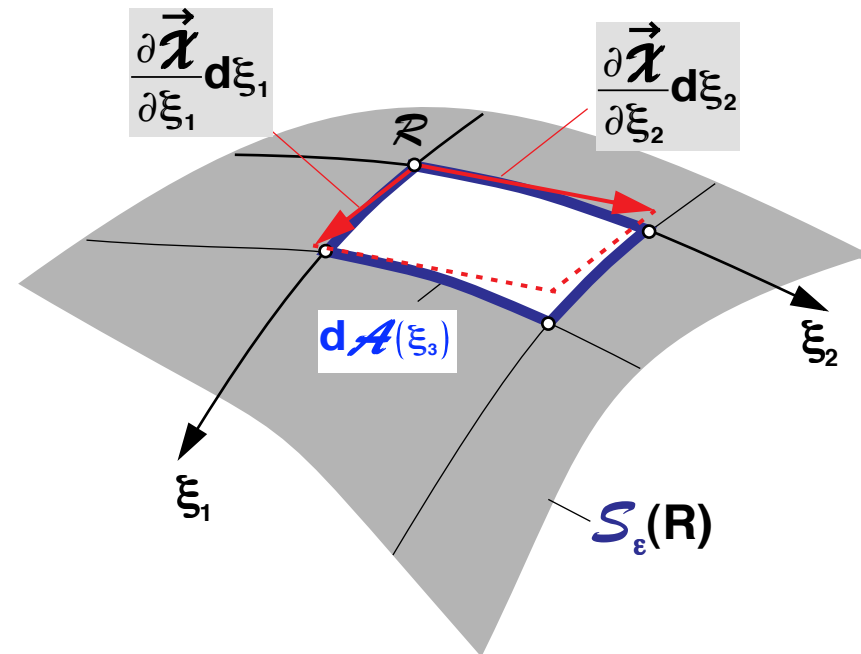
- During deformation, the differential surface area $d\mathbf{A}(\xi_3)$ shown in the previous figure becomes $d\mathcal{A}(\xi_3) = \mathcal{D}_t(d\mathbf{A}(\xi_3))$

- To a first approximation in differentials, $d\mathcal{A}(\xi_3)$ is given by

$$d\mathcal{A}(\xi_3) = \left| \frac{\partial \vec{\mathcal{X}}}{\partial \xi_1} d\xi_1 \times \frac{\partial \vec{\mathcal{X}}}{\partial \xi_2} d\xi_2 \right|$$

- Using $\vec{g}_\alpha = \frac{\partial \vec{\mathcal{X}}}{\partial \xi_\alpha}$ gives

$$d\mathcal{A}(\xi_3) = \left| \vec{g}_1 \times \vec{g}_2 \right| d\xi_1 d\xi_2$$



DIFFERENTIAL AREAS AND VOLUMES FOR $\mathcal{R}_t$

CONTINUED

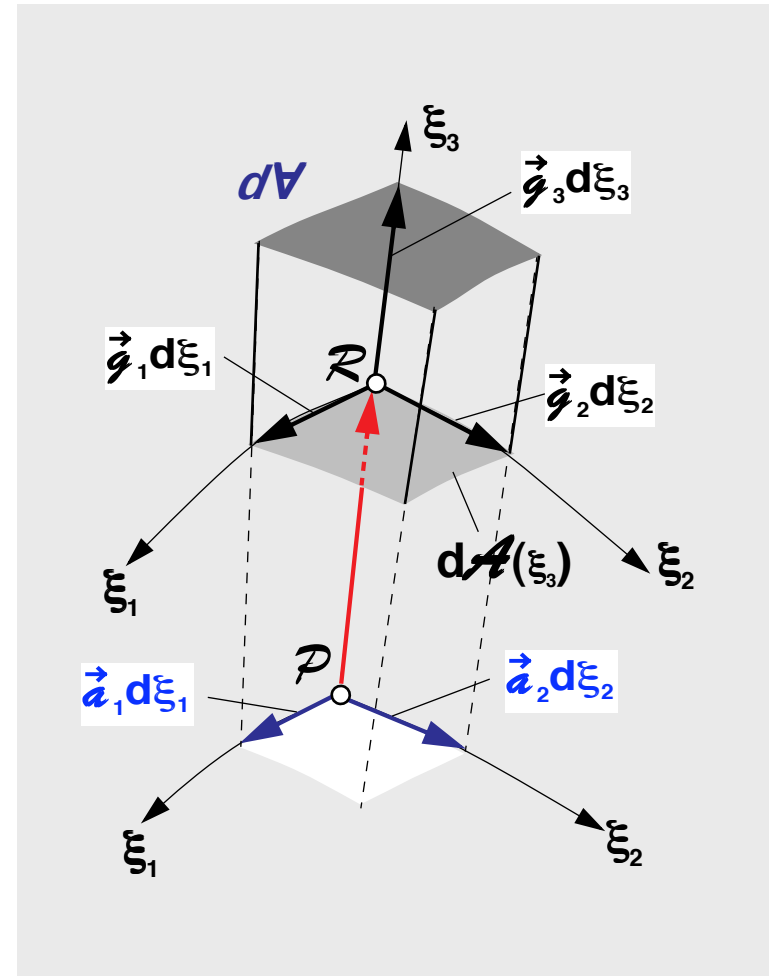
- The differential area of the face of the differential element given by $\xi_2 = 0$ is given by

$$d\mathcal{A}_{(2)} = \left| \frac{\partial \vec{\mathcal{X}}}{\partial \xi_1} d\xi_1 \times \frac{\partial \vec{\mathcal{X}}}{\partial \xi_3} d\xi_3 \right|,$$

to a first approximation in differentials

- Substituting $\vec{g}_k = \frac{\partial \vec{\mathcal{X}}}{\partial \xi_k}$ into the previous expression gives

$$d\mathcal{A}_{(2)} = \left| \vec{g}_1 \times \vec{g}_3 \right| d\xi_1 d\xi_3$$



DIFFERENTIAL AREAS AND VOLUMES FOR $\mathcal{R}_t$

CONTINUED

- Similarly, the differential area of the face of the differential element given by $\xi_1 = 0$ is given by

$$d\mathcal{A}_{\langle 1 \rangle} = \left| \frac{\partial \vec{\mathcal{X}}}{\partial \xi_2} d\xi_2 \times \frac{\partial \vec{\mathcal{X}}}{\partial \xi_3} d\xi_3 \right| = \left| \vec{\mathcal{G}}_2 \times \vec{\mathcal{G}}_3 \right| d\xi_2 d\xi_3$$

and the volume of the differential element is given by

$$dV = \left(\vec{\mathcal{G}}_1 \times \vec{\mathcal{G}}_2 \right) \cdot \vec{\mathcal{G}}_3 d\xi_1 d\xi_2 d\xi_3$$

- From $\vec{\mathcal{G}}_k = H_{(k)} \left(\hat{\mathbf{g}}_k + \frac{1}{H_{(k)}} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_k} \right)$ and $\frac{1}{H_{(p)}} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_p} \equiv \mathbf{U}_{, \langle p \rangle}^{(k)} \hat{\mathbf{g}}_k$, it is seen that areas and volume of the differential element of the deformed shell are extremely complicated expressions, when expressed in terms of the displacement-field derivatives $\mathbf{U}_{, \langle p \rangle}^{(k)}$

DIFFERENTIAL AREAS AND VOLUMES FOR $\mathcal{R}_t$

CONTINUED

- More insightful forms for the differential areas and volumes are obtained by using $|\vec{g}_k| = \mathcal{H}_k$ and

$$|\vec{g}_1 \times \vec{g}_2| = \mathcal{H}_1 \mathcal{H}_2 |\hat{g}_1 \times \hat{g}_2| = \mathcal{H}_1 \mathcal{H}_2 |\sin \Theta_{12} \hat{g}^3| = \mathcal{H}_1 \mathcal{H}_2 \sin \Theta_{12}$$

$$|\vec{g}_1 \times \vec{g}_3| = \mathcal{H}_1 \mathcal{H}_3 |\hat{g}_1 \times \hat{g}_3| = \mathcal{H}_1 \mathcal{H}_3 |\sin \Theta_{13} \hat{g}^2| = \mathcal{H}_1 \mathcal{H}_3 \sin \Theta_{13}$$

$$|\vec{g}_2 \times \vec{g}_3| = \mathcal{H}_2 \mathcal{H}_3 |\hat{g}_2 \times \hat{g}_3| = \mathcal{H}_2 \mathcal{H}_3 |\sin \Theta_{23} \hat{g}^1| = \mathcal{H}_2 \mathcal{H}_3 \sin \Theta_{23}$$

$$(\vec{g}_1 \times \vec{g}_2) \cdot \vec{g}_3 = \mathcal{H} \quad \text{where} \quad \mathcal{H}^2 = \begin{vmatrix} \mathcal{H}_1^2 & \mathcal{H}_1 \mathcal{H}_2 \cos \Theta_{12} & \mathcal{H}_1 \mathcal{H}_3 \cos \Theta_{13} \\ \mathcal{H}_1 \mathcal{H}_2 \cos \Theta_{12} & \mathcal{H}_2^2 & \mathcal{H}_2 \mathcal{H}_3 \cos \Theta_{23} \\ \mathcal{H}_1 \mathcal{H}_3 \cos \Theta_{13} & \mathcal{H}_2 \mathcal{H}_3 \cos \Theta_{23} & \mathcal{H}_3^2 \end{vmatrix}$$

$$\text{or} \quad \mathcal{H} = \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \sqrt{\sin^2 \Theta_{12} + \sin^2 \Theta_{13} + \sin^2 \Theta_{23} + 2(\cos \Theta_{12} \cos \Theta_{13} \cos \Theta_{23} - 1)}$$

DIFFERENTIAL AREAS AND VOLUMES FOR $\mathcal{R}_t$

CONCLUDED

- With these results,

$$d\mathcal{A}_{(\xi_3)} = \left| \vec{g}_1 \times \vec{g}_2 \right| d\xi_1 d\xi_2 \quad \text{becomes} \quad d\mathcal{A}_{(\xi_3)} = \mathcal{H}_1 \mathcal{H}_2 \sin \theta_{12} d\xi_1 d\xi_2$$

$$d\mathcal{A}_{\langle 2 \rangle} = \left| \vec{g}_1 \times \vec{g}_3 \right| d\xi_1 d\xi_3 \quad \text{becomes} \quad d\mathcal{A}_{\langle 2 \rangle} = \mathcal{H}_1 \mathcal{H}_3 \sin \theta_{13} d\xi_1 d\xi_3$$

$$d\mathcal{A}_{\langle 1 \rangle} = \left| \vec{g}_2 \times \vec{g}_3 \right| d\xi_2 d\xi_3 \quad \text{becomes} \quad d\mathcal{A}_{\langle 1 \rangle} = \mathcal{H}_2 \mathcal{H}_3 \sin \theta_{23} d\xi_2 d\xi_3$$

$$d\mathcal{V} = \left(\vec{g}_1 \times \vec{g}_2 \right) \cdot \vec{g}_3 d\xi_1 d\xi_2 d\xi_3 \quad \text{becomes} \quad d\mathcal{V} = \mathcal{H} d\xi_1 d\xi_2 d\xi_3$$

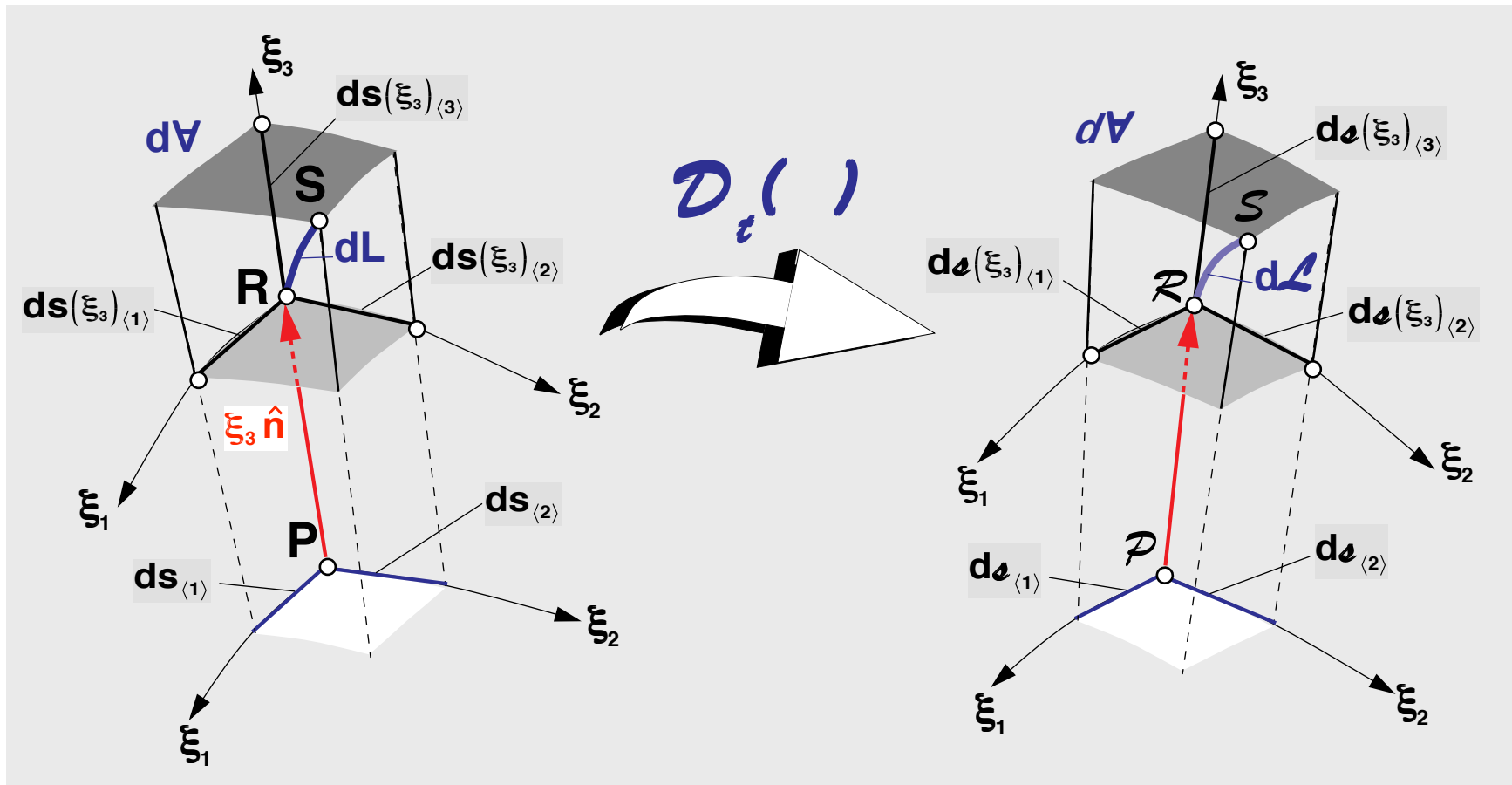
CHARACTERIZATION OF SHELL DEFORMATIONS

ELONGATION AND SHEAR OF THE SHELL

- **Elongation** (and contraction) and **shear** are two fundamental characteristics of deformation
- In the general theory of the mechanics of solids, it is proven that elongation of any differential arc composed of material points can be described completely in terms of the **Green-Lagrange strains**
- Shearing action is characterized by the change in angle between the tangent lines of any two differential arcs, composed of material points, that emanate from the same material point
- Shearing action can also be described completely in terms of the **Green-Lagrange strains**

ELONGATION AND SHEAR OF THE SHELL - CONTINUED

- Consider the differential arc $dL_{(\mu)} = \widehat{RS} \subset \mathcal{R}_0$ shown in the figure



ELONGATION AND SHEAR OF THE SHELL - CONTINUED

- The arc is part of a curve $C \subset \mathcal{R}_0$ that is defined parametrically by

$$\vec{X}(\mu) = \mathbf{X}_k(\xi_1(\mu), \xi_2(\mu), \xi_3(\mu)) \hat{\mathbf{i}}_k \quad \text{where } \mu \text{ is a parameter}$$

- During deformation, arc $\widehat{RS} \subset \mathcal{R}_0$ generally changes spatial orientation and **elongates or contracts** (negative elongation) into $\widehat{RS} \subset \mathcal{R}_t$

- The **elongation** $e_{\hat{\mathbf{T}}(\mu)}(\xi_1(\mu), \xi_2(\mu), \xi_3(\mu), t)$ of the differential arc $dL(\mu) = \widehat{RS}$ as it deforms into the differential arc $d\mathcal{L}(\mu) = \widehat{RS}$ is defined by

$$e_{\hat{\mathbf{T}}(\mu)} \equiv \frac{d\mathcal{L} - dL}{dL}$$

ELONGATION AND SHEAR OF THE SHELL - CONTINUED

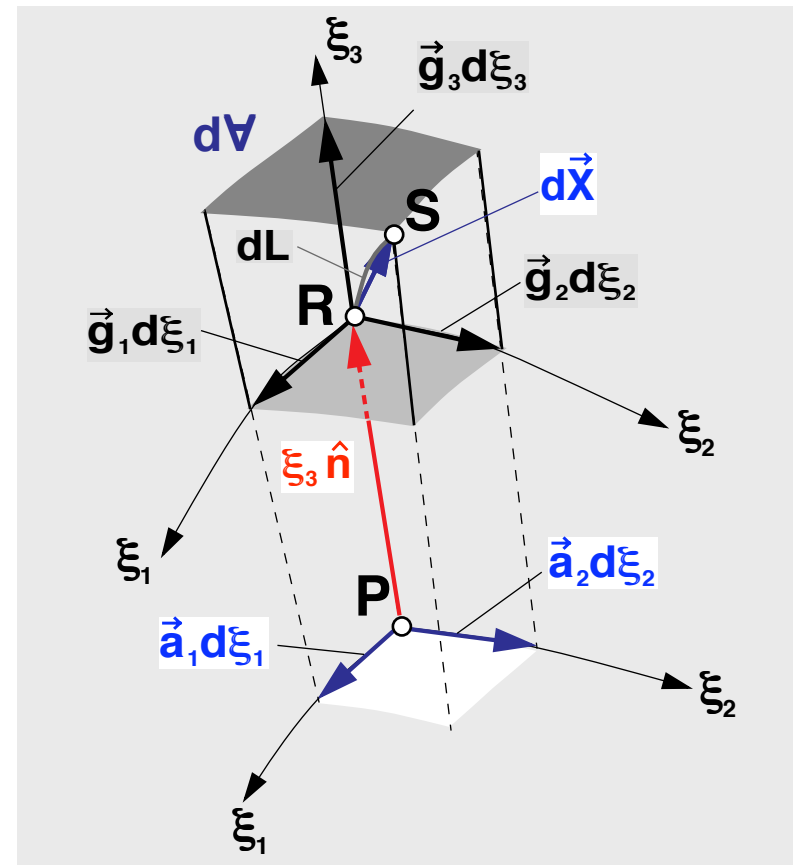
- In this definition of the elongation, the **unit-magnitude vector field**

$$\hat{\mathbf{T}}_{(\mu)} \equiv \frac{d\vec{\mathbf{X}}}{dL} \quad \text{defines the orientation of} \quad dL_{(\mu)} = \widehat{RS} \subset \mathcal{R}_0$$

- From $e_{\hat{T}} = \frac{d\mathcal{L} - dL}{dL}$, it follows that

$$d\mathcal{L} = \left(1 + \mathbf{e}_{\hat{\mathbf{T}}}\right) d\mathbf{L} \quad \text{and}$$

$$\left(1 + \mathbf{e}_{\hat{\mathbf{T}}}\right)^2 - 1 = \frac{\mathrm{d}\mathcal{L}^2 - \mathrm{d}\mathbf{L}^2}{\mathrm{d}\mathbf{L}^2}$$



ELONGATION AND SHEAR OF THE SHELL - CONTINUED

- Using $d\mathcal{L}^2 = (\vec{\mathcal{g}}_j \cdot \vec{\mathcal{g}}_k) d\xi_j d\xi_k$ and $dL^2 = (\vec{g}_j \cdot \vec{g}_k) d\xi_j d\xi_k$ gives

$$\left(1 + e_{\hat{T}}\right)^2 - 1 = \left[(\vec{\mathcal{g}}_j \cdot \vec{\mathcal{g}}_k) - (\vec{g}_j \cdot \vec{g}_k)\right] \frac{d\xi_j}{dL} \frac{d\xi_k}{dL}$$

- The direct dependence on $\hat{T}_{(\mu)}$ is seen by noting that

$$\hat{T} \equiv \frac{d\vec{X}}{dL} = \frac{\partial \vec{X}}{\partial \xi_j} \frac{d\xi_j}{dL} = \vec{g}_j \frac{d\xi_j}{dL} \quad \text{and} \quad \vec{g}^m \cdot \vec{g}_n = \delta_n^m \quad \text{give} \quad \frac{d\xi_j}{dL} = \hat{T} \cdot \vec{g}^j$$

- Thus, $\left(1 + e_{\hat{T}}\right)^2 - 1 = \left[(\vec{\mathcal{g}}_j \cdot \vec{\mathcal{g}}_k) - (\vec{g}_j \cdot \vec{g}_k)\right] (\hat{T} \cdot \vec{g}^j)(\hat{T} \cdot \vec{g}^k)$

- The **six independent quantities** $\epsilon_{jk}(\xi_1, \xi_2, \xi_3, t)$, defined such that

$$\left[(\vec{\mathcal{g}}_j \cdot \vec{\mathcal{g}}_k) - (\vec{g}_j \cdot \vec{g}_k)\right] \frac{d\xi_j}{dL} \frac{d\xi_k}{dL} \equiv 2\epsilon_{jk}(\xi_1, \xi_2, \xi_3, t) \frac{H_{(j)} d\xi_j}{dL} \frac{H_{(k)} d\xi_k}{dL}$$

ELONGATION AND SHEAR OF THE SHELL-CONCLUDED

completely defined the elongation $e_{\hat{T}}(\xi_1, \xi_2, \xi_3, t)$; that is

$$e_{\hat{T}} = \sqrt{1 + 2\varepsilon_{jk} H_{(j)} H_{(k)} (\hat{T} \cdot \vec{g}^j) (\hat{T} \cdot \vec{g}^k)} - 1$$

- Also, $\left(1 + e_{\hat{T}}\right)^2 - 1 = \frac{d\mathcal{L}^2 - dL^2}{dL^2}$ gives

$$\frac{d\mathcal{L}^2 - dL^2}{dL^2} = 2\varepsilon_{jk} H_{(j)} H_{(k)} (\hat{T} \cdot \vec{g}^j) (\hat{T} \cdot \vec{g}^k)$$

- This last expression is also written as

$$d\mathcal{L}^2 - dL^2 = 2\varepsilon_{jk} H_{(j)} H_{(k)} d\xi_j d\xi_k$$

NONLINEAR SHELL STRAINS

- The physical components of the nonlinear, **Green-Lagrange strain tensor** for an arbitrary point of the shell are given by

$$2\varepsilon_{ij}(H_{(i)}d\xi_i)(H_{(j)}d\xi_j) = d\mathcal{L}^2 - dL^2$$

where parentheses around the subscript are used in order not to violate the notation rules for the repeated-index summation convention

- In this expression, $d\mathcal{L} = \mathcal{D}_t(dL)$ is the deformed image of the infinitesimal arc length dL , which is also infinitesimal
- Previously, it was shown for general nonorthogonal coordinates that

$$d\mathcal{L}^2 = (\mathcal{H}_1 d\xi_1)^2 + 2\mathcal{H}_1\mathcal{H}_2 \cos\Theta_{12} d\xi_1 d\xi_2 + (\mathcal{H}_2 d\xi_2)^2 + 2\mathcal{H}_1\mathcal{H}_3 \cos\Theta_{13} d\xi_1 d\xi_3 + 2\mathcal{H}_2\mathcal{H}_3 \cos\Theta_{23} d\xi_2 d\xi_3 + (\mathcal{H}_3 d\xi_3)^2 \quad \text{and}$$

$$dL^2 = (H_1 d\xi_1)^2 + 2H_1H_2 \cos\Theta_{12} d\xi_1 d\xi_2 + (H_2 d\xi_2)^2 + 2H_1H_3 \cos\Theta_{13} d\xi_1 d\xi_3 + 2H_2H_3 \cos\Theta_{23} d\xi_2 d\xi_3 + (H_3 d\xi_3)^2$$

NONLINEAR SHELL STRAINS - CONTINUED

- Thus,

$$\begin{aligned} d\mathcal{L}^2 - dL^2 = & \left(\mathcal{H}_1^2 - H_1^2 \right) d\xi_1 d\xi_1 + 2 \left(\mathcal{H}_1 \mathcal{H}_2 \cos \Theta_{12} - H_1 H_2 \cos \Theta_{12} \right) d\xi_1 d\xi_2 + \\ & \left(\mathcal{H}_2^2 - H_2^2 \right) d\xi_2 d\xi_2 + 2 \left(\mathcal{H}_1 \mathcal{H}_3 \cos \Theta_{13} - H_1 H_3 \cos \Theta_{13} \right) d\xi_1 d\xi_3 + \\ & \left(\mathcal{H}_3^2 - H_3^2 \right) d\xi_3 d\xi_3 + 2 \left(\mathcal{H}_2 \mathcal{H}_3 \cos \Theta_{23} - H_2 H_3 \cos \Theta_{23} \right) d\xi_2 d\xi_3 \end{aligned}$$

- Using the definition $2\varepsilon_{ij} (H_{(i)} d\xi_i) (H_{(j)} d\xi_j) \equiv d\mathcal{L}^2 - dL^2$ gives

$$2\varepsilon_{11} = \left(\frac{\mathcal{H}_1}{H_1} \right)^2 - 1$$

$$2\varepsilon_{22} = \left(\frac{\mathcal{H}_2}{H_2} \right)^2 - 1$$

$$2\varepsilon_{33} = \left(\frac{\mathcal{H}_3}{H_3} \right)^2 - 1$$

$$2\varepsilon_{12} = 2 \left(\frac{\mathcal{H}_1 \mathcal{H}_2}{H_1 H_2} \cos \Theta_{12} - \cos \Theta_{12} \right)$$

$$2\varepsilon_{13} = 2 \left(\frac{\mathcal{H}_1 \mathcal{H}_3}{H_1 H_3} \cos \Theta_{13} - \cos \Theta_{13} \right)$$

$$2\varepsilon_{23} = 2 \left(\frac{\mathcal{H}_2 \mathcal{H}_3}{H_2 H_3} \cos \Theta_{23} - \cos \Theta_{23} \right)$$

NONLINEAR SHELL STRAINS - CONTINUED

- The strains are expressed in terms of the displacement field by using

$$\mathcal{H}_1 = H_1 \sqrt{1 + 2 \left(\hat{\mathbf{g}}_1 \cdot \frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \right) + \left(\frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \cdot \frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \right)}$$

$$\mathcal{H}_2 = H_2 \sqrt{1 + 2 \left(\hat{\mathbf{g}}_2 \cdot \frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \right) + \left(\frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \cdot \frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \right)}$$

$$\mathcal{H}_3 = H_3 \sqrt{1 + 2 \left(\hat{\mathbf{g}}_3 \cdot \frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \right) + \left(\frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \cdot \frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \right)}$$

$$\frac{\mathcal{H}_1 \mathcal{H}_2}{H_1 H_2} \cos \Theta_{12} - \cos \Theta_{12} = \left(\hat{\mathbf{g}}_1 \cdot \frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \right) + \left(\hat{\mathbf{g}}_2 \cdot \frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \right) + \left(\frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \cdot \frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \right)$$

$$\frac{\mathcal{H}_1 \mathcal{H}_3}{H_1 H_3} \cos \Theta_{13} - \cos \Theta_{13} = \left(\hat{\mathbf{g}}_1 \cdot \frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \right) + \left(\hat{\mathbf{g}}_3 \cdot \frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \right) + \left(\frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \cdot \frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \right)$$

$$\frac{\mathcal{H}_2 \mathcal{H}_3}{H_2 H_3} \cos \Theta_{23} - \cos \Theta_{23} = \left(\hat{\mathbf{g}}_2 \cdot \frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \right) + \left(\hat{\mathbf{g}}_3 \cdot \frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \right) + \left(\frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \cdot \frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \right)$$

NONLINEAR SHELL STRAINS - CONTINUED

- Therefore,
$$\epsilon_{11} = \left(\hat{\mathbf{g}}_1 \cdot \frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \right) + \frac{1}{2} \left(\frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \cdot \frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \right)$$

$$\epsilon_{22} = \left(\hat{\mathbf{g}}_2 \cdot \frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \right) + \frac{1}{2} \left(\frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \cdot \frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \right)$$

$$\epsilon_{33} = \left(\hat{\mathbf{g}}_3 \cdot \frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \right) + \frac{1}{2} \left(\frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \cdot \frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \right)$$

$$2\epsilon_{12} = \left(\hat{\mathbf{g}}_1 \cdot \frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \right) + \left(\hat{\mathbf{g}}_2 \cdot \frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \right) + \left(\frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \cdot \frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \right)$$

$$2\epsilon_{13} = \left(\hat{\mathbf{g}}_1 \cdot \frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \right) + \left(\hat{\mathbf{g}}_3 \cdot \frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \right) + \left(\frac{1}{H_1} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_1} \cdot \frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \right)$$

$$2\epsilon_{23} = \left(\hat{\mathbf{g}}_2 \cdot \frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \right) + \left(\hat{\mathbf{g}}_3 \cdot \frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \right) + \left(\frac{1}{H_2} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_2} \cdot \frac{1}{H_3} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_3} \right)$$

NONLINEAR SHELL STRAINS - CONTINUED

or

$$2\varepsilon_{ij} = \left(\hat{\mathbf{g}}_i \cdot \frac{1}{H_{(j)}} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_j} \right) + \left(\frac{1}{H_{(i)}} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_i} \cdot \hat{\mathbf{g}}_j \right) + \left(\frac{1}{H_{(i)}} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_i} \cdot \frac{1}{H_{(j)}} \frac{\partial \vec{\mathbf{U}}}{\partial \xi_j} \right)$$

- With these expressions, it follows that the geometries of $\mathcal{R}_0$ and $\mathcal{R}_t$ are related by

$$\mathcal{H}_1 = H_1 \sqrt{1 + 2\varepsilon_{11}}$$

$$\frac{\mathcal{H}_1 \mathcal{H}_2}{H_1 H_2} \cos \Theta_{12} - \cos \Theta_{12} = 2\varepsilon_{12}$$

$$\mathcal{H}_2 = H_2 \sqrt{1 + 2\varepsilon_{22}}$$

$$\frac{\mathcal{H}_1 \mathcal{H}_3}{H_1 H_3} \cos \Theta_{13} - \cos \Theta_{13} = 2\varepsilon_{13}$$

$$\mathcal{H}_3 = H_3 \sqrt{1 + 2\varepsilon_{33}}$$

$$\frac{\mathcal{H}_2 \mathcal{H}_3}{H_2 H_3} \cos \Theta_{23} - \cos \Theta_{23} = 2\varepsilon_{23}$$

NONLINEAR SHELL STRAINS - CONTINUED

- By using these expressions, the cosines and sines of the angles between intersecting convected coordinate curves are found to be

$$\cos \Theta_{12} = \frac{2\varepsilon_{12} + \cos \Theta_{12}}{\sqrt{1 + 2\varepsilon_{11}} \sqrt{1 + 2\varepsilon_{22}}}$$

$$\sin \Theta_{12} = \sqrt{1 - \frac{(2\varepsilon_{12} + \cos \Theta_{12})^2}{(1 + 2\varepsilon_{11})(1 + 2\varepsilon_{22})}}$$

$$\cos \Theta_{13} = \frac{2\varepsilon_{13} + \cos \Theta_{13}}{\sqrt{1 + 2\varepsilon_{11}} \sqrt{1 + 2\varepsilon_{33}}}$$

$$\sin \Theta_{13} = \sqrt{1 - \frac{(2\varepsilon_{13} + \cos \Theta_{13})^2}{(1 + 2\varepsilon_{11})(1 + 2\varepsilon_{33})}}$$

$$\cos \Theta_{23} = \frac{2\varepsilon_{23} + \cos \Theta_{23}}{\sqrt{1 + 2\varepsilon_{22}} \sqrt{1 + 2\varepsilon_{33}}}$$

$$\sin \Theta_{23} = \sqrt{1 - \frac{(2\varepsilon_{23} + \cos \Theta_{23})^2}{(1 + 2\varepsilon_{22})(1 + 2\varepsilon_{33})}}$$

- The differential area $d\mathcal{A}^{(\xi_3)} = H_1 H_2 \sin \Theta_{12} d\xi_1 d\xi_2$ becomes

$$d\mathcal{A}^{(\xi_3)} = H_1 H_2 \sqrt{(1 + 2\varepsilon_{11})(1 + 2\varepsilon_{22}) - (2\varepsilon_{12} + \cos \Theta_{12})^2} d\xi_1 d\xi_2$$

NONLINEAR SHELL STRAINS - CONTINUED

- Likewise, $d\mathcal{A}_{\langle 2 \rangle} = \mathcal{H}_1 \mathcal{H}_3 \sin \Theta_{13} d\xi_1 d\xi_3$ becomes

$$d\mathcal{A}_{\langle 2 \rangle} = H_1 H_3 \sqrt{(1 + 2\varepsilon_{11})(1 + 2\varepsilon_{33}) - (2\varepsilon_{13} + \cos\Theta_{13})^2} d\xi_1 d\xi_3$$

- and $d\mathcal{A}_{\langle 1 \rangle} = \mathcal{H}_2 \mathcal{H}_3 \sin \Theta_{23} d\xi_2 d\xi_3$ becomes

$$d\mathcal{A}_{\langle 1 \rangle} = H_2 H_3 \sqrt{(1 + 2\varepsilon_{22})(1 + 2\varepsilon_{33}) - (2\varepsilon_{23} + \cos\Theta_{23})^2} d\xi_2 d\xi_3$$

- Next,

$$\mathcal{H} = \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \sqrt{\sin^2 \Theta_{12} + \sin^2 \Theta_{13} + \sin^2 \Theta_{23} + 2(\cos \Theta_{12} \cos \Theta_{13} \cos \Theta_{23} - 1)}$$

becomes

$$\begin{aligned} \frac{\mathcal{H}^2}{H_1^2 H_2^2 H_3^2} = & (1 + 2\varepsilon_{11})(1 + 2\varepsilon_{22})(1 + 2\varepsilon_{33}) + 2(2\varepsilon_{12} + \cos\Theta_{12})(2\varepsilon_{13} + \cos\Theta_{13})(2\varepsilon_{23} + \cos\Theta_{23}) \\ & - (1 + 2\varepsilon_{33})(2\varepsilon_{12} + \cos\Theta_{12})^2 - (1 + 2\varepsilon_{22})(2\varepsilon_{13} + \cos\Theta_{13})^2 - (1 + 2\varepsilon_{11})(2\varepsilon_{23} + \cos\Theta_{23})^2 \end{aligned}$$

NONLINEAR SHELL STRAINS - CONCLUDED

- The differential volume is then obtained from $dV = J d\xi_1 d\xi_2 d\xi_3$
- For the shell coordinate system defined with respect to undeformed parallel surfaces, recall that $\Theta_{13} = \Theta_{23} = \frac{\pi}{2}$ and

$$\cos\Theta_{12} = \frac{\left(1 + \frac{\xi_3}{r_{11}}\right)\left(1 + \frac{\xi_3}{r_{22}}\right)\cos\theta_{12} + \left[\left(1 + \frac{\xi_3}{r_{11}}\right)\frac{\xi_3}{r_{21}} - \left(1 + \frac{\xi_3}{r_{22}}\right)\frac{\xi_3}{r_{12}}\right]\sin\theta_{12} + \frac{\xi_3}{r_{12}}\frac{\xi_3}{r_{21}}\cos\theta_{12}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{12}}\right)^2} \sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}}$$

“SMALL-STRAIN” APPROXIMATIONS

- For “**small strains**,” the magnitude of each of the nonlinear **Green-Lagrange strains** is presumed to be of second order, compared to unity
- This property is denoted herein by $\mathcal{O}(\varepsilon^2)$, with $\varepsilon^2 \ll 1$
- This presumption permits use of the **binomial theorem** and **Taylor series** for simplifying various strain expressions
- Applying the binomial theorem yields the following simplifications

$$\sqrt{1 + 2\varepsilon_{k(k)}} = 1 + \varepsilon_{k(k)} + \mathcal{O}(\varepsilon^4)$$

$$\frac{1}{1 - \varepsilon_{k(k)}} = 1 + \varepsilon_{k(k)} + \mathcal{O}(\varepsilon^4)$$

$$\frac{1}{\sqrt{1 + 2\varepsilon_{k(k)}}} = 1 - \varepsilon_{k(k)} + \mathcal{O}(\varepsilon^4)$$

$$\frac{1}{1 + \varepsilon_{k(k)}} = 1 - \varepsilon_{k(k)} + \mathcal{O}(\varepsilon^4)$$

“SMALL-STRAIN” APPROXIMATIONS - CONTINUED

- Applying these simplifications to the metric coefficients for the deformed shell yields $\mathcal{H}_k = H_k \left[1 + \varepsilon_{(kk)} + \mathcal{O}(\varepsilon^4) \right]$
- By definition, the **shearing strains**, denoted by γ_{ij} , are given by the changes in angles; that is, $\gamma_{ij} \equiv \Theta_{ij} - \Theta_{ij}$

- Thus,

$$\cos \Theta_{ij} = \cos(\Theta_{ij} - \gamma_{ij}) = \cos \gamma_{ij} \cos \Theta_{(ij)} + \sin \gamma_{ij} \sin \Theta_{(ij)}$$

$$\sin \Theta_{ij} = \sin(\Theta_{ij} - \gamma_{ij}) = \sin \Theta_{(ij)} \cos \gamma_{ij} - \cos \Theta_{(ij)} \sin \gamma_{ij}$$

- For “small” shearing strains, the Taylor series expansions of the sine and cosine functions with small-magnitude arguments yields

$$\cos \gamma_{ij} = 1 + \mathcal{O}(\varepsilon^4) \quad \text{and} \quad \sin \gamma_{ij} = \gamma_{ij} + \mathcal{O}(\varepsilon^4)$$

“SMALL-STRAIN” APPROXIMATIONS - CONTINUED

- Applying these simplifications gives

$$\cos \Theta_{ij} = \cos \Theta_{ij} + \gamma_{ij} \sin \Theta_{(ij)} + \mathcal{O}(\varepsilon^4) \quad \text{and}$$

$$\sin \Theta_{ij} = \sin \Theta_{ij} - \gamma_{ij} \cos \Theta_{(ij)} + \mathcal{O}(\varepsilon^4)$$

- For “small” elongations, $e_{\hat{T}}$ is presumed to also be $\mathcal{O}(\varepsilon^2)$ such that

$$\left(1 + e_{\hat{T}}\right)^2 - 1 = \frac{d\mathcal{L}^2 - dL^2}{dL^2} \quad \text{becomes}$$

$$2e_{\hat{T}} = \frac{d\mathcal{L}^2 - dL^2}{dL^2} = 2\varepsilon_{jk} \frac{H_{(j)} d\xi_j}{dL} \frac{H_{(k)} d\xi_k}{dL} + \mathcal{O}(\varepsilon^4)$$

“SMALL-STRAIN” APPROXIMATIONS - CONTINUED

- Next, $\cos \Theta_{ij} = \cos \Theta_{ij} + \gamma_{ij} \sin \Theta_{(ij)} + \mathcal{O}(\varepsilon^4)$ and $\mathcal{H}_k = H_k [1 + \varepsilon_{(kk)} + \mathcal{O}(\varepsilon^4)]$

are substituted into $\cos \Theta_{12} = \frac{2\varepsilon_{12} + \cos \Theta_{12}}{\sqrt{1 + 2\varepsilon_{11}} \sqrt{1 + 2\varepsilon_{22}}}$ to get

$$\gamma_{12} = \frac{2\varepsilon_{12} - (\varepsilon_{11} + \varepsilon_{22}) \cos \Theta_{12}}{\sin \Theta_{12}} + \mathcal{O}(\varepsilon^4)$$

- Likewise,

$$\gamma_{13} = \frac{2\varepsilon_{13} - (\varepsilon_{11} + \varepsilon_{33}) \cos \Theta_{13}}{\sin \Theta_{13}} + \mathcal{O}(\varepsilon^4)$$

$$\gamma_{23} = \frac{2\varepsilon_{23} - (\varepsilon_{22} + \varepsilon_{33}) \cos \Theta_{23}}{\sin \Theta_{23}} + \mathcal{O}(\varepsilon^4)$$

“SMALL-STRAIN” APPROXIMATIONS - CONTINUED

- “Small” strain approximations to the differential areas and differential volume are obtained as follows
- Substituting $\mathcal{H}_k = H_k [1 + \varepsilon_{(kk)} + \mathcal{O}(\varepsilon^4)]$, $\sin \Theta_{ij} = \sin \Theta_{ij} - \gamma_{ij} \cos \Theta_{(ij)} + \mathcal{O}(\varepsilon^4)$, and $\cos \Theta_{ij} = \cos \Theta_{ij} + \gamma_{ij} \sin \Theta_{(ij)} + \mathcal{O}(\varepsilon^4)$ into $d\mathcal{A}(\xi_3) = \mathcal{H}_1 \mathcal{H}_2 \sin \Theta_{12} d\xi_1 d\xi_2$ and simplifying yields

$$d\mathcal{A}(\xi_3) = H_1 H_2 \left[\sin \Theta_{12} (1 + \varepsilon_{11} + \varepsilon_{22}) - \gamma_{12} \cos \Theta_{12} \right] d\xi_1 d\xi_2 + \mathcal{O}(\varepsilon^4)$$

- Then, $\gamma_{12} = \frac{2\varepsilon_{12} - (\varepsilon_{11} + \varepsilon_{22}) \cos \Theta_{12}}{\sin \Theta_{12}} + \mathcal{O}(\varepsilon^4)$ is used to get

$$d\mathcal{A}(\xi_3) = H_1 H_2 \left[\sin \Theta_{12} + (\varepsilon_{11} + \varepsilon_{22}) \csc \Theta_{12} - 2\varepsilon_{12} \cot \Theta_{12} + \mathcal{O}(\varepsilon^4) \right] d\xi_1 d\xi_2$$

“SMALL-STRAIN” APPROXIMATIONS - CONTINUED

- Similarly, $d\mathcal{A}_{\langle 2 \rangle} = \mathcal{H}_1 \mathcal{H}_3 \sin \Theta_{13} d\xi_1 d\xi_3$ and $d\mathcal{A}_{\langle 1 \rangle} = \mathcal{H}_2 \mathcal{H}_3 \sin \Theta_{23} d\xi_2 d\xi_3$ become

$$d\mathcal{A}_{\langle 2 \rangle} = H_1 H_3 \left[\sin \Theta_{13} + (\epsilon_{11} + \epsilon_{33}) \csc \Theta_{13} - 2\epsilon_{13} \cot \Theta_{13} + \mathcal{O}(\epsilon^4) \right]$$

$$d\mathcal{A}_{\langle 1 \rangle} = H_2 H_3 \left[\sin \Theta_{23} + (\epsilon_{22} + \epsilon_{33}) \csc \Theta_{23} - 2\epsilon_{23} \cot \Theta_{23} + \mathcal{O}(\epsilon^4) \right]$$

- For the coordinate system defined with respect to undeformed parallel surfaces, $\Theta_{13} = \Theta_{23} = \frac{\pi}{2}$ and, as a result,

$$\gamma_{12} = 2\epsilon_{12} \csc \Theta_{12} - (\epsilon_{11} + \epsilon_{22}) \cot \Theta_{12} + \mathcal{O}(\epsilon^4)$$

$$\gamma_{13} = 2\epsilon_{13} + \mathcal{O}(\epsilon^4)$$

$$\gamma_{23} = 2\epsilon_{23} + \mathcal{O}(\epsilon^4)$$

“SMALL-STRAIN” APPROXIMATIONS - CONTINUED

- In addition,

$$d\mathcal{A}_{\langle 1 \rangle} = H_2 H_3 \left[1 + \varepsilon_{22} + \varepsilon_{33} + \mathcal{O}(\varepsilon^4) \right]$$

$$d\mathcal{A}_{\langle 2 \rangle} = H_1 H_3 \left[1 + \varepsilon_{11} + \varepsilon_{33} + \mathcal{O}(\varepsilon^4) \right]$$

$$d\mathcal{A}_{(\xi_3)} = H_1 H_2 \left[\sin\Theta_{12} + (\varepsilon_{11} + \varepsilon_{22}) \csc\Theta_{12} - 2\varepsilon_{12} \cot\Theta_{12} + \mathcal{O}(\varepsilon^4) \right]$$

- To simplify

$$\mathcal{H} = \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \sqrt{\sin^2\Theta_{12} + \sin^2\Theta_{13} + \sin^2\Theta_{23} + 2(\cos\Theta_{12}\cos\Theta_{13}\cos\Theta_{23} - 1)}$$

it is noted that $\sin^2\Theta_{ij} = \sin^2\Theta_{ij} - 2\gamma_{ij} \cos\Theta_{(ij)} + \mathcal{O}(\varepsilon^4)$ and that

$$\begin{aligned} \cos\Theta_{12}\cos\Theta_{13}\cos\Theta_{23} &= \cos\Theta_{12}\cos\Theta_{13}\cos\Theta_{23} + \gamma_{13} \cos\Theta_{12}\sin\Theta_{13}\cos\Theta_{23} \\ &+ \gamma_{12} \sin\Theta_{12}\cos\Theta_{13}\cos\Theta_{23} + \gamma_{23} \cos\Theta_{12}\cos\Theta_{13}\sin\Theta_{23} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

“SMALL-STRAIN” APPROXIMATIONS - CONTINUED

- For the coordinate system defined with respect to undeformed parallel surfaces, $\Theta_{13} = \Theta_{23} = \frac{\pi}{2}$, and the following simplifications are obtained

$$\sin^2 \Theta_{12} + \sin^2 \Theta_{13} + \sin^2 \Theta_{23} = 2 + \sin^2 \Theta_{12} - 2\gamma_{12} \cos \Theta_{12} + \mathcal{O}(\varepsilon^4)$$

$$2(\cos \Theta_{12} \cos \Theta_{13} \cos \Theta_{23} - 1) = -2 + \mathcal{O}(\varepsilon^4)$$

$$\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 = H_1 H_2 (1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

$$\mathcal{H} = H_1 H_2 H_3 \sin \Theta_{12} (1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \sqrt{1 - 2\gamma_{12} \cot \Theta_{12} \csc \Theta_{12} + \mathcal{O}(\varepsilon^4)}$$

$$\mathcal{H} = H_1 H_2 H_3 \sin \Theta_{12} (1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \left[1 - \gamma_{12} \cot \Theta_{12} \csc \Theta_{12} + \mathcal{O}(\varepsilon^4) \right]$$

$$\mathcal{H} = H_1 H_2 H_3 \sin \Theta_{12} \left(1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} - \gamma_{12} \cot \Theta_{12} \csc \Theta_{12} + \mathcal{O}(\varepsilon^4) \right)$$

“SMALL-STRAIN” APPROXIMATIONS - CONCLUDED

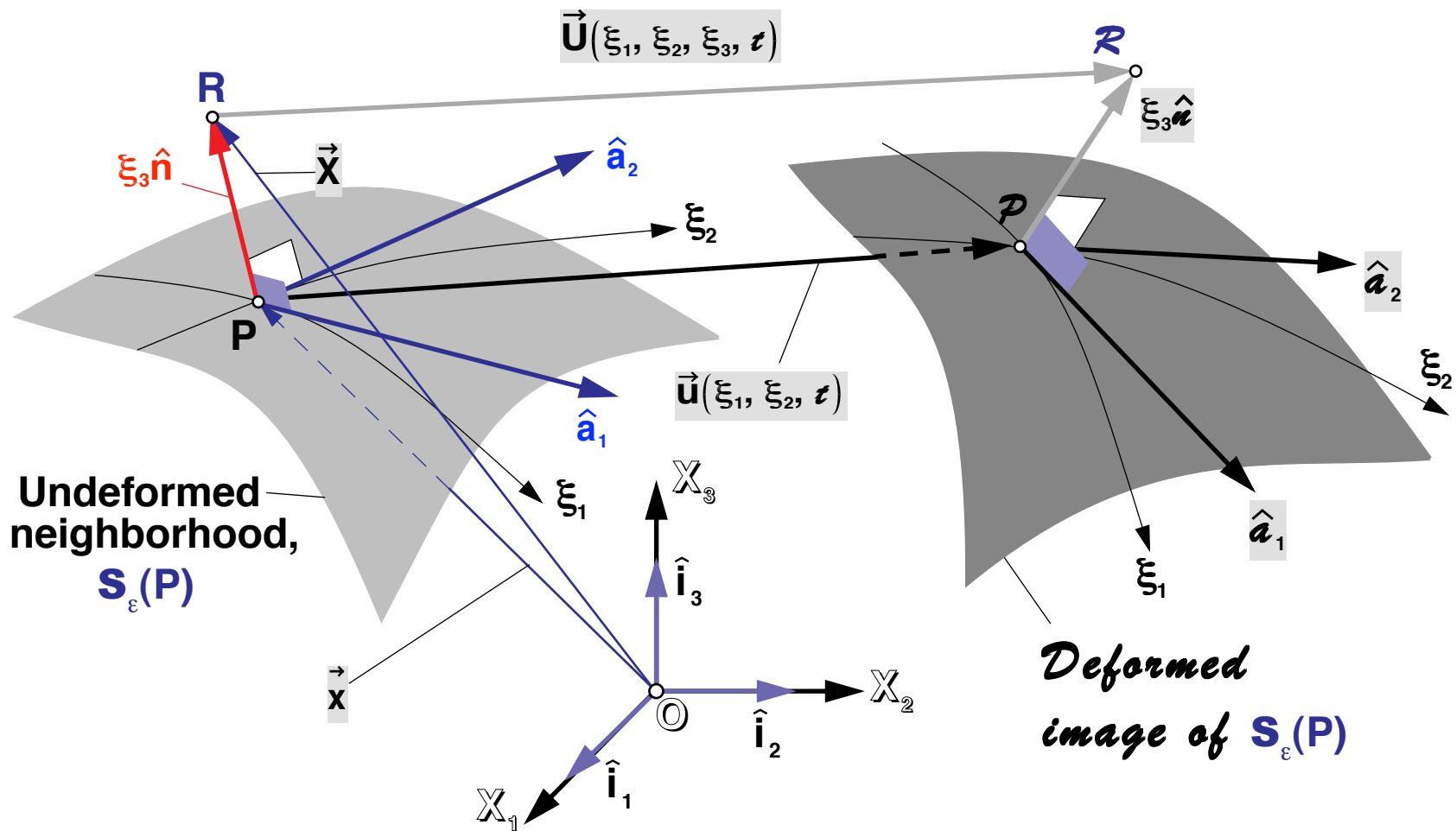
- Using $\gamma_{12} = \frac{2\varepsilon_{12} - (\varepsilon_{11} + \varepsilon_{22})\cos\Theta_{12}}{\sin\Theta_{12}} + \mathcal{O}(\varepsilon^4)$ gives

$$\mathcal{H} = H_1 H_2 H_3 \sin\Theta_{12} \left(1 + (\varepsilon_{11} + \varepsilon_{22}) \left(1 + \csc\Theta_{12} \cot^2\Theta_{12} \right) - 2\varepsilon_{12} \csc^2\Theta_{12} \cot\Theta_{12} + \varepsilon_{33} + \mathcal{O}(\varepsilon^4) \right)$$

THE KINEMATIC HYPOTHESIS

- Up to this point in the present study, no relationship between the displacement-vector field $\vec{u}(\xi_1, \xi_2, t)$, for points of the undeformed reference surface, and the corresponding displacement-vector field $\vec{U}(\xi_1, \xi_2, \xi_3, t)$ for points of a corresponding undeformed parallel surface has been given
- A **kinematic hypothesis** is required to make this connection
- Let **R** be a point of a parallel surface given by $\vec{X} = \vec{x}(\xi_1, \xi_2) + \xi_3 \hat{n}(\xi_1, \xi_2)$ and **P** be the corresponding point of the reference surface, as shown in the following figure
- For the **Kirchhoff hypothesis** of deformation, used in **classical shell theory**, the particle $\mathcal{R} = \mathcal{D}_t(\mathbf{R})$ is always located on the line normal to the tangent plane spanned by $\hat{a}_1$ and $\hat{a}_2$ during deformation and it also remains at the same distance from the tangent plane

THE KINEMATIC HYPOTHESIS - CONTINUED



THE KINEMATIC HYPOTHESIS - CONTINUED

- The deformed image of $\mathbf{R}$ is indicated as $\mathcal{R}$ in the figure and is given by

$$\vec{\mathbf{U}}(\xi_1, \xi_2, \xi_3, \boldsymbol{\tau}) = \vec{\mathbf{u}}(\xi_1, \xi_2, \boldsymbol{\tau}) + \xi_3 \left[\hat{\boldsymbol{n}}(\xi_1, \xi_2, \boldsymbol{\tau}) - \hat{\mathbf{n}}(\xi_1, \xi_2) \right]$$

where, $\hat{\boldsymbol{n}}(\xi_1, \xi_2, \boldsymbol{\tau})$ is the field of unit-magnitude vectors that are perpendicular the corresponding tangent plane of the deformed shell

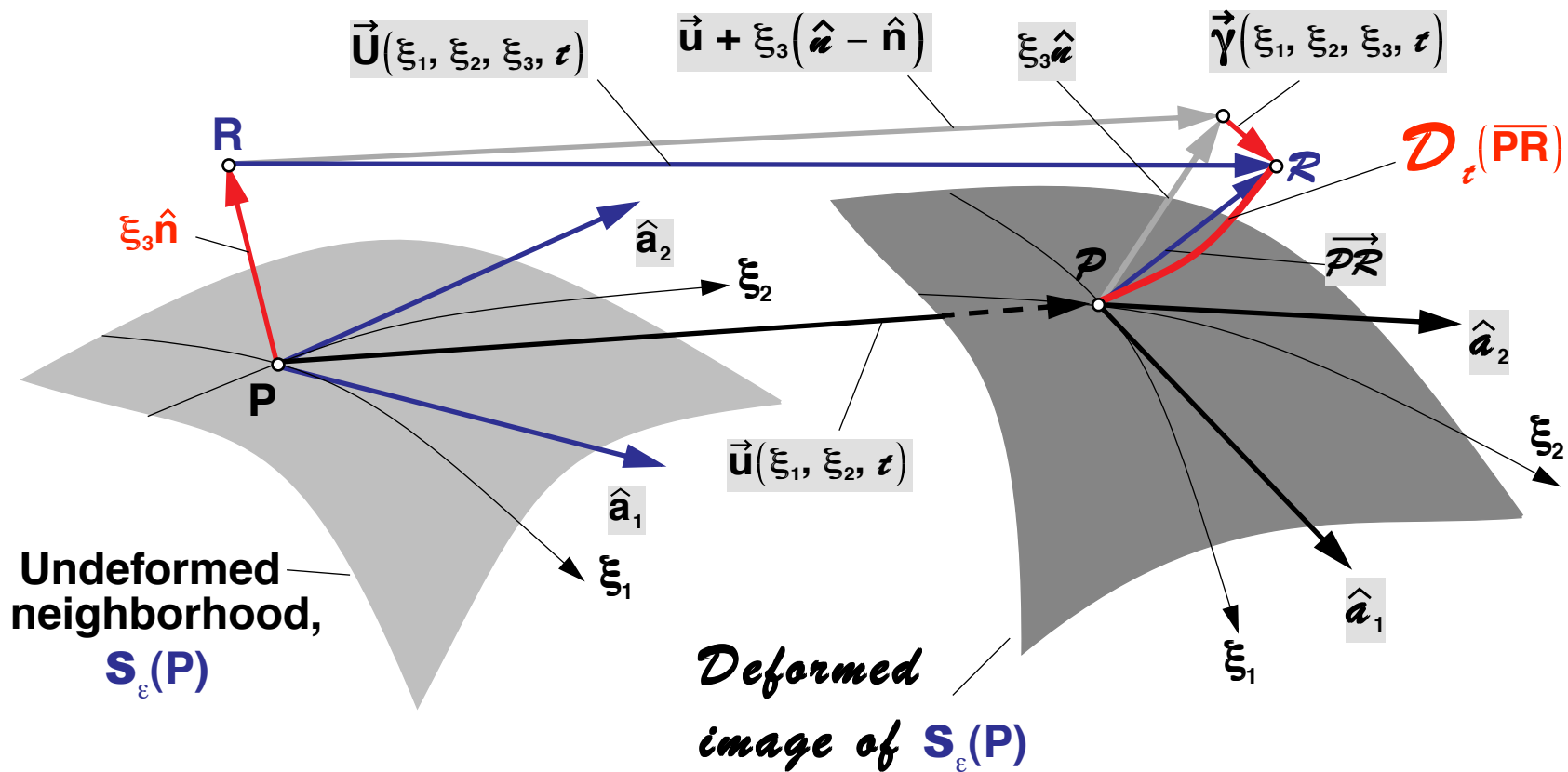
- It is convenient to introduce the **difference-vector field** $\vec{\Phi}$ given by

$$\vec{\Phi}(\xi_1, \xi_2, \boldsymbol{\tau}) \equiv \hat{\boldsymbol{n}}(\xi_1, \xi_2, \boldsymbol{\tau}) - \hat{\mathbf{n}}(\xi_1, \xi_2) \quad \text{such that} \quad \vec{\mathbf{U}} = \vec{\mathbf{u}} + \xi_3 \vec{\Phi}$$

- Recall that, in general, rotation of $\hat{\mathbf{n}}$ into $\hat{\boldsymbol{n}}$ is characterized by an **orthogonal transformation** of the form $\hat{\boldsymbol{n}} = \mathcal{R}(\hat{\mathbf{n}})$; thus, $\vec{\Phi} = \mathcal{R}(\hat{\mathbf{n}}) - \hat{\mathbf{n}}$

THE KINEMATIC HYPOTHESIS - CONTINUED

- A more general class of deformations, depicted in the following figure, is considered in the present study



THE KINEMATIC HYPOTHESIS - CONTINUED

- For this type of deformation, the material point **R** moves to the position **R**, which is located by the vector $\overrightarrow{\mathcal{P}\mathcal{R}}$
- Moreover, the material line element $\overline{\mathcal{P}\mathcal{R}}$ deforms into the curve $\mathcal{D}_t(\overline{\mathcal{P}\mathcal{R}})$
- The vector field $\vec{\gamma}(\xi_1, \xi_2, \xi_3, t)$ shown in the figure is given by
 $\vec{\gamma} = \overline{\mathcal{P}\mathcal{R}} - \xi_3 \hat{\mathcal{A}}$ and locates the point $\mathcal{R} \in \mathcal{D}_t(\overline{\mathcal{P}\mathcal{R}})$
- The basic kinematic hypothesis used in the present study to represent this type of deformation is given by

$$\vec{\mathcal{U}} = \vec{\mathcal{U}} + \xi_3 \left(\hat{\mathcal{A}} - \hat{\mathcal{n}} \right) + \vec{\gamma}$$

and includes that of **classical shell theory** as a well-defined special case

THE KINEMATIC HYPOTHESIS - CONTINUED

- In particular, the term $\xi_3 \left(\hat{\boldsymbol{n}} - \hat{\boldsymbol{n}} \right)$ corresponds to the following deformations presumed in **classical Love-Kirchhoff shell theory**
- **A line of material particles that are initially perpendicular to the tangent plane at a given point of the reference surface remains so during deformation**
- **Extension of this perpendicular material line element is negligible during deformation**
- The vector field $\vec{\gamma}(\xi_1, \xi_2, \xi_3, t)$ defines a general nonlinear distribution of transverse-shearing deformations across the shell thickness, as shown in the previous **figure**
- The specific functional form through the shell thickness is user defined and should be guided by practical considerations and experimental evidence

THE KINEMATIC HYPOTHESIS - CONCLUDED

- With a kinematic hypothesis such as

$$\vec{U} = \vec{u} + \xi_3 \left(\hat{n} - \hat{n} \right) + \vec{\gamma} = \vec{u} + \xi_3 \vec{\Phi} + \vec{\gamma}$$

given, general expressions for the Green-Lagrange shell strains in terms of the components of $\vec{u}(\xi_1, \xi_2, \tau)$, $\vec{\Phi}(\xi_1, \xi_2, \tau)$, and $\vec{\gamma}(\xi_1, \xi_2, \xi_3, \tau)$ can be obtained directly from the nonlinear strain-displacement relations

$$2\varepsilon_{ij} = \left(\hat{g}_i \cdot \frac{1}{H_{(j)}} \frac{\partial \vec{U}}{\partial \xi_j} \right) + \left(\frac{1}{H_{(i)}} \frac{\partial \vec{U}}{\partial \xi_i} \cdot \hat{g}_j \right) + \left(\frac{1}{H_{(i)}} \frac{\partial \vec{U}}{\partial \xi_i} \cdot \frac{1}{H_{(j)}} \frac{\partial \vec{U}}{\partial \xi_j} \right)$$

- **However, for the kinematic hypothesis of the present study, an alternate approach is presented that shows how the differences in the geometries of the deformed and undeformed reference surfaces affect the shell strains**

ALTERNATE FORMULATION OF THE SHELL STRAINS

ALTERNATE FORMULATION OF SHELL STRAINS

- To see clearly how the geometric parameters of the undeformed and deformed shell configurations contribute to the shell strains, an **alternate form of the kinematic hypothesis** proposed previously herein is used; that is,

$$\vec{\mathcal{X}}(\xi_1, \xi_2, \xi_3, t) = \vec{\mathcal{X}}(\xi_1, \xi_2, t) + \xi_3 \hat{\mathcal{a}}(\xi_1, \xi_2, t) + \vec{\gamma}(\xi_1, \xi_2, \xi_3, t)$$

- Substituting this expression into $\vec{g}_k = \frac{\partial \vec{\mathcal{X}}}{\partial \xi_k}$ and using $\hat{\mathcal{a}}_\alpha = \frac{1}{\mathcal{A}_{(\alpha)}} \frac{\partial \vec{\mathcal{X}}}{\partial \xi_\alpha}$ gives

$$\vec{g}_\alpha = \mathcal{A}_{(\alpha)} \left[\hat{\mathcal{a}}_\alpha + \xi_3 \frac{1}{\mathcal{A}_{(\alpha)}} \frac{\partial \hat{\mathcal{a}}}{\partial \xi_\alpha} + \frac{1}{\mathcal{A}_{(\alpha)}} \frac{\partial \vec{\gamma}}{\partial \xi_\alpha} \right] \quad \text{and} \quad \vec{g}_3 = \hat{\mathcal{a}} + \frac{\partial \vec{\gamma}}{\partial \xi_3}$$

ALTERNATE FORMULATION OF SHELL STRAINS

CONTINUED

- Previously, it was shown herein that the Green-Lagrange shell strains are given by

$$2\varepsilon_{ij} H_{(i)} H_{(j)} = \left(\vec{\mathcal{G}}_i \cdot \vec{\mathcal{G}}_j \right) - \left(\vec{g}_i \cdot \vec{g}_j \right)$$

- These strains are expressed more explicitly as

$$2\varepsilon_{\alpha\beta} = \frac{A_{(\alpha)} A_{(\beta)}}{H_{(\alpha)} H_{(\beta)}} \left[\left(\frac{\vec{\mathcal{G}}_\alpha}{A_{(\alpha)}} \cdot \frac{\vec{\mathcal{G}}_\beta}{A_{(\beta)}} \right) - \left(\frac{\vec{g}_\alpha}{A_{(\alpha)}} \cdot \frac{\vec{g}_\beta}{A_{(\beta)}} \right) \right]$$

$$2\varepsilon_{\alpha 3} H_{(\alpha)} = \left(\vec{\mathcal{G}}_\alpha \cdot \vec{\mathcal{G}}_3 \right) - \left(\vec{g}_\alpha \cdot \vec{g}_3 \right)$$

$$2\varepsilon_{33} = \left(\vec{\mathcal{G}}_3 \cdot \vec{\mathcal{G}}_3 \right) - \left(\vec{g}_3 \cdot \vec{g}_3 \right)$$

where it is noted that the shell metric coefficient $H_3 = 1$

ALTERNATE FORMULATION OF SHELL STRAINS

CONCLUDED

- In the kinematic hypothesis $\vec{\mathcal{X}} = \vec{x} + \xi_3 \hat{a} + \vec{\gamma}$, deformation is envisioned as “small” transverse shearing deformations superimposed on a classical Love-Kirchhoff-type “small” strain - “large” rotation deformation state, in the deformed configuration

- The position vector $\vec{x}(\xi_1, \xi_2, t)$ accounts for the reference-surface displacements and $\hat{a}(\xi_1, \xi_2, t)$ is associated with the rigid-body rotation of $\hat{n}(\xi_1, \xi_2)$, and the corresponding reference-surface tangent plane, into the deformed configuration

- The vector field $\vec{\gamma}(\xi_1, \xi_2, \xi_3, t)$ is presumed to have the form

$$\vec{\gamma} = f_1(\xi_3) \Lambda_1 \hat{a}_1 + f_2(\xi_3) \Lambda_2 \hat{a}_2 + f_3(\xi_3) \Lambda_3 \hat{a}$$

where the unknown functions $\Lambda_k = \Lambda_k(\xi_1, \xi_2, t)$ and $\Lambda_k(\xi_1, \xi_2, 0) = 0$

THROUGH-THE-THICKNESS NORMAL STRAIN

- First, consider the transverse, through-the-thickness normal strain

$$2\varepsilon_{33} = (\vec{g}_3 \cdot \vec{g}_3) - (\vec{g}_3 \cdot \vec{g}_3)$$

- Substituting $\vec{g}_3 = \hat{n} + \frac{\partial \vec{\gamma}}{\partial \xi_3}$ into this strain expression, and noting that

$$\vec{g}_3 = \hat{n} \quad \text{and} \quad \hat{n} \cdot \hat{n} = 1, \quad \text{gives} \quad \varepsilon_{33} = \left(\hat{n} \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} \right) + \frac{1}{2} \left| \frac{\partial \vec{\gamma}}{\partial \xi_3} \right|^2$$

- Herein, it is presumed that the transverse, through-the-thickness normal strain ε_{33} is negligible compared to the other strains

- Thus, $\left| \frac{\partial \vec{\gamma}}{\partial \xi_3} \right|$ is presumed to be of $\mathcal{O}(\varepsilon^2)$ and $\hat{n} \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} = f_3'(\xi_3) \Lambda_3 = 0$

such that $\varepsilon_{33} = 0 + \mathcal{O}(\varepsilon^4)$ and $\vec{\gamma} = f_1(\xi_3) \Lambda_1 \hat{a}_1 + f_2(\xi_3) \Lambda_2 \hat{a}_2$

TRANSVERSE SHEARING STRAINS

- In the expression $\mathbf{f}_3'(\xi_3)$, the **prime mark denotes differentiation with respect to the independent variable ξ_3**
- In the present study, the transverse shearing deformations and strains are presumed to be relatively “small”
- Thus, the functions $\Lambda_\alpha(\xi_1, \xi_2, \tau)$, which completely characterize the transverse shearing deformations of the shell, are presumed to be of $\mathcal{O}(\varepsilon^2)$

- First, substituting $\vec{\mathcal{G}}_3 = \hat{\mathbf{u}} + \frac{\partial \vec{\gamma}}{\partial \xi_3}$ into $2\varepsilon_{\alpha 3} H_{(\alpha)} = (\vec{\mathcal{G}}_\alpha \cdot \vec{\mathcal{G}}_3) - (\vec{\mathbf{g}}_\alpha \cdot \vec{\mathbf{g}}_3)$

and noting that $\vec{\mathbf{g}}_\alpha \cdot \vec{\mathbf{g}}_3 = 0$ gives

$$2\varepsilon_{13} \frac{H_1}{A_1} = \frac{\vec{\mathcal{G}}_1}{A_1} \cdot \left(\hat{\mathbf{u}} + \frac{\partial \vec{\gamma}}{\partial \xi_3} \right) \quad \text{and} \quad 2\varepsilon_{23} \frac{H_2}{A_2} = \frac{\vec{\mathcal{G}}_2}{A_2} \cdot \left(\hat{\mathbf{u}} + \frac{\partial \vec{\gamma}}{\partial \xi_3} \right)$$

TRANSVERSE SHEARING STRAINS

CONTINUED

- Then, using $\vec{g}_\alpha = \mathcal{A}_{(\alpha)} \left[\hat{a}_\alpha + \xi_3 \frac{1}{\mathcal{A}_{(\alpha)}} \frac{\partial \hat{u}}{\partial \xi_\alpha} + \frac{1}{\mathcal{A}_{(\alpha)}} \frac{\partial \vec{\gamma}}{\partial \xi_\alpha} \right]$ gives

$$2\varepsilon_{13} \frac{H_1}{\mathcal{A}_1} = \left[\hat{a}_1 + \xi_3 \frac{1}{\mathcal{A}_1} \frac{\partial \hat{u}}{\partial \xi_1} + \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right] \cdot \left(\hat{u} + \frac{\partial \vec{\gamma}}{\partial \xi_3} \right) \quad \text{and}$$

$$2\varepsilon_{23} \frac{H_2}{\mathcal{A}_2} = \left[\hat{a}_2 + \xi_3 \frac{1}{\mathcal{A}_2} \frac{\partial \hat{u}}{\partial \xi_2} + \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \right] \cdot \left(\hat{u} + \frac{\partial \vec{\gamma}}{\partial \xi_3} \right)$$

- Noting that $\hat{a}_\alpha \cdot \hat{u} = 0$, and that $\hat{u} \cdot \hat{u} = 1$ leads to $\frac{1}{\mathcal{A}_{(\alpha)}} \frac{\partial \hat{u}}{\partial \xi_\alpha} \cdot \hat{u} = 0$,
gives

TRANSVERSE SHEARING STRAINS

CONTINUED

$$2\varepsilon_{13} \frac{H_1}{\mathcal{A}_1} = \left(\hat{\mathbf{a}}_1 \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} \right) + \xi_3 \left(\frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} \right) + \left(\frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \cdot \hat{\mathbf{u}} \right) + \left(\frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} \right)$$

and

$$2\varepsilon_{23} \frac{H_2}{\mathcal{A}_2} = \left(\hat{\mathbf{a}}_2 \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} \right) + \xi_3 \left(\frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} \right) + \left(\frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \cdot \hat{\mathbf{u}} \right) + \left(\frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} \right)$$

- The nonlinear terms given by the last term in each of these two strain equations are presumed to be of $\mathcal{O}(\varepsilon^4)$, and are neglected

- In addition, differentiating $\hat{\mathbf{u}} \cdot \vec{\gamma} = 0$ gives $\frac{1}{\mathcal{A}_{(\alpha)}} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_\alpha} \cdot \vec{\gamma} = - \frac{1}{\mathcal{A}_{(\alpha)}} \frac{\partial \vec{\gamma}}{\partial \xi_\alpha} \cdot \hat{\mathbf{u}}$

TRANSVERSE SHEARING STRAINS

CONTINUED

- Together, these results produce

$$2\varepsilon_{13} \frac{H_1}{\mathcal{A}_1} = \left(\hat{\mathbf{a}}_1 \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} \right) + \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_1} \cdot \left(\xi_3 \frac{\partial \vec{\gamma}}{\partial \xi_3} - \vec{\gamma} \right) + \mathcal{O}(\varepsilon^4)$$

$$2\varepsilon_{23} \frac{H_2}{\mathcal{A}_2} = \left(\hat{\mathbf{a}}_2 \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} \right) + \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_2} \cdot \left(\xi_3 \frac{\partial \vec{\gamma}}{\partial \xi_3} - \vec{\gamma} \right) + \mathcal{O}(\varepsilon^4)$$

- For convenience, let $\vec{\Gamma} = \xi_3 \frac{\partial \vec{\gamma}}{\partial \xi_3} - \vec{\gamma}$ such that

$$2\varepsilon_{13} \frac{H_1}{\mathcal{A}_1} = \left(\hat{\mathbf{a}}_1 \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} \right) + \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_1} \cdot \vec{\Gamma} + \mathcal{O}(\varepsilon^4) \quad \text{and}$$

$$2\varepsilon_{23} \frac{H_2}{\mathcal{A}_2} = \left(\hat{\mathbf{a}}_2 \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} \right) + \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_2} \cdot \vec{\Gamma} + \mathcal{O}(\varepsilon^4)$$

TRANSVERSE SHEARING STRAINS

CONTINUED

- Expressing $\vec{\Gamma} = p_1(\xi_3)\Lambda_1\hat{a}_1 + p_2(\xi_3)\Lambda_2\hat{a}_2$ and using $\vec{\Gamma} = \xi_3 \frac{\partial \vec{\gamma}}{\partial \xi_3} - \vec{\gamma}$, it follows that

$$p_1(\xi_3) = \xi_3 f_1'(\xi_3) - f_1(\xi_3) \quad \text{and} \quad p_2(\xi_3) = \xi_3 f_2'(\xi_3) - f_2(\xi_3)$$

- Next, using $\vec{a}_\alpha = \mathcal{A}_{(\alpha)} \hat{a}_\alpha$ and $\mathcal{A}_1 \mathcal{A}_2 \cos \theta_{12} = \vec{a}_1 \cdot \vec{a}_2$ with

$$\vec{\gamma} = f_1(\xi_3)\Lambda_1\hat{a}_1 + f_2(\xi_3)\Lambda_2\hat{a}_2 \quad \text{and} \quad \vec{\Gamma} = p_1(\xi_3)\Lambda_1\hat{a}_1 + p_2(\xi_3)\Lambda_2\hat{a}_2 \quad \text{gives}$$

$$\hat{a}_1 \cdot \vec{\gamma} = f_1(\xi_3)\Lambda_1 + f_2(\xi_3)\Lambda_2 \cos \theta_{12}$$

$$\hat{a}_2 \cdot \vec{\gamma} = f_2(\xi_3)\Lambda_2 + f_1(\xi_3)\Lambda_1 \cos \theta_{12}$$

$$\hat{a}_1 \cdot \vec{\Gamma} = p_1(\xi_3)\Lambda_1 + p_2(\xi_3)\Lambda_2 \cos \theta_{12}$$

$$\hat{a}_2 \cdot \vec{\Gamma} = p_2(\xi_3)\Lambda_2 + p_1(\xi_3)\Lambda_1 \cos \theta_{12}$$

TRANSVERSE SHEARING STRAINS

CONTINUED

- Then, using

$$\cos\theta_{12} = 2\varepsilon_{12}^o + \cos\theta_{12}\left(1 - \varepsilon_{11}^o - \varepsilon_{22}^o\right) + \mathcal{O}(\varepsilon^4),$$

for “small” strains, with the last four equations yields

$$\hat{\mathbf{a}}_1 \cdot \vec{\gamma} = \mathbf{f}_1(\xi_3)\Lambda_1 + \mathbf{f}_2(\xi_3)\Lambda_2\cos\theta_{12} + \mathcal{O}(\varepsilon^4)$$

$$\hat{\mathbf{a}}_2 \cdot \vec{\gamma} = \mathbf{f}_2(\xi_3)\Lambda_2 + \mathbf{f}_1(\xi_3)\Lambda_1\cos\theta_{12} + \mathcal{O}(\varepsilon^4)$$

$$\hat{\mathbf{a}}_1 \cdot \vec{\Gamma} = \mathbf{p}_1(\xi_3)\Lambda_1 + \mathbf{p}_2(\xi_3)\Lambda_2\cos\theta_{12} + \mathcal{O}(\varepsilon^4)$$

$$\hat{\mathbf{a}}_2 \cdot \vec{\Gamma} = \mathbf{p}_2(\xi_3)\Lambda_2 + \mathbf{p}_1(\xi_3)\Lambda_1\cos\theta_{12} + \mathcal{O}(\varepsilon^4)$$

TRANSVERSE SHEARING STRAINS

CONTINUED

- Further simplification of the equations

$$2\varepsilon_{13} \frac{H_1}{\mathcal{A}_1} = \left(\hat{\mathbf{a}}_1 \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} \right) + \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \cdot \vec{\Gamma} + \mathcal{O}(\varepsilon^4) \quad \text{and}$$

$$2\varepsilon_{23} \frac{H_2}{\mathcal{A}_2} = \left(\hat{\mathbf{a}}_2 \cdot \frac{\partial \vec{\gamma}}{\partial \xi_3} \right) + \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \cdot \vec{\Gamma} + \mathcal{O}(\varepsilon^4)$$

is obtained by using the equations for the derivatives of $\hat{\mathbf{u}}(\xi_1, \xi_2, \tau)$ for a general set of nonorthogonal curvilinear Gaussian coordinates

- The derivatives are given by

$$\frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} = \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) \hat{\mathbf{a}}_1 - \frac{\csc \theta_{12}}{r_{12}} \hat{\mathbf{a}}_2$$

$$\frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} = \frac{\csc \theta_{12}}{r_{21}} \hat{\mathbf{a}}_1 + \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right) \hat{\mathbf{a}}_2$$

TRANSVERSE SHEARING STRAINS CONTINUED

- Thus,

$$\frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_1} \cdot \vec{\Gamma} = \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) (\hat{\mathbf{a}}_1 \cdot \vec{\Gamma}) - \frac{\csc \theta_{12}}{r_{12}} (\hat{\mathbf{a}}_2 \cdot \vec{\Gamma}) \quad \text{and}$$

$$\frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_2} \cdot \vec{\Gamma} = \frac{\csc \theta_{12}}{r_{21}} (\hat{\mathbf{a}}_1 \cdot \vec{\Gamma}) + \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right) (\hat{\mathbf{a}}_2 \cdot \vec{\Gamma})$$

- Then, using

$$\hat{\mathbf{a}}_1 \cdot \vec{\Gamma} = p_1(\xi_3) \Lambda_1 + p_2(\xi_3) \Lambda_2 \cos \theta_{12} + \mathcal{O}(\varepsilon^4)$$

$$\hat{\mathbf{a}}_2 \cdot \vec{\Gamma} = p_2(\xi_3) \Lambda_2 + p_1(\xi_3) \Lambda_1 \cos \theta_{12} + \mathcal{O}(\varepsilon^4)$$

$$\cot \theta_{12} = \cot \theta_{12} + 2\varepsilon_{12}^0 \csc^3 \theta_{12} - \left(\varepsilon_{11}^0 + \varepsilon_{22}^0 \right) \csc^2 \theta_{12} \cot \theta_{12} + \mathcal{O}(\varepsilon^4) \quad \text{and}$$

$$\csc \theta_{12} = \csc \theta_{12} \left[1 + 2\varepsilon_{12}^0 \cot \theta_{12} \csc \theta_{12} - \cot^2 \theta_{12} \left(\varepsilon_{11}^0 + \varepsilon_{22}^0 \right) \right] + \mathcal{O}(\varepsilon^4) \quad \text{gives}$$

TRANSVERSE SHEARING STRAINS

CONTINUED

$$\frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \cdot \vec{\Gamma} = \frac{\mathbf{p}_1(\xi_3) \Lambda_1 + \mathbf{p}_2(\xi_3) \Lambda_2 \cos \theta_{12}}{\mathbf{r}_{11}} - \frac{\mathbf{p}_2(\xi_3) \Lambda_2 \sin \theta_{12}}{\mathbf{r}_{12}} + \mathcal{O}(\varepsilon^4)$$

$$\frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \cdot \vec{\Gamma} = \frac{\mathbf{p}_1(\xi_3) \Lambda_1 \sin \theta_{12}}{\mathbf{r}_{21}} + \frac{\mathbf{p}_2(\xi_3) \Lambda_2 + \mathbf{p}_1(\xi_3) \Lambda_1 \cos \theta_{12}}{\mathbf{r}_{22}} + \mathcal{O}(\varepsilon^4)$$

for “small” strains

- Applying these results, and

$$\hat{\mathbf{a}}_1 \cdot \vec{\gamma} = \mathbf{f}_1(\xi_3) \Lambda_1 + \mathbf{f}_2(\xi_3) \Lambda_2 \cos \theta_{12} + \mathcal{O}(\varepsilon^4)$$

$$\hat{\mathbf{a}}_2 \cdot \vec{\gamma} = \mathbf{f}_2(\xi_3) \Lambda_2 + \mathbf{f}_1(\xi_3) \Lambda_1 \cos \theta_{12} + \mathcal{O}(\varepsilon^4)$$

to the last expressions given for “small” transverse shearing strains yields

TRANSVERSE SHEARING STRAINS

CONTINUED

$$2\varepsilon_{13} \frac{H_1}{\mathcal{A}_1} = f_1'(\xi_3)\Lambda_1 + f_2'(\xi_3)\Lambda_2 \cos\theta_{12} + \frac{p_1(\xi_3)\Lambda_1 + p_2(\xi_3)\Lambda_2 \cos\theta_{12}}{r_{11}} - \frac{p_2(\xi_3)\Lambda_2 \sin\theta_{12}}{r_{12}} + \mathcal{O}(\varepsilon^4)$$

and

$$2\varepsilon_{23} \frac{H_2}{\mathcal{A}_2} = f_2'(\xi_3)\Lambda_2 + f_1'(\xi_3)\Lambda_1 \cos\theta_{12} + \frac{p_1(\xi_3)\Lambda_1 \sin\theta_{12}}{r_{21}} + \frac{p_2(\xi_3)\Lambda_2 + p_1(\xi_3)\Lambda_1 \cos\theta_{12}}{r_{22}} + \mathcal{O}(\varepsilon^4)$$

- Next, $\mathcal{A}_\alpha = A_\alpha \left[1 + \varepsilon_{\alpha(\alpha)}^o + \mathcal{O}(\varepsilon^4) \right]$ is used to get

TRANSVERSE SHEARING STRAINS CONTINUED

$$2\varepsilon_{13} \frac{H_1}{A_1} = f_1'(\xi_3)\Lambda_1 + f_2'(\xi_3)\Lambda_2 \cos\theta_{12} \\ + \frac{p_1(\xi_3)\Lambda_1 + p_2(\xi_3)\Lambda_2 \cos\theta_{12}}{r_{11}} - \frac{p_2(\xi_3)\Lambda_2 \sin\theta_{12}}{r_{12}} + \mathcal{O}(\varepsilon^4)$$

and

$$2\varepsilon_{23} \frac{H_2}{A_2} = f_2'(\xi_3)\Lambda_2 + f_1'(\xi_3)\Lambda_1 \cos\theta_{12} \\ + \frac{p_1(\xi_3)\Lambda_1 \sin\theta_{12}}{r_{21}} + \frac{p_2(\xi_3)\Lambda_2 + p_1(\xi_3)\Lambda_1 \cos\theta_{12}}{r_{22}} + \mathcal{O}(\varepsilon^4)$$

- Note that it was shown previously that

$$\frac{H_1}{A_1} = \sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{12}}\right)^2} \quad \text{and}$$

TRANSVERSE SHEARING STRAINS

CONTINUED

$$\frac{H_2}{A_2} = \sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}}\right)^2}$$

- Next, $\vec{\mathcal{X}}(\xi_1, \xi_2, 0, t) = \vec{\mathbf{x}}(\xi_1, \xi_2, t)$ is enforced in $\vec{\mathcal{X}} = \vec{\mathbf{x}} + \xi_3 \hat{\mathbf{a}} + \vec{\gamma}$, with $\vec{\gamma} = f_1(\xi_3) \Lambda_1 \hat{\mathbf{a}}_1 + f_2(\xi_3) \Lambda_2 \hat{\mathbf{a}}_2$ to get the requirements $f_1(0) = f_2(0) = 0$
- Then, the **reference-surface strain definitions** $2\epsilon_{\alpha 3}^0 \equiv 2\epsilon_{\alpha 3}(\xi_1, \xi_2, 0, t)$ and $f_1(0) = f_2(0) = 0$ are used to get

$$2\epsilon_{13}^0 = f_1'(0) \Lambda_1 + f_2'(0) \Lambda_2 \cos \theta_{12} + \mathcal{O}(\epsilon^4) \quad \text{and}$$

$$2\epsilon_{23}^0 = f_2'(0) \Lambda_2 + f_1'(0) \Lambda_1 \cos \theta_{12} + \mathcal{O}(\epsilon^4)$$

TRANSVERSE SHEARING STRAINS

CONTINUED

- Thus,

$$\Lambda_1 = \frac{2\varepsilon_{13}^o - 2\varepsilon_{23}^o \cos\theta_{12}}{f_1'(0) \sin^2\theta_{12}} + \mathcal{O}(\varepsilon^4) \quad \text{and} \quad \Lambda_2 = \frac{2\varepsilon_{23}^o - 2\varepsilon_{13}^o \cos\theta_{12}}{f_2'(0) \sin^2\theta_{12}} + \mathcal{O}(\varepsilon^4)$$

- Now, it is convenient to introduce the functions $F_\alpha(\xi_3) = \frac{f_\alpha(\xi_3)}{f_{(\alpha)}'(0)}$ such that

$$f_1(\xi_3)\Lambda_1 = F_1(\xi_3) \frac{2\varepsilon_{13}^o - 2\varepsilon_{23}^o \cos\theta_{12}}{\sin^2\theta_{12}} + \mathcal{O}(\varepsilon^4),$$

$$f_2(\xi_3)\Lambda_2 = F_2(\xi_3) \frac{2\varepsilon_{23}^o - 2\varepsilon_{13}^o \cos\theta_{12}}{\sin^2\theta_{12}} + \mathcal{O}(\varepsilon^4), \text{ and}$$

$$\vec{\gamma} = \csc^2\theta_{12} \left[F_1(\xi_3) \left(2\varepsilon_{13}^o - 2\varepsilon_{23}^o \cos\theta_{12} \right) \hat{a}_1 + F_2(\xi_3) \left(2\varepsilon_{23}^o - 2\varepsilon_{13}^o \cos\theta_{12} \right) \hat{a}_2 \right] + \overrightarrow{\mathcal{O}(\varepsilon^4)}$$

TRANSVERSE SHEARING STRAINS

CONTINUED

- To facilitate simplification, let $\gamma_1 = F_1(\xi_3)\gamma_1^\circ$ and $\gamma_2 = F_2(\xi_3)\gamma_2^\circ$, where

$$\gamma_1^\circ = \csc^2 \theta_{12} (2\varepsilon_{13}^\circ - 2\varepsilon_{23}^\circ \cos \theta_{12}) \quad \text{and} \quad \gamma_2^\circ = \csc^2 \theta_{12} (2\varepsilon_{23}^\circ - 2\varepsilon_{13}^\circ \cos \theta_{12})$$

such that

$$\vec{\gamma} = \gamma_1 \hat{a}_1 + \gamma_2 \hat{a}_2, \quad f_1(\xi_3)\Lambda_1 = F_1(\xi_3)\gamma_1^\circ + \mathcal{O}(\varepsilon^4), \quad \text{and} \quad f_2(\xi_3)\Lambda_2 = F_2(\xi_3)\gamma_2^\circ + \mathcal{O}(\varepsilon^4)$$

- With this notation, $\vec{\mathcal{X}} = \vec{x} + \xi_3 \hat{a} + \vec{\gamma}$ becomes

$$\vec{\mathcal{X}} = \vec{x} + \xi_3 \hat{a} + F_1(\xi_3)\gamma_1^\circ \hat{a}_1 + F_2(\xi_3)\gamma_2^\circ \hat{a}_2 + \overrightarrow{\mathcal{O}(\varepsilon^4)}$$

for “small” strains

TRANSVERSE SHEARING STRAINS

CONTINUED

- In these expressions, and subsequent expressions,

$$\epsilon_{13}^0(\xi_1, \xi_2, \tau) \quad \text{and} \quad \epsilon_{23}^0(\xi_1, \xi_2, \tau)$$

are used as **the primary unknowns that characterize the transverse shear deformations of the shell**

- However, $\gamma_1^0(\xi_1, \xi_2, \tau)$ and $\gamma_2^0(\xi_1, \xi_2, \tau)$ can also be used

- Next, differentiating $F_\alpha(\xi_3) = \frac{f_\alpha(\xi_3)}{f_{(\alpha)}'(0)}$ gives $F_\alpha'(\xi_3) = \frac{f_\alpha'(\xi_3)}{f_{(\alpha)}'(0)}$

- Thus, $f_1'(\xi_3)\Lambda_1 = F_1'(\xi_3)\gamma_1^0 + \mathcal{O}(\varepsilon^4)$ and $f_2'(\xi_3)\Lambda_2 = F_2'(\xi_3)\gamma_2^0 + \mathcal{O}(\varepsilon^4)$

TRANSVERSE SHEARING STRAINS

CONTINUED

- Moreover, from $\mathbf{p}_1(\xi_3) = \xi_3 \mathbf{f}_1'(\xi_3) - \mathbf{f}_1(\xi_3)$ and $\mathbf{p}_2(\xi_3) = \xi_3 \mathbf{f}_2'(\xi_3) - \mathbf{f}_2(\xi_3)$ it follows that

$$\mathbf{p}_1(\xi_3) \Lambda_1 = [\xi_3 \mathbf{F}_1'(\xi_3) - \mathbf{F}_1(\xi_3)] \gamma_1^{\circ} + \mathcal{O}(\varepsilon^4) \quad \text{and}$$

$$\mathbf{p}_2(\xi_3) \Lambda_2 = [\xi_3 \mathbf{F}_2'(\xi_3) - \mathbf{F}_2(\xi_3)] \gamma_2^{\circ} + \mathcal{O}(\varepsilon^4)$$

- For convenience, $\mathbf{P}_1(\xi_3) \equiv \xi_3 \mathbf{F}_1'(\xi_3) - \mathbf{F}_1(\xi_3)$ and $\mathbf{P}_2(\xi_3) \equiv \xi_3 \mathbf{F}_2'(\xi_3) - \mathbf{F}_2(\xi_3)$ such that

$$\mathbf{p}_1(\xi_3) \Lambda_1 = \mathbf{P}_1(\xi_3) \gamma_1^{\circ} + \mathcal{O}(\varepsilon^4) \quad \text{and} \quad \mathbf{p}_2(\xi_3) \Lambda_2 = \mathbf{P}_2(\xi_3) \gamma_2^{\circ} + \mathcal{O}(\varepsilon^4)$$

TRANSVERSE SHEARING STRAINS

CONTINUED

- The transverse shearing strains are now expressed as

$$2\varepsilon_{13} \frac{H_1}{A_1} = \left[F_1'(\xi_3) + \frac{P_1(\xi_3)}{r_{11}} \right] \gamma_1^o + \left[\left(F_2'(\xi_3) + \frac{P_2(\xi_3)}{r_{11}} \right) \cos\theta_{12} - \frac{P_2(\xi_3) \sin\theta_{12}}{r_{12}} \right] \gamma_2^o + \mathcal{O}(\varepsilon^4)$$

and

$$2\varepsilon_{23} \frac{H_2}{A_2} = \left[F_2'(\xi_3) + \frac{P_2(\xi_3)}{r_{22}} \right] \gamma_2^o + \left[\left(F_1'(\xi_3) + \frac{P_1(\xi_3)}{r_{22}} \right) \cos\theta_{12} + \frac{P_1(\xi_3) \sin\theta_{12}}{r_{21}} \right] \gamma_1^o + \mathcal{O}(\varepsilon^4)$$

TRANSVERSE SHEARING STRAINS

CONTINUED

- Previously herein, it was shown that

$$\frac{1}{\epsilon_{11}} = \frac{1}{r_{11}} + K_{11}^{\circ} \quad \frac{1}{\epsilon_{22}} = \frac{1}{r_{22}} + K_{22}^{\circ}$$

$$\frac{1}{\epsilon_{21}} = K_{12}^{\circ} + \frac{\cot\theta_{12}}{2} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} + K_{11}^{\circ} - K_{22}^{\circ} \right) + \frac{1}{2} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right)$$

$$-\frac{1}{\epsilon_{12}} = K_{12}^{\circ} - \frac{\cot\theta_{12}}{2} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} + K_{11}^{\circ} - K_{22}^{\circ} \right) + \frac{1}{2} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right)$$

$$\cot\theta_{12} = \cot\theta_{12} + 2\epsilon_{12}^{\circ} \csc^3\theta_{12} - \left(\epsilon_{11}^{\circ} + \epsilon_{22}^{\circ} \right) \csc^2\theta_{12} \cot\theta_{12} + \mathcal{O}(\epsilon^4)$$

- Thus, $\frac{2\epsilon_{13}^{\circ}}{\epsilon_{11}} = \frac{2\epsilon_{13}^{\circ}}{r_{11}} + \mathcal{O}(\epsilon^4)$ and $\frac{2\epsilon_{23}^{\circ}}{\epsilon_{22}} = \frac{2\epsilon_{23}^{\circ}}{r_{22}} + \mathcal{O}(\epsilon^4)$

TRANSVERSE SHEARING STRAINS

CONTINUED

- In addition,

$$\frac{2\varepsilon_{\alpha\beta}^{\circ}}{\tau_{12}} = \varepsilon_{\alpha\beta}^{\circ} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \cot\theta_{12} - \varepsilon_{\alpha\beta}^{\circ} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) + \mathcal{O}(\varepsilon^4) \quad \text{and}$$

$$\frac{2\varepsilon_{\alpha\beta}^{\circ}}{\tau_{21}} = \varepsilon_{\alpha\beta}^{\circ} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \cot\theta_{12} + \varepsilon_{\alpha\beta}^{\circ} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) + \mathcal{O}(\varepsilon^4)$$

where products of $2\varepsilon_{\alpha\beta}^{\circ}$ and $\kappa_{\alpha\beta}^{\circ}$ are presumed to be $\mathcal{O}(\varepsilon^4)$

- Using the identity $\frac{1}{r_{12}} + \frac{1}{r_{21}} = \cot\theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$ gives

$$\frac{2\varepsilon_{\alpha\beta}^{\circ}}{\tau_{12}} = \frac{2\varepsilon_{\alpha\beta}^{\circ}}{r_{12}} + \mathcal{O}(\varepsilon^4) \quad \text{and} \quad \frac{2\varepsilon_{\alpha\beta}^{\circ}}{\tau_{21}} = \frac{2\varepsilon_{\alpha\beta}^{\circ}}{r_{21}} + \mathcal{O}(\varepsilon^4)$$

TRANSVERSE SHEARING STRAINS

CONTINUED

- Using these results gives $\frac{\gamma_1^o}{r_{21}} = \frac{\gamma_1^o}{r_{21}} + \mathcal{O}(\varepsilon^4)$ and $\frac{\gamma_2^o}{r_{12}} = \frac{\gamma_2^o}{r_{12}} + \mathcal{O}(\varepsilon^4)$
- Applying these “**small-strain**” simplifications to the last expressions given for the transverse shearing strains yields

$$2\varepsilon_{13} \frac{H_1}{A_1} = \left[F_1'(\xi_3) + \frac{P_1(\xi_3)}{r_{11}} \right] \gamma_1^o + \left[\left(F_2'(\xi_3) + \frac{P_2(\xi_3)}{r_{11}} \right) \cos\theta_{12} - \frac{P_2(\xi_3) \sin\theta_{12}}{r_{12}} \right] \gamma_2^o + \mathcal{O}(\varepsilon^4)$$

$$2\varepsilon_{23} \frac{H_2}{A_2} = \left[F_2'(\xi_3) + \frac{P_2(\xi_3)}{r_{22}} \right] \gamma_2^o + \left[\left(F_1'(\xi_3) + \frac{P_1(\xi_3)}{r_{22}} \right) \cos\theta_{12} + \frac{P_1(\xi_3) \sin\theta_{12}}{r_{21}} \right] \gamma_1^o + \mathcal{O}(\varepsilon^4)$$

where $P_1(\xi_3) \equiv \xi_3 F_1'(\xi_3) - F_1(\xi_3)$, $P_2(\xi_3) \equiv \xi_3 F_2'(\xi_3) - F_2(\xi_3)$, and $F_1(\xi_3)$ and $F_2(\xi_3)$ are user defined and their selection should be guided by practical considerations and experimental evidence

TRANSVERSE SHEARING STRAINS

CONTINUED

- Also, note that the requirements $f_1(0) = f_2(0) = 0$ stated earlier herein become the requirements $F_1(0) = F_2(0) = 0$
- Additional conditions that can be used to influence the choice of $F_1(\xi_3)$ and $F_2(\xi_3)$ are obtained from the traction boundary conditions on the bounding surfaces of the shell
- To illustrate this process, consider shells that are free of tangential surface tractions on the bounding surfaces
 - For this case, the transverse shearing stresses vanish at the bounding surfaces
 - Also, in many approximate shell theories, nonzero surface tractions are replaced with statically equivalent distributed loads and couples that are applied to the shell reference surface

TRANSVERSE SHEARING STRAINS

CONTINUED

- Enforcing these traction-free boundary conditions generally involves the use of constitutive equations and yields eight homogeneous, linear algebraic equations in terms of $F_1(\xi_3)$ and $F_2(\xi_3)$ and their first derivatives, evaluated at each bounding surface
- An alternate and less cumbersome approach is used subsequently in which the traction-free boundary conditions are satisfied by requiring the transverse shearing strains at the bounding surfaces to vanish
- Inspection of the transverse-shearing-strain equations indicates that these strains vanish at the bounding surfaces of the shell, provided that

$$F_1'(\mathbf{c}_3^+) = F_1'(\mathbf{c}_3^-) = F_2'(\mathbf{c}_3^+) = F_2'(\mathbf{c}_3^-) = 0 \quad \text{and}$$

$$F_1(\mathbf{c}_3^+) = F_1(\mathbf{c}_3^-) = F_2(\mathbf{c}_3^+) = F_2(\mathbf{c}_3^-) = 0$$

where it is noted that $\mathbf{c}_3^-(\xi_1, \xi_2) \leq \xi_3 \leq \mathbf{c}_3^+(\xi_1, \xi_2)$

TRANSVERSE SHEARING STRAINS

CONTINUED

- These two conditions cannot be satisfied exactly unless the shell has a uniform thickness or a piecewise uniform thickness since $F_\alpha = F_\alpha(\xi_3)$
- However, the underlying presumption that transverse through-the-thickness strains are negligible places limitations on how rapidly $c_3^-(\xi_1, \xi_2)$ and $c_3^+(\xi_1, \xi_2)$ can vary with (ξ_1, ξ_2)
- Thus, using average values for c_3^- and c_3^+ is likely to produce meaningful results for global response quantities
- It is significant to note that the choice $F_1(\xi_3) = F_2(\xi_3) = \xi_3$ produces uniform through-the-thickness transverse shearing strains
- $F_1(\xi_3) = F_2(\xi_3) = \xi_3$ corresponds to a first-order transverse-shear deformation theory, which uses a “shear correction factor” to account for the fact that the transverse-shearing strains do not vanish on the bounding surfaces of the shell

TRANSVERSE SHEARING STRAINS

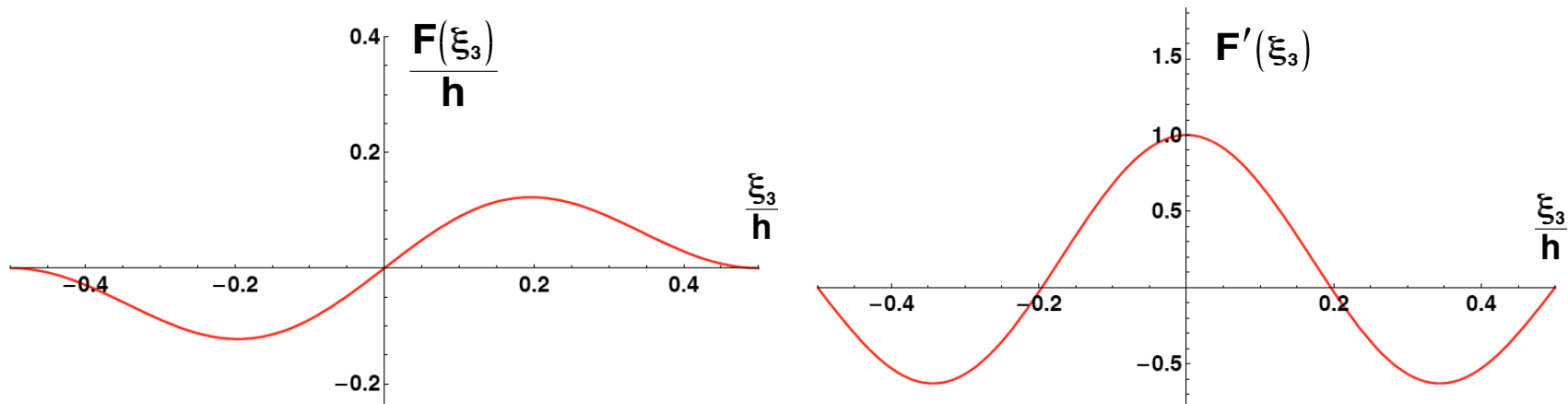
CONTINUED

- It is reiterated that the specific functional forms of $F_\alpha(\xi_3)$ are user defined and their selection should be guided by practical considerations and experimental evidence

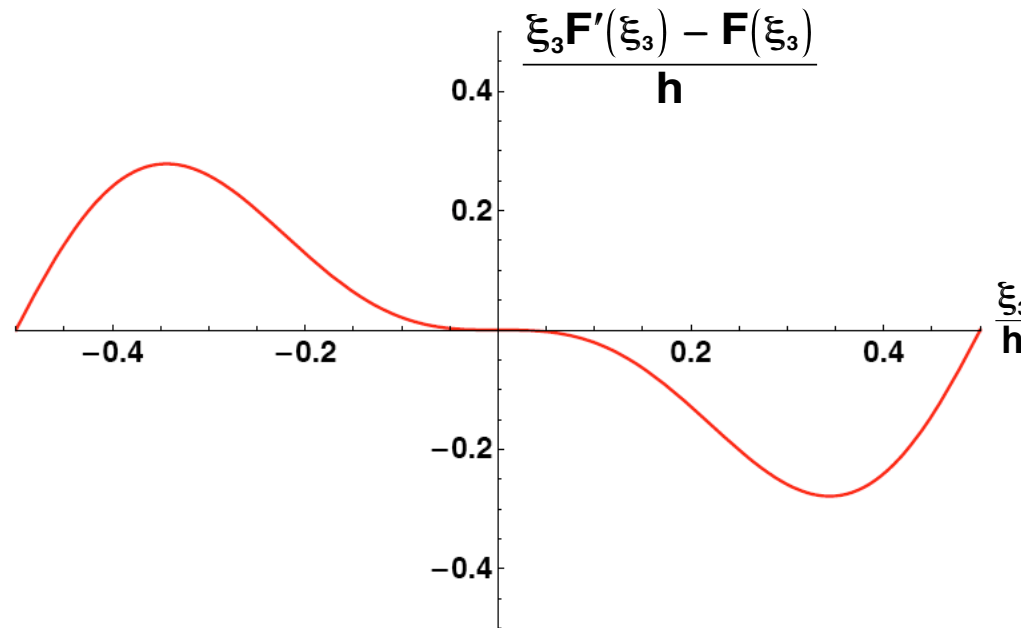
- For the practical case of shells with a uniform thickness h and

$-\frac{h}{2} \leq \xi_3 \leq +\frac{h}{2}$, consider $F_\alpha(\xi_3) = F(\xi_3) = \frac{h}{2\pi} \left(\cos \frac{\pi \xi_3}{h} \sin \frac{2\pi \xi_3}{h} \right)$ which

satisfy $F(0) = 0$, $F\left(\frac{h}{2}\right) = F\left(-\frac{h}{2}\right) = 0$, and $F'\left(\frac{h}{2}\right) = F'\left(-\frac{h}{2}\right) = 0$



TRANSVERSE SHEARING STRAINS - CONTINUED



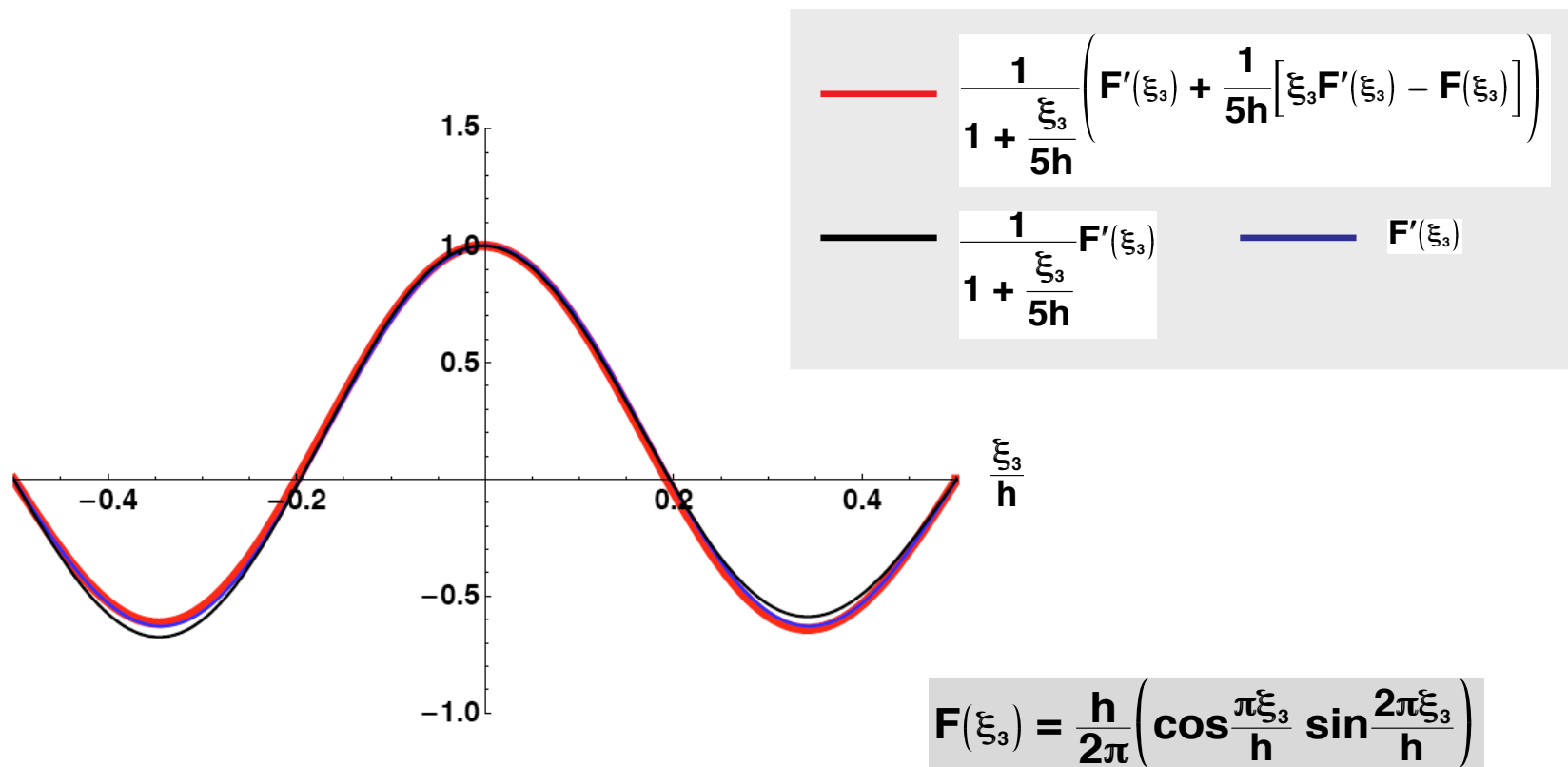
- To get an estimate of the through-the-thickness variation in the transverse-shearing strains, let $\theta_{12} = \frac{\pi}{2}$, $\frac{1}{r_{12}} = 0$, and $r_{11} = 5h$

- These values yield
$$\frac{2\varepsilon_{13}}{2\varepsilon_{13}^o} \approx \frac{1}{1 + \frac{\xi_3}{5h}} \left(F'(\xi_3) + \frac{1}{5h} [\xi_3 F'(\xi_3) - F(\xi_3)] \right)$$

TRANSVERSE SHEARING STRAINS

CONTINUED

- Note that $r_{11} = 5h$ corresponds to a very thick shell

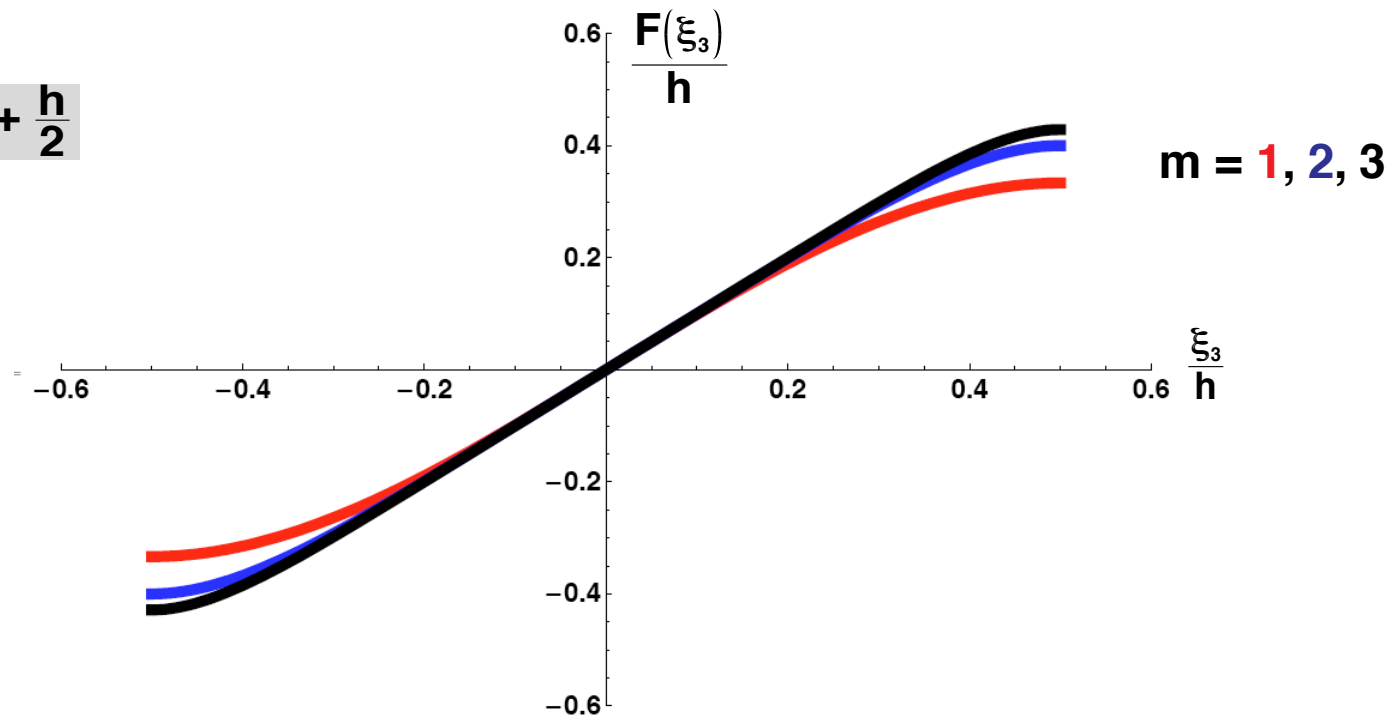


TRANSVERSE SHEARING STRAINS CONTINUED

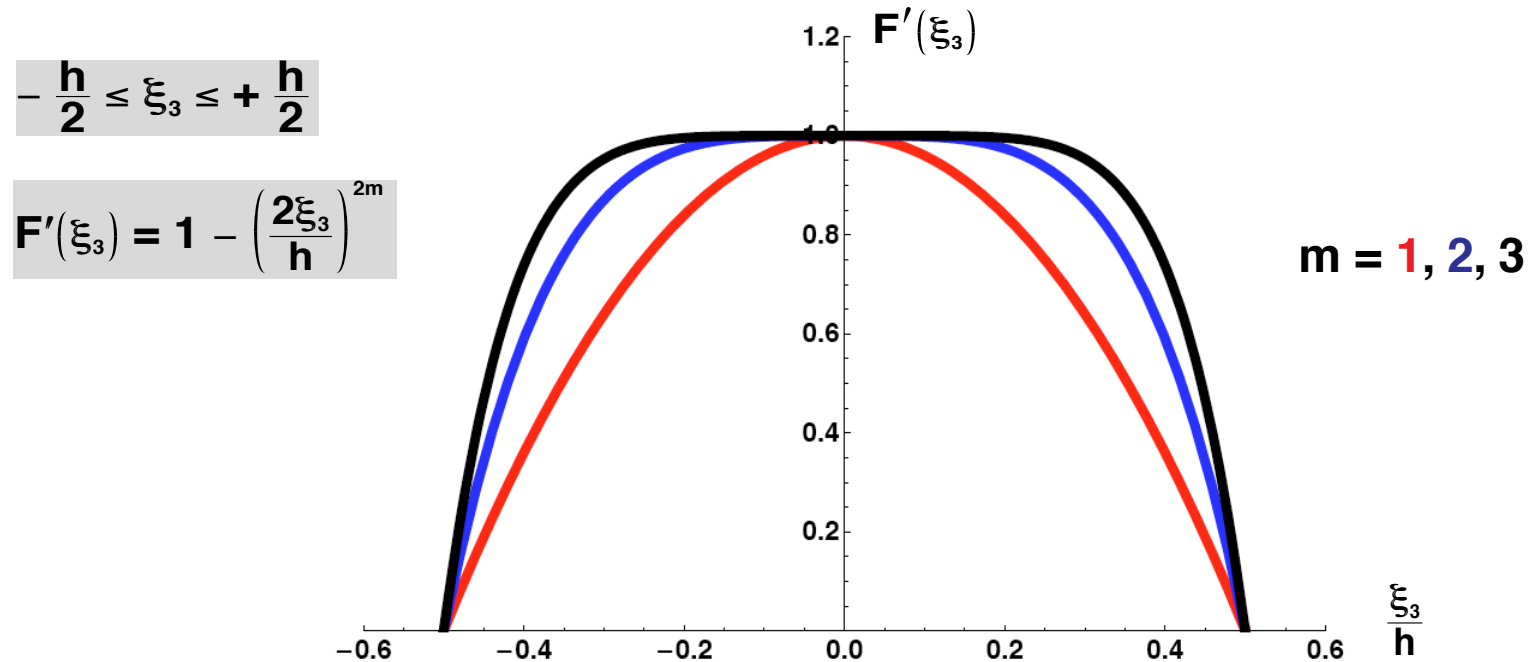
- Now consider the family of polynomials given by

$$F(\xi_3) = \xi_3 \left[1 - \frac{1}{2m+1} \left(\frac{2\xi_3}{h} \right)^{2m} \right] \quad \text{with} \quad F'(\xi_3) = 1 - \left(\frac{2\xi_3}{h} \right)^{2m}, \quad \text{for } m = 1, 2, \dots$$

$$-\frac{h}{2} \leq \xi_3 \leq +\frac{h}{2}$$



TRANSVERSE SHEARING STRAINS CONTINUED



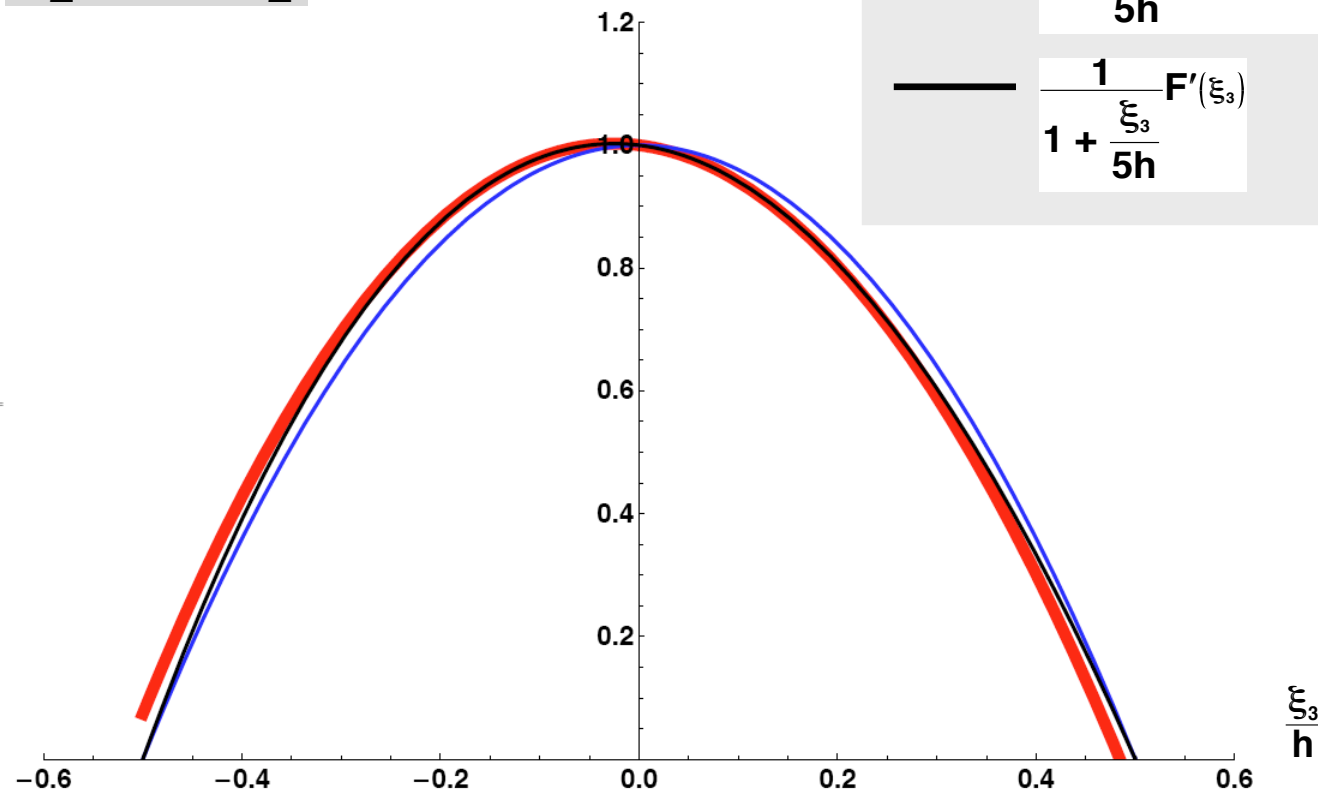
- Note that these polynomials do not satisfy $F\left(\frac{h}{2}\right) = F\left(-\frac{h}{2}\right) = 0$

TRANSVERSE SHEARING STRAINS

CONTINUED

$$-\frac{h}{2} \leq \xi_3 \leq +\frac{h}{2}$$

$$m = 1$$



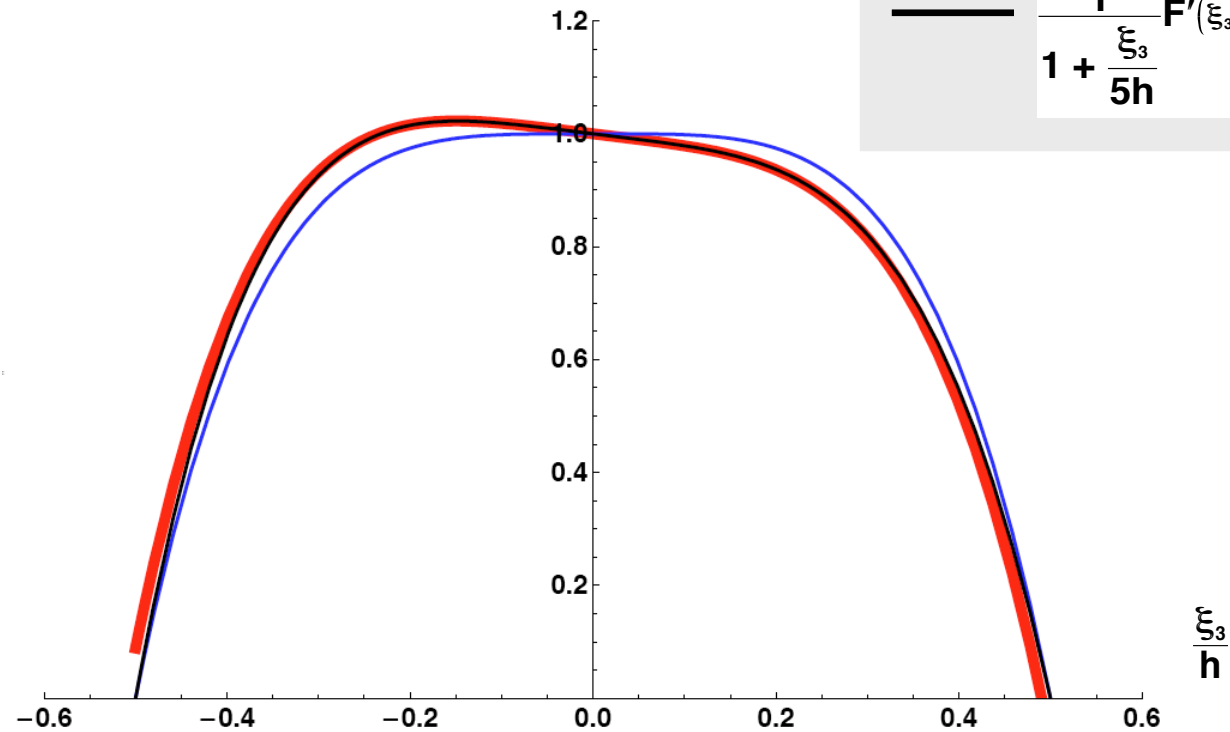
$$\begin{aligned} & \text{Red line: } \frac{1}{1 + \frac{\xi_3}{5h}} \left(F'(\xi_3) + \frac{1}{5h} [\xi_3 F'(\xi_3) - F(\xi_3)] \right) \\ & \text{Black line: } \frac{1}{1 + \frac{\xi_3}{5h}} F'(\xi_3) \\ & \text{Blue line: } F'(\xi_3) \end{aligned}$$

TRANSVERSE SHEARING STRAINS

CONTINUED

$$-\frac{h}{2} \leq \xi_3 \leq +\frac{h}{2}$$

$$m = 2$$



—	$\frac{1}{1 + \frac{\xi_3}{5h}} \left(F'(\xi_3) + \frac{1}{5h} [\xi_3 F'(\xi_3) - F(\xi_3)] \right)$	
—	$\frac{1}{1 + \frac{\xi_3}{5h}} F'(\xi_3)$	— $F'(\xi_3)$

TRANSVERSE SHEARING STRAINS

CONCLUDED

- These plots suggest that the requirements on $F(\xi_3)$ and $F'(\xi_3)$ can be relaxed to

$$F(0) = 0 \quad \text{and} \quad F'\left(\frac{h}{2}\right) = F'\left(-\frac{h}{2}\right) = 0$$

without any serious loss in accuracy in the analysis of global response quantities like displacements, buckling loads, and vibration frequencies

PARALLEL-SURFACE TANGENTIAL STRAINS

- The next step in the analysis is to simplify the strain expressions

$$2\epsilon_{\alpha\beta} = \frac{\mathbf{A}_{(\alpha)}\mathbf{A}_{(\beta)}}{\mathbf{H}_{(\alpha)}\mathbf{H}_{(\beta)}} \left[\left(\frac{\vec{\mathcal{G}}_{\alpha}}{\mathbf{A}_{(\alpha)}} \cdot \frac{\vec{\mathcal{G}}_{\beta}}{\mathbf{A}_{(\beta)}} \right) - \left(\frac{\vec{\mathbf{g}}_{\alpha}}{\mathbf{A}_{(\alpha)}} \cdot \frac{\vec{\mathbf{g}}_{\beta}}{\mathbf{A}_{(\beta)}} \right) \right]$$

by first simplifying the dot products, where

$$\vec{\mathcal{G}}_{\alpha} = \mathcal{A}_{(\alpha)} \left[\hat{\mathbf{a}}_{\alpha} + \xi_3 \frac{1}{\mathcal{A}_{(\alpha)}} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_{\alpha}} + \frac{1}{\mathcal{A}_{(\alpha)}} \frac{\partial \vec{\gamma}}{\partial \xi_{\alpha}} \right]$$

- These strains constitute the extensional and shearing strains acting in the parallel surface passing through a given point of the shell, at that point

- First, recall that $\frac{\vec{\mathbf{g}}_1}{\mathbf{A}_1} = \hat{\mathbf{a}}_1 + \xi_3 \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_1}$ and $\frac{\vec{\mathbf{g}}_2}{\mathbf{A}_2} = \hat{\mathbf{a}}_2 + \xi_3 \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_2}$

PARALLEL-SURFACE TANGENTIAL STRAINS

CONTINUED

- By using the derivative expressions

$$\frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} = \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) \hat{a}_1 - \frac{\csc \theta_{12}}{r_{12}} \hat{a}_2 \quad \text{and}$$

$$\frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} = \frac{\csc \theta_{12}}{r_{21}} \hat{a}_1 + \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right) \hat{a}_2$$

it follows that:

$$\frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \cdot \hat{a}_1 = \frac{1}{r_{11}} \quad \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} \cdot \hat{a}_2 = \frac{1}{r_{22}}$$

$$\xi_3 \left(\frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} \cdot \hat{a}_1 + \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \cdot \hat{a}_2 \right) = \left(\frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cos \theta_{12} + \left(\frac{\xi_3}{r_{21}} - \frac{\xi_3}{r_{12}} \right) \sin \theta_{12}$$

$$\left(\xi_3 \right)^2 \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} = \left(\frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2$$

$$\left(\xi_3 \right)^2 \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} = \left(\frac{\xi_3}{r_{22}} \right)^2 + \left(\frac{\xi_3}{r_{21}} \right)^2$$

PARALLEL-SURFACE TANGENTIAL STRAINS

CONTINUED

$$\left(\xi_3\right)^2 \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} = \left(\frac{\xi_3}{r_{11}} \frac{\xi_3}{r_{21}} - \frac{\xi_3}{r_{12}} \frac{\xi_3}{r_{22}} \right) \sin \theta_{12} + \left(\frac{\xi_3}{r_{12}} \frac{\xi_3}{r_{21}} + \frac{\xi_3}{r_{11}} \frac{\xi_3}{r_{22}} \right) \cos \theta_{12}$$

- These expressions yield

$$\frac{\vec{g}_1}{A_1} \cdot \frac{\vec{g}_1}{A_1} = 1 + 2 \frac{\xi_3}{r_{11}} + \left(\frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2$$

$$\frac{\vec{g}_2}{A_2} \cdot \frac{\vec{g}_2}{A_2} = 1 + 2 \frac{\xi_3}{r_{22}} + \left(\frac{\xi_3}{r_{21}} \right)^2 + \left(\frac{\xi_3}{r_{22}} \right)^2$$

$$\begin{aligned} \frac{\vec{g}_1}{A_1} \cdot \frac{\vec{g}_2}{A_2} = & \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cos \theta_{12} + \left(\frac{\xi_3}{r_{21}} - \frac{\xi_3}{r_{12}} \right) \sin \theta_{12} \\ & + \left(\frac{\xi_3}{r_{11}} \frac{\xi_3}{r_{21}} - \frac{\xi_3}{r_{12}} \frac{\xi_3}{r_{22}} \right) \sin \theta_{12} + \left(\frac{\xi_3}{r_{12}} \frac{\xi_3}{r_{21}} + \frac{\xi_3}{r_{11}} \frac{\xi_3}{r_{22}} \right) \cos \theta_{12} \end{aligned}$$

PARALLEL-SURFACE TANGENTIAL STRAINS

CONTINUED

- Analogously, it follows that

$$\frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} = \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) \hat{\mathbf{a}}_1 - \frac{\csc \theta_{12}}{r_{12}} \hat{\mathbf{a}}_2 \quad \text{and}$$

$$\frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} = \frac{\csc \theta_{12}}{r_{21}} \hat{\mathbf{a}}_1 + \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right) \hat{\mathbf{a}}_2 \quad \text{yield}$$

$$\frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \cdot \hat{\mathbf{a}}_1 = \frac{1}{r_{11}}$$

$$\left(\xi_3 \right)^2 \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} = \left(\frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2$$

$$\frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \cdot \hat{\mathbf{a}}_2 = \frac{1}{r_{22}}$$

$$\left(\xi_3 \right)^2 \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} = \left(\frac{\xi_3}{r_{21}} \right)^2 + \left(\frac{\xi_3}{r_{22}} \right)^2$$

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- In addition,

$$\xi_3 \left(\frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \cdot \hat{\mathbf{a}}_1 + \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \cdot \hat{\mathbf{a}}_2 \right) = \left(\frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cos \theta_{12} + \left(\frac{\xi_3}{r_{21}} - \frac{\xi_3}{r_{12}} \right) \sin \theta_{12}$$

$$(\xi_3)^2 \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} = \left(\frac{\xi_3}{r_{11}} \frac{\xi_3}{r_{21}} - \frac{\xi_3}{r_{12}} \frac{\xi_3}{r_{22}} \right) \sin \theta_{12} + \left(\frac{\xi_3}{r_{12}} \frac{\xi_3}{r_{21}} + \frac{\xi_3}{r_{11}} \frac{\xi_3}{r_{22}} \right) \cos \theta_{12}$$

- To obtain additional simplifications, **it is presumed that the transverse-shearing strains and their derivatives with respect to the reference-surface Gaussian coordinates are small enough that nonlinear terms associated with their products are negligible**
- To gain some physical insight, let $\gamma = \gamma_{\max} \sin\left(\frac{2\pi\tilde{h}}{\ell} \bar{x}\right)$ represent a transverse-shearing response that oscillates along the deformed shell reference surface with a wavelength ℓ in a small neighborhood of an arbitrary point of the reference surface

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- The symbol $\tilde{h}$ denotes the corresponding average thickness and $\bar{x}$ represents any arc-length coordinate for the reference surface that has been nondimensionalized by $\tilde{h}$
- This type of nondimensionalization is consistent with localized kinking or wrinkling that might be observed for an actual shell and that is easily characterized in terms of the nearby shell thickness

- The derivative is given by $\gamma' = 2\pi\gamma_{\max}\left(\frac{\tilde{h}}{\ell}\right) \cos\left(\frac{2\pi\tilde{h}}{\ell} \bar{x}\right)$

- Inspection of these equations reveals that

$$|\gamma_{\max}| \ll 1 \quad \text{and} \quad |\gamma_{\max}| \frac{\tilde{h}}{\ell} \ll 1$$

for a “small” transverse shearing strain and its derivative

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- Therefore, $\frac{\tilde{h}}{\ell} \leq 1$ which means that the wavelength of the response must be larger than the average thickness
- Thus, this presumption of “small” transverse-shearing strains and their derivatives is consistent with the absence of **extreme short wavelength transverse-shearing action** along the reference surface
- For this presumption, it follows that the nonlinear terms given by

$$\frac{1}{\mathcal{A}_{(\alpha)}} \frac{\partial \vec{\gamma}}{\partial \xi_{\alpha}} \cdot \frac{1}{\mathcal{A}_{(\beta)}} \frac{\partial \vec{\gamma}}{\partial \xi_{\beta}} \leq \left| \frac{1}{\mathcal{A}_{(\alpha)}} \frac{\partial \vec{\gamma}}{\partial \xi_{\alpha}} \right| \left| \frac{1}{\mathcal{A}_{(\beta)}} \frac{\partial \vec{\gamma}}{\partial \xi_{\beta}} \right| = 0 + \mathcal{O}(\varepsilon^4)$$

are negligible

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- The remaining dot products needed are expressed as

$$2\xi_3 \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_1} \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} = 2 \left(\frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right) - 2 \frac{\xi_3 \csc \theta_{12}}{r_{12}} \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right),$$

$$2\xi_3 \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_2} \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} = 2 \left(\frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \right) + 2 \frac{\xi_3 \csc \theta_{12}}{r_{21}} \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \right), \text{ and}$$

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$$\begin{aligned} \xi_3 \left(\frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_1} \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} + \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_2} \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right) = \\ \frac{\xi_3 \csc \theta_{12}}{r_{21}} \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right) + \left(\frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \right) \\ + \left(\frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right) - \frac{\xi_3 \csc \theta_{12}}{r_{12}} \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \right) \end{aligned}$$

PARALLEL-SURFACE TANGENTIAL STRAINS

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- With these simplifications, the inner products associated with the convected basis are written as

$$\frac{\vec{g}_1}{A_1} \cdot \frac{\vec{g}_1}{A_1} = \left(\frac{A_1}{A_1} \right)^2 \left[1 + \frac{2\xi_3}{r_{11}} + \left(\frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right] + 2\Gamma_{11} + \mathcal{O}(\varepsilon^4)$$

$$\frac{\vec{g}_2}{A_2} \cdot \frac{\vec{g}_2}{A_2} = \left(\frac{A_2}{A_2} \right)^2 \left[1 + \frac{2\xi_3}{r_{22}} + \left(\frac{\xi_3}{r_{21}} \right)^2 + \left(\frac{\xi_3}{r_{22}} \right)^2 \right] + 2\Gamma_{22} + \mathcal{O}(\varepsilon^4)$$

$$\begin{aligned} \frac{\vec{g}_1}{A_1} \cdot \frac{\vec{g}_2}{A_2} &= \frac{A_1 A_2}{A_1 A_2} \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cos \theta_{12} + \left(\frac{\xi_3}{r_{21}} - \frac{\xi_3}{r_{12}} \right) \sin \theta_{12} \right. \\ &\quad \left. + \left(\frac{\xi_3}{r_{11}} \frac{\xi_3}{r_{21}} - \frac{\xi_3}{r_{12}} \frac{\xi_3}{r_{22}} \right) \sin \theta_{12} + \left(\frac{\xi_3}{r_{12}} \frac{\xi_3}{r_{21}} + \frac{\xi_3}{r_{11}} \frac{\xi_3}{r_{22}} \right) \cos \theta_{12} + 2\Gamma_{12} + \mathcal{O}(\varepsilon^4) \right] \end{aligned}$$

where

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$$\Gamma_{11} = \left(\frac{\mathcal{A}_1}{\mathbf{A}_1} \right)^2 \left[- \frac{\xi_3 \mathbf{csc} \theta_{12}}{r_{12}} \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right) + \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \mathbf{cot} \theta_{12}}{r_{12}} \right) \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right) \right] + \mathcal{O}(\varepsilon^4)$$

$$\Gamma_{22} = \left(\frac{\mathcal{A}_2}{\mathbf{A}_2} \right)^2 \left[\frac{\xi_3 \mathbf{csc} \theta_{12}}{r_{21}} \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \right) + \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \mathbf{cot} \theta_{12}}{r_{21}} \right) \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \right) \right] + \mathcal{O}(\varepsilon^4)$$

PARALLEL-SURFACE TANGENTIAL STRAINS

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$$2\Gamma_{12} = \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathbf{A}_1 \mathbf{A}_2} \left[\frac{\xi_3 \csc \theta_{12}}{r_{21}} \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right) - \frac{\xi_3 \csc \theta_{12}}{r_{12}} \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \right) \right. \\ \left. + \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \right) \right. \\ \left. + \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right) \right] + \mathcal{O}(\varepsilon^4)$$

- The next step in the analysis is to simplify Γ_{11} , Γ_{22} , and $2\Gamma_{12}$ for “small” strains and their derivatives

- First, consider the terms $\hat{\mathbf{a}}_\alpha \cdot \frac{1}{\mathcal{A}_{(\beta)}} \frac{\partial \vec{\gamma}}{\partial \xi_\beta}$

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- By direct analogy, the components of the derivatives of the deformed-surface vector field

$$\vec{u}(\xi_1, \xi_2) = u_1 \hat{a}_1 + u_2 \hat{a}_2 + u_3 \hat{n}$$

for general nonorthogonal Gaussian reference-surface coordinates, are given by

$$\hat{a}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{u}}{\partial \xi_1} = \frac{1}{\mathcal{A}_1} \frac{\partial}{\partial \xi_1} (u_1 + u_2 \cos \theta_{12}) - \frac{u_2 \sin \theta_{12}}{\rho_{11}} + \frac{u_3}{r_{11}}$$

$$\hat{a}_2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{u}}{\partial \xi_1} = \left(\frac{1}{\mathcal{A}_1} \frac{\partial u_1}{\partial \xi_1} + \frac{u_3}{r_{11}} \right) \cos \theta_{12} + \frac{1}{\mathcal{A}_1} \frac{\partial u_2}{\partial \xi_1} + \left(\frac{u_1}{\rho_{11}} - \frac{u_3}{r_{12}} \right) \sin \theta_{12}$$

$$\hat{n} \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{u}}{\partial \xi_1} = \frac{1}{\mathcal{A}_1} \frac{\partial u_3}{\partial \xi_1} - \frac{u_1}{r_{11}} + u_2 \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right)$$

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$$\hat{a}_1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{u}}{\partial \xi_2} = \frac{1}{\mathcal{A}_2} \frac{\partial u_1}{\partial \xi_2} + \left(\frac{1}{\mathcal{A}_2} \frac{\partial u_2}{\partial \xi_2} + \frac{u_3}{r_{22}} \right) \cos \theta_{12} + \left(\frac{u_3}{r_{21}} - \frac{u_2}{\rho_{22}} \right) \sin \theta_{12}$$

$$\hat{a}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{u}}{\partial \xi_2} = \frac{1}{\mathcal{A}_2} \frac{\partial}{\partial \xi_2} \left(u_2 + u_1 \cos \theta_{12} \right) + \frac{u_1 \sin \theta_{12}}{\rho_{22}} + \frac{u_3}{r_{22}}$$

$$\hat{a}_3 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{u}}{\partial \xi_2} = \frac{1}{\mathcal{A}_2} \frac{\partial u_3}{\partial \xi_2} - u_1 \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) - \frac{u_2}{r_{22}}$$

- In these expressions,

$$\frac{1}{\rho_{11}} = \frac{\csc \theta_{12}}{\mathcal{A}_1 \mathcal{A}_2} \left(\frac{\partial}{\partial \xi_1} \left[\mathcal{A}_2 \cos \theta_{12} \right] - \frac{\partial \mathcal{A}_1}{\partial \xi_2} \right) \quad \text{and}$$

$$\frac{1}{\rho_{22}} = - \frac{\csc \theta_{12}}{\mathcal{A}_1 \mathcal{A}_2} \left(\frac{\partial}{\partial \xi_2} \left[\mathcal{A}_1 \cos \theta_{12} \right] - \frac{\partial \mathcal{A}_2}{\partial \xi_1} \right)$$

PARALLEL-SURFACE TANGENTIAL STRAINS

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- Previously, it was shown that $\vec{\gamma} = \gamma_1 \hat{a}_1 + \gamma_2 \hat{a}_2$, where

$$\gamma_1 = F_1(\xi_3) \left(2\varepsilon_{13}^0 - 2\varepsilon_{23}^0 \cos\theta_{12} \right) \csc^2\theta_{12} + \mathcal{O}(\varepsilon^4) = F_1(\xi_3) \gamma_1^0 + \mathcal{O}(\varepsilon^4) \quad \text{and}$$

$$\gamma_2 = F_2(\xi_3) \left(2\varepsilon_{23}^0 - 2\varepsilon_{13}^0 \cos\theta_{12} \right) \csc^2\theta_{12} + \mathcal{O}(\varepsilon^4) = F_2(\xi_3) \gamma_2^0 + \mathcal{O}(\varepsilon^4)$$

- Thus, the derivative components of $\vec{\gamma} = \gamma_1 \hat{a}_1 + \gamma_2 \hat{a}_2$ are given by

$$\hat{a}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} = \frac{1}{\mathcal{A}_1} \frac{\partial}{\partial \xi_1} \left(\gamma_1 + \gamma_2 \cos\theta_{12} \right) - \frac{\gamma_2 \sin\theta_{12}}{\rho_{11}}$$

$$\hat{a}_2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} = \frac{1}{\mathcal{A}_1} \frac{\partial \gamma_1}{\partial \xi_1} \cos\theta_{12} + \frac{1}{\mathcal{A}_1} \frac{\partial \gamma_2}{\partial \xi_1} + \frac{\gamma_1}{\rho_{11}} \sin\theta_{12}$$

PARALLEL-SURFACE TANGENTIAL STRAINS

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$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} = \frac{1}{\mathcal{A}_2} \frac{\partial \gamma_1}{\partial \xi_2} + \frac{1}{\mathcal{A}_2} \frac{\partial \gamma_2}{\partial \xi_2} \cos \theta_{12} - \frac{\gamma_2}{\rho_{22}} \sin \theta_{12}$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} = \frac{1}{\mathcal{A}_2} \frac{\partial}{\partial \xi_2} (\gamma_2 + \gamma_1 \cos \theta_{12}) + \frac{\gamma_1 \sin \theta_{12}}{\rho_{22}}$$

- Thus, the components $\hat{\mathbf{a}}_\alpha \cdot \frac{1}{\mathcal{A}_{(\beta)}} \frac{\partial \vec{\gamma}}{\partial \xi_\beta}$ involve $2\varepsilon_{13}^\circ$, $2\varepsilon_{23}^\circ$, and their derivatives
- As a result, these components are presumed $\mathcal{O}(\varepsilon^2)$, consistent with the presumption of “small” transverse-shearing strains and their derivatives

PARALLEL-SURFACE TANGENTIAL STRAINS

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- By using $\cos\theta_{12} = 2\varepsilon_{12}^{\circ} + \cos\theta_{12}^{\circ}\left(1 - \varepsilon_{11}^{\circ} - \varepsilon_{22}^{\circ}\right) + \mathcal{O}(\varepsilon^4)$ and

$\sin\theta_{12} = \sin\theta_{12}^{\circ} - \cos\theta_{12}^{\circ}\gamma_{12}^{\circ} + \mathcal{O}(\varepsilon^4)$, it follows that

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} = \frac{1}{\mathcal{A}_1} \frac{\partial}{\partial \xi_1} \left(\gamma_1 + \gamma_2 \cos\theta_{12} \right) - \frac{\gamma_2 \sin\theta_{12}}{\rho_{11}} + \mathcal{O}(\varepsilon^4)$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} = \frac{1}{\mathcal{A}_1} \frac{\partial \gamma_1}{\partial \xi_1} \cos\theta_{12} + \frac{1}{\mathcal{A}_1} \frac{\partial \gamma_2}{\partial \xi_1} + \frac{\gamma_1}{\rho_{11}} \sin\theta_{12} + \mathcal{O}(\varepsilon^4)$$

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} = \frac{1}{\mathcal{A}_2} \frac{\partial \gamma_1}{\partial \xi_2} + \frac{1}{\mathcal{A}_2} \frac{\partial \gamma_2}{\partial \xi_2} \cos\theta_{12} - \frac{\gamma_2}{\rho_{22}} \sin\theta_{12} + \mathcal{O}(\varepsilon^4)$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} = \frac{1}{\mathcal{A}_2} \frac{\partial}{\partial \xi_2} \left(\gamma_2 + \gamma_1 \cos\theta_{12} \right) + \frac{\gamma_1 \sin\theta_{12}}{\rho_{22}} + \mathcal{O}(\varepsilon^4)$$

PARALLEL-SURFACE TANGENTIAL STRAINS

CONTINUED

- Similarly, using $\frac{1}{\mathcal{A}_1} = \frac{1}{A_1} \left(1 - \varepsilon_{11}^o + \mathcal{O}(\varepsilon^4) \right)$ and $\frac{1}{\mathcal{A}_2} = \frac{1}{A_2} \left(1 - \varepsilon_{22}^o + \mathcal{O}(\varepsilon^4) \right)$ yields

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\gamma_1 + \gamma_2 \cos \theta_{12} \right) - \frac{\gamma_2 \sin \theta_{12}}{\rho_{11}} + \mathcal{O}(\varepsilon^4)$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} = \frac{1}{A_1} \frac{\partial \gamma_1}{\partial \xi_1} \cos \theta_{12} + \frac{1}{A_1} \frac{\partial \gamma_2}{\partial \xi_1} + \frac{\gamma_1}{\rho_{11}} \sin \theta_{12} + \mathcal{O}(\varepsilon^4)$$

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} = \frac{1}{A_2} \frac{\partial \gamma_1}{\partial \xi_2} + \frac{1}{A_2} \frac{\partial \gamma_2}{\partial \xi_2} \cos \theta_{12} - \frac{\gamma_2}{\rho_{22}} \sin \theta_{12} + \mathcal{O}(\varepsilon^4)$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\gamma_2 + \gamma_1 \cos \theta_{12} \right) + \frac{\gamma_1 \sin \theta_{12}}{\rho_{22}} + \mathcal{O}(\varepsilon^4)$$

PARALLEL-SURFACE TANGENTIAL STRAINS

CONTINUED

- Now consider the expressions

$$\frac{1}{\rho_{11}} = \frac{\csc \theta_{12}}{\mathcal{A}_1 \mathcal{A}_2} \left(\frac{\partial}{\partial \xi_1} [\mathcal{A}_2 \cos \theta_{12}] - \frac{\partial \mathcal{A}_1}{\partial \xi_2} \right) \quad \text{and}$$

$$\frac{1}{\rho_{22}} = - \frac{\csc \theta_{12}}{\mathcal{A}_1 \mathcal{A}_2} \left(\frac{\partial}{\partial \xi_2} [\mathcal{A}_1 \cos \theta_{12}] - \frac{\partial \mathcal{A}_2}{\partial \xi_1} \right)$$

- Using

$$\mathcal{A}_1 = A_1 \left(1 + \varepsilon_{11}^{\circ} + \mathcal{O}(\varepsilon^4) \right) \quad \mathcal{A}_2 = A_2 \left(1 + \varepsilon_{22}^{\circ} + \mathcal{O}(\varepsilon^4) \right)$$

$$\cos \theta_{12} = 2\varepsilon_{12}^{\circ} + \cos \theta_{12} \left(1 - \varepsilon_{11}^{\circ} - \varepsilon_{22}^{\circ} \right) + \mathcal{O}(\varepsilon^4)$$

$$\csc \theta_{12} = \csc \theta_{12} \left[1 + 2\varepsilon_{12}^{\circ} \cot \theta_{12} \csc \theta_{12} - \cot^2 \theta_{12} \left(\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ} \right) \right] + \mathcal{O}(\varepsilon^4) \quad \text{gives}$$

PARALLEL-SURFACE TANGENTIAL STRAINS CONTINUED

$$\frac{1}{\rho_{11}} = \frac{1}{\rho_{11}} \left[1 + 2\varepsilon_{12}^{\circ} \cot\theta_{12} \csc\theta_{12} - \csc^2\theta_{12} (\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}) \right] + \frac{\csc\theta_{12}}{A_1 A_2} \left[\frac{\partial}{\partial \xi_1} (A_2 [2\varepsilon_{12}^{\circ} - \varepsilon_{11}^{\circ} \cos\theta_{12}]) - \frac{\partial}{\partial \xi_2} [A_1 \varepsilon_{11}^{\circ}] \right] + \mathcal{O}(\varepsilon^4)$$

and

$$\frac{1}{\rho_{22}} = \frac{1}{\rho_{22}} \left[1 + 2\varepsilon_{12}^{\circ} \cot\theta_{12} \csc\theta_{12} - \csc^2\theta_{12} (\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}) \right] - \frac{\csc\theta_{12}}{A_1 A_2} \left[\frac{\partial}{\partial \xi_2} (A_1 [2\varepsilon_{12}^{\circ} - \cos\theta_{12} \varepsilon_{22}^{\circ}]) - \frac{\partial}{\partial \xi_1} [A_2 \varepsilon_{22}^{\circ}] \right] + \mathcal{O}(\varepsilon^4)$$

for the geodesic curvatures of the deformed reference surface, for “small” strains and their derivatives

PARALLEL-SURFACE TANGENTIAL STRAINS

CONTINUED

- As one might expect, the **changes in reference-surface geodesic curvature** due to deformation depends only on the tangential reference-surface strains and their derivatives
- The presumption that the derivatives of the tangential reference-surface strains are “small” is **consistent with the absence of strain concentrations** near a cutout, or other discontinuities, that exhibit extreme localized spatial variations

PARALLEL-SURFACE TANGENTIAL STRAINS

CONTINUED

- Using geodesic curvatures of the deformed reference surface gives

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} = \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} (\gamma_1 + \gamma_2 \cos \theta_{12}) - \frac{\gamma_2 \sin \theta_{12}}{\rho_{11}} + \mathcal{O}(\epsilon^4)$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} = \frac{1}{\mathbf{A}_1} \frac{\partial \gamma_1}{\partial \xi_1} \cos \theta_{12} + \frac{1}{\mathbf{A}_1} \frac{\partial \gamma_2}{\partial \xi_1} + \frac{\gamma_1}{\rho_{11}} \sin \theta_{12} + \mathcal{O}(\epsilon^4)$$

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} = \frac{1}{\mathbf{A}_2} \frac{\partial \gamma_1}{\partial \xi_2} + \frac{1}{\mathbf{A}_2} \frac{\partial \gamma_2}{\partial \xi_2} \cos \theta_{12} - \frac{\gamma_2}{\rho_{22}} \sin \theta_{12} + \mathcal{O}(\epsilon^4)$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} = \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} (\gamma_2 + \gamma_1 \cos \theta_{12}) + \frac{\gamma_1 \sin \theta_{12}}{\rho_{22}} + \mathcal{O}(\epsilon^4)$$

which are expressed entirely in terms of reference-surface geometric parameters and transverse shearing strains

PARALLEL-SURFACE TANGENTIAL STRAINS

CONTINUED

- Now, $\mathcal{A}_1 = A_1 \left(1 + \varepsilon_{11}^o + \mathcal{O}(\varepsilon^4) \right)$ and $\mathcal{A}_2 = A_2 \left(1 + \varepsilon_{22}^o + \mathcal{O}(\varepsilon^4) \right)$ are used to get

$$\left(\frac{\mathcal{A}_1}{A_1} \right)^2 = 1 + 2\varepsilon_{11}^o + \mathcal{O}(\varepsilon^4)$$

$$\left(\frac{\mathcal{A}_2}{A_2} \right)^2 = 1 + 2\varepsilon_{22}^o + \mathcal{O}(\varepsilon^4)$$

$$\frac{\mathcal{A}_1 \mathcal{A}_2}{A_1 A_2} = 1 + \varepsilon_{11}^o + \varepsilon_{22}^o + \mathcal{O}(\varepsilon^4)$$

- Hence,

$$\left(\frac{\mathcal{A}_1}{A_1} \right)^2 \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right) = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\gamma_1 + \gamma_2 \cos \theta_{12} \right) - \frac{\gamma_2 \sin \theta_{12}}{\rho_{11}} + \mathcal{O}(\varepsilon^4)$$

$$\left(\frac{\mathcal{A}_1}{A_1} \right)^2 \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right) = \frac{1}{A_1} \frac{\partial \gamma_1}{\partial \xi_1} \cos \theta_{12} + \frac{1}{A_1} \frac{\partial \gamma_2}{\partial \xi_1} + \frac{\gamma_1}{\rho_{11}} \sin \theta_{12} + \mathcal{O}(\varepsilon^4)$$

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$$\left(\frac{\mathcal{A}_2}{\mathbf{A}_2}\right)^2 \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \right) = \frac{1}{\mathbf{A}_2} \frac{\partial \gamma_1}{\partial \xi_2} + \frac{1}{\mathbf{A}_2} \frac{\partial \gamma_2}{\partial \xi_2} \cos \theta_{12} - \frac{\gamma_2}{\rho_{22}} \sin \theta_{12} + \mathcal{O}(\varepsilon^4)$$

$$\left(\frac{\mathcal{A}_2}{\mathbf{A}_2}\right)^2 \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \right) = \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} \left(\gamma_2 + \gamma_1 \cos \theta_{12} \right) + \frac{\gamma_1 \sin \theta_{12}}{\rho_{22}} + \mathcal{O}(\varepsilon^4)$$

$$\frac{\mathcal{A}_1 \mathcal{A}_2}{\mathbf{A}_1 \mathbf{A}_2} \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right) = \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\gamma_1 + \gamma_2 \cos \theta_{12} \right) - \frac{\gamma_2 \sin \theta_{12}}{\rho_{11}} + \mathcal{O}(\varepsilon^4)$$

$$\frac{\mathcal{A}_1 \mathcal{A}_2}{\mathbf{A}_1 \mathbf{A}_2} \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \vec{\gamma}}{\partial \xi_1} \right) = \frac{1}{\mathbf{A}_1} \frac{\partial \gamma_1}{\partial \xi_1} \cos \theta_{12} + \frac{1}{\mathbf{A}_1} \frac{\partial \gamma_2}{\partial \xi_1} + \frac{\gamma_1}{\rho_{11}} \sin \theta_{12} + \mathcal{O}(\varepsilon^4)$$

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$$\frac{\mathcal{A}_1 \mathcal{A}_2}{\mathbf{A}_1 \mathbf{A}_2} \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \right) = \frac{1}{\mathbf{A}_2} \frac{\partial \gamma_1}{\partial \xi_2} + \frac{1}{\mathbf{A}_2} \frac{\partial \gamma_2}{\partial \xi_2} \cos \theta_{12} - \frac{\gamma_2}{\rho_{22}} \sin \theta_{12} + \mathcal{O}(\epsilon^4)$$

$$\frac{\mathcal{A}_1 \mathcal{A}_2}{\mathbf{A}_1 \mathbf{A}_2} \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \vec{\gamma}}{\partial \xi_2} \right) = \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} \left(\gamma_2 + \gamma_1 \cos \theta_{12} \right) + \frac{\gamma_1 \sin \theta_{12}}{\rho_{22}} + \mathcal{O}(\epsilon^4)$$

- Using these results with

$$\cot \theta_{12} = \cot \theta_{12} + 2\epsilon_{12}^{\circ} \csc^3 \theta_{12} - \left(\epsilon_{11}^{\circ} + \epsilon_{22}^{\circ} \right) \csc^2 \theta_{12} \cot \theta_{12} + \mathcal{O}(\epsilon^4) \quad \text{and}$$

$$\csc \theta_{12} = \csc \theta_{12} \left[1 + 2\epsilon_{12}^{\circ} \cot \theta_{12} \csc \theta_{12} - \cot^2 \theta_{12} \left(\epsilon_{11}^{\circ} + \epsilon_{22}^{\circ} \right) \right] + \mathcal{O}(\epsilon^4) \quad \text{leads to}$$

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$$\Gamma_{11} = -\frac{\xi_3}{r_{12}} \left(\frac{\cot\theta_{12}}{A_1} \frac{\partial\gamma_1}{\partial\xi_1} + \frac{\csc\theta_{12}}{A_1} \frac{\partial\gamma_2}{\partial\xi_1} + \frac{\gamma_1}{\rho_{11}} \right) + \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}} \right) \left(\frac{1}{A_1} \frac{\partial}{\partial\xi_1} (\gamma_1 + \gamma_2 \cos\theta_{12}) - \frac{\gamma_2 \sin\theta_{12}}{\rho_{11}} \right) + \mathcal{O}(\varepsilon^4)$$

$$\Gamma_{22} = \frac{\xi_3}{r_{21}} \left(\frac{\csc\theta_{12}}{A_2} \frac{\partial\gamma_1}{\partial\xi_2} + \frac{\cot\theta_{12}}{A_2} \frac{\partial\gamma_2}{\partial\xi_2} - \frac{\gamma_2}{\rho_{22}} \right) + \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}} \right) \left(\frac{1}{A_2} \frac{\partial}{\partial\xi_2} (\gamma_2 + \gamma_1 \cos\theta_{12}) + \frac{\gamma_1 \sin\theta_{12}}{\rho_{22}} \right) + \mathcal{O}(\varepsilon^4)$$

PARALLEL-SURFACE TANGENTIAL STRAINS

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$$\begin{aligned}
 2\Gamma_{12} = & \frac{\xi_3}{r_{21}} \left(\frac{\csc\theta_{12}}{A_1} \frac{\partial}{\partial \xi_1} (\gamma_1 + \gamma_2 \cos\theta_{12}) - \frac{\gamma_2}{\rho_{11}} \right) \\
 & - \frac{\xi_3}{r_{12}} \left(\frac{\csc\theta_{12}}{A_2} \frac{\partial}{\partial \xi_2} (\gamma_2 + \gamma_1 \cos\theta_{12}) + \frac{\gamma_1}{\rho_{22}} \right) \\
 & + \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}} \right) \left(\frac{1}{A_2} \frac{\partial \gamma_1}{\partial \xi_2} + \frac{1}{A_2} \frac{\partial \gamma_2}{\partial \xi_2} \cos\theta_{12} - \frac{\gamma_2}{\rho_{22}} \sin\theta_{12} \right) \\
 & + \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}} \right) \left(\frac{1}{A_1} \frac{\partial \gamma_1}{\partial \xi_1} \cos\theta_{12} + \frac{1}{A_1} \frac{\partial \gamma_2}{\partial \xi_1} + \frac{\gamma_1}{\rho_{11}} \sin\theta_{12} \right) + \mathcal{O}(\varepsilon^4)
 \end{aligned}$$

- Next, using $\cot\theta_{12} = \cot\theta_{12}^0 + 2\varepsilon_{12}^0 \csc^3\theta_{12} - (\varepsilon_{11}^0 + \varepsilon_{22}^0) \csc^2\theta_{12} \cot\theta_{12} + \mathcal{O}(\varepsilon^4)$

and the identity $\frac{1}{r_{12}} + \frac{1}{r_{21}} = \cot\theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$ with

PARALLEL-SURFACE TANGENTIAL STRAINS

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$$-\frac{1}{r_{12}} = K_{12}^{\circ} - \frac{\cot\theta_{12}}{2} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} + K_{11}^{\circ} - K_{22}^{\circ} \right) + \frac{1}{2} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right)$$

$$\frac{1}{r_{21}} = K_{12}^{\circ} + \frac{\cot\theta_{12}}{2} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} + K_{11}^{\circ} - K_{22}^{\circ} \right) + \frac{1}{2} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \quad \text{gives}$$

$$-\frac{1}{r_{12}} = -\frac{1}{r_{12}} + K_{12}^{\circ} - \frac{\cot\theta_{12}}{2} (K_{11}^{\circ} - K_{22}^{\circ})$$

$$- \frac{1}{2} \left[2\varepsilon_{12}^{\circ} \csc^3\theta_{12} - (\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}) \csc^2\theta_{12} \cot\theta_{12} \right] \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) + \mathcal{O}(\varepsilon^4) \quad \text{and}$$

$$\frac{1}{r_{21}} = \frac{1}{r_{21}} + K_{12}^{\circ} + \frac{\cot\theta_{12}}{2} (K_{11}^{\circ} - K_{22}^{\circ})$$

$$+ \frac{1}{2} \left(2\varepsilon_{12}^{\circ} \csc^3\theta_{12} - (\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}) \csc^2\theta_{12} \cot\theta_{12} \right) \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) + \mathcal{O}(\varepsilon^4)$$

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- Using these expressions with $\frac{1}{r_{11}} = \frac{1}{r_{11}} + K_{11}^o$ and $\frac{1}{r_{22}} = \frac{1}{r_{22}} + K_{22}^o$ yields

$$\Gamma_{11} = \left(1 + \frac{\xi_3}{r_{11}}\right) \frac{1}{A_1} \frac{\partial \gamma_1}{\partial \xi_1} - \frac{\xi_3}{r_{12}} \frac{\gamma_1}{\rho_{11}} - \frac{\xi_3}{r_{12}} \left(\frac{\csc \theta_{12}}{A_1} \frac{\partial \gamma_2}{\partial \xi_1} \right) + \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right) \left(\frac{1}{A_1} \frac{\partial}{\partial \xi_1} (\gamma_2 \cos \theta_{12}) - \frac{\gamma_2 \sin \theta_{12}}{\rho_{11}} \right) + \mathcal{O}(\epsilon^4)$$

$$\Gamma_{22} = \left(1 + \frac{\xi_3}{r_{22}}\right) \frac{1}{A_2} \frac{\partial \gamma_2}{\partial \xi_2} - \frac{\xi_3}{r_{21}} \frac{\gamma_2}{\rho_{22}} + \frac{\xi_3}{r_{21}} \left(\frac{\csc \theta_{12}}{A_2} \frac{\partial \gamma_1}{\partial \xi_2} \right) + \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right) \left(\frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\gamma_1 \cos \theta_{12}) + \frac{\gamma_1 \sin \theta_{12}}{\rho_{22}} \right) + \mathcal{O}(\epsilon^4)$$

PARALLEL-SURFACE TANGENTIAL STRAINS

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$$\begin{aligned}
 2\Gamma_{12} = & \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right) \frac{1}{A_2} \frac{\partial \gamma_1}{\partial \xi_2} + \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right) \left(\frac{\cos \theta_{12}}{A_1} \frac{\partial \gamma_1}{\partial \xi_1} + \frac{\gamma_1 \sin \theta_{12}}{\rho_{11}} \right) \\
 & + \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right) \frac{1}{A_1} \frac{\partial \gamma_2}{\partial \xi_1} + \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right) \left(\frac{\cos \theta_{12}}{A_2} \frac{\partial \gamma_2}{\partial \xi_2} - \frac{\gamma_2 \sin \theta_{12}}{\rho_{22}} \right) \\
 & + \frac{\xi_3}{r_{21}} \left(\frac{\csc \theta_{12}}{A_1} \frac{\partial \gamma_1}{\partial \xi} \right) - \frac{\xi_3}{r_{12}} \left(\frac{\csc \theta_{12}}{A_2} \frac{\partial}{\partial \xi_2} (\gamma_1 \cos \theta_{12}) + \frac{\gamma_1}{\rho_{22}} \right) \\
 & + \frac{\xi_3}{r_{21}} \left(\frac{\csc \theta_{12}}{A_1} \frac{\partial}{\partial \xi_1} (\gamma_2 \cos \theta_{12}) - \frac{\gamma_2}{\rho_{11}} \right) - \frac{\xi_3}{r_{12}} \left(\frac{\csc \theta_{12}}{A_2} \frac{\partial \gamma_2}{\partial \xi_2} \right) + \mathcal{O}(\epsilon^4)
 \end{aligned}$$

PARALLEL-SURFACE TANGENTIAL STRAINS CONTINUED

- Next, $\gamma_1 = F_1(\xi_3)\gamma_1^\circ$ and $\gamma_2 = F_2(\xi_3)\gamma_2^\circ$ are used to express Γ_{11} , Γ_{22} , and $2\Gamma_{12}$ in terms of the primary unknowns as follows

$$\begin{aligned} \Gamma_{11} = & F_2(\xi_3) \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(\frac{1}{A_1} \frac{\partial}{\partial \xi_1} (\gamma_2^\circ \cos \theta_{12}) - \frac{\gamma_2^\circ \sin \theta_{12}}{\rho_{11}} \right) \\ & - F_2(\xi_3) \frac{\xi_3}{r_{12}} \left(\frac{\csc \theta_{12}}{A_1} \frac{\partial \gamma_2^\circ}{\partial \xi_1} \right) + F_1(\xi_3) \left[\left(1 + \frac{\xi_3}{r_{11}} \right) \frac{1}{A_1} \frac{\partial \gamma_1^\circ}{\partial \xi_1} - \frac{\xi_3}{r_{12}} \frac{\gamma_1^\circ}{\rho_{11}} \right] + \mathcal{O}(\epsilon^4) \end{aligned}$$

$$\begin{aligned} \Gamma_{22} = & F_1(\xi_3) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \left(\frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\gamma_1^\circ \cos \theta_{12}) + \frac{\gamma_1^\circ \sin \theta_{12}}{\rho_{22}} \right) \\ & + F_1(\xi_3) \frac{\xi_3}{r_{21}} \left(\frac{\csc \theta_{12}}{A_2} \frac{\partial \gamma_1^\circ}{\partial \xi_2} \right) + F_2(\xi_3) \left[\left(1 + \frac{\xi_3}{r_{22}} \right) \frac{1}{A_2} \frac{\partial \gamma_2^\circ}{\partial \xi_2} - \frac{\xi_3}{r_{21}} \frac{\gamma_2^\circ}{\rho_{22}} \right] + \mathcal{O}(\epsilon^4) \end{aligned}$$

PARALLEL-SURFACE TANGENTIAL STRAINS

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$$\begin{aligned}
 2\Gamma_{12} = & F_1(\xi_3) \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \frac{1}{A_2} \frac{\partial \gamma_1^o}{\partial \xi_2} + F_2(\xi_3) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \frac{1}{A_1} \frac{\partial \gamma_2^o}{\partial \xi_1} \\
 & + F_1(\xi_3) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \left(\frac{\cos \theta_{12}}{A_1} \frac{\partial \gamma_1^o}{\partial \xi_1} + \frac{\gamma_1^o \sin \theta_{12}}{\rho_{11}} \right) \\
 & + F_2(\xi_3) \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(\frac{\cos \theta_{12}}{A_2} \frac{\partial \gamma_2^o}{\partial \xi_2} - \frac{\gamma_2^o \sin \theta_{12}}{\rho_{22}} \right) \\
 & + F_1(\xi_3) \left[\frac{\xi_3}{r_{21}} \left(\frac{\csc \theta_{12}}{A_1} \frac{\partial \gamma_1^o}{\partial \xi} \right) - \frac{\xi_3}{r_{12}} \left(\frac{\csc \theta_{12}}{A_2} \frac{\partial}{\partial \xi_2} (\gamma_1^o \cos \theta_{12}) + \frac{\gamma_1^o}{\rho_{22}} \right) \right] \\
 & + F_2(\xi_3) \left[\frac{\xi_3}{r_{21}} \left(\frac{\csc \theta_{12}}{A_1} \frac{\partial}{\partial \xi_1} (\gamma_2^o \cos \theta_{12}) - \frac{\gamma_2^o}{\rho_{11}} \right) - \frac{\xi_3}{r_{12}} \left(\frac{\csc \theta_{12}}{A_2} \frac{\partial \gamma_2^o}{\partial \xi_2} \right) \right] + \mathcal{O}(\varepsilon^4)
 \end{aligned}$$

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- Now, consider the **simplification of the shell-strain expression**

$$2\varepsilon_{\alpha\beta} = \frac{\mathbf{A}_{(\alpha)}\mathbf{A}_{(\beta)}}{\mathbf{H}_{(\alpha)}\mathbf{H}_{(\beta)}} \left[\left(\frac{\vec{\mathcal{G}}_{\alpha}}{\mathbf{A}_{(\alpha)}} \cdot \frac{\vec{\mathcal{G}}_{\beta}}{\mathbf{A}_{(\beta)}} \right) - \left(\frac{\vec{\mathbf{g}}_{\alpha}}{\mathbf{A}_{(\alpha)}} \cdot \frac{\vec{\mathbf{g}}_{\beta}}{\mathbf{A}_{(\beta)}} \right) \right]$$

by using the “small-strain” approximations; that is

$$\frac{\vec{\mathcal{G}}_1}{\mathbf{A}_1} \cdot \frac{\vec{\mathcal{G}}_1}{\mathbf{A}_1} = \left(\frac{\mathcal{A}_1}{\mathbf{A}_1} \right)^2 \left[1 + \frac{2\xi_3}{\mathbf{r}_{11}} + \left(\frac{\xi_3}{\mathbf{r}_{11}} \right)^2 + \left(\frac{\xi_3}{\mathbf{r}_{12}} \right)^2 \right] + 2\Gamma_{11} + \mathcal{O}(\varepsilon^4)$$

$$\frac{\vec{\mathcal{G}}_2}{\mathbf{A}_2} \cdot \frac{\vec{\mathcal{G}}_2}{\mathbf{A}_2} = \left(\frac{\mathcal{A}_2}{\mathbf{A}_2} \right)^2 \left[1 + \frac{2\xi_3}{\mathbf{r}_{22}} + \left(\frac{\xi_3}{\mathbf{r}_{21}} \right)^2 + \left(\frac{\xi_3}{\mathbf{r}_{22}} \right)^2 \right] + 2\Gamma_{22} + \mathcal{O}(\varepsilon^4)$$

$$\begin{aligned} \frac{\vec{\mathcal{G}}_1}{\mathbf{A}_1} \cdot \frac{\vec{\mathcal{G}}_2}{\mathbf{A}_2} &= \frac{\mathcal{A}_1\mathcal{A}_2}{\mathbf{A}_1\mathbf{A}_2} \left[\left(1 + \frac{\xi_3}{\mathbf{r}_{11}} + \frac{\xi_3}{\mathbf{r}_{22}} \right) \cos\theta_{12} + \left(\frac{\xi_3}{\mathbf{r}_{21}} - \frac{\xi_3}{\mathbf{r}_{12}} \right) \sin\theta_{12} \right. \\ &\quad \left. + \left(\frac{\xi_3}{\mathbf{r}_{11}} \frac{\xi_3}{\mathbf{r}_{21}} - \frac{\xi_3}{\mathbf{r}_{12}} \frac{\xi_3}{\mathbf{r}_{22}} \right) \sin\theta_{12} + \left(\frac{\xi_3}{\mathbf{r}_{12}} \frac{\xi_3}{\mathbf{r}_{21}} + \frac{\xi_3}{\mathbf{r}_{11}} \frac{\xi_3}{\mathbf{r}_{22}} \right) \cos\theta_{12} + 2\Gamma_{12} + \mathcal{O}(\varepsilon^4) \right] \end{aligned}$$

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are used with

$$\frac{\vec{g}_1}{A_1} \cdot \frac{\vec{g}_1}{A_1} = 1 + 2\frac{\xi_3}{r_{11}} + \left(\frac{\xi_3}{r_{11}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2$$

$$\frac{\vec{g}_2}{A_2} \cdot \frac{\vec{g}_2}{A_2} = 1 + 2\frac{\xi_3}{r_{22}} + \left(\frac{\xi_3}{r_{21}}\right)^2 + \left(\frac{\xi_3}{r_{22}}\right)^2$$

$$\begin{aligned} \frac{\vec{g}_1}{A_1} \cdot \frac{\vec{g}_2}{A_2} = & \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}}\right) \cos\theta_{12} + \left(\frac{\xi_3}{r_{21}} - \frac{\xi_3}{r_{12}}\right) \sin\theta_{12} \\ & + \left(\frac{\xi_3}{r_{11}} \frac{\xi_3}{r_{21}} - \frac{\xi_3}{r_{12}} \frac{\xi_3}{r_{22}}\right) \sin\theta_{12} + \left(\frac{\xi_3}{r_{12}} \frac{\xi_3}{r_{21}} + \frac{\xi_3}{r_{11}} \frac{\xi_3}{r_{22}}\right) \cos\theta_{12} \end{aligned}$$

to get

$$\begin{aligned} 2\varepsilon_{11} \left(\frac{H_1}{A_1}\right)^2 = & \left(\frac{\mathcal{A}_1}{A_1}\right)^2 \left[1 + \frac{2\xi_3}{r_{11}} + \left(\frac{\xi_3}{r_{11}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2 \right] \\ & - \left[1 + \frac{2\xi_3}{r_{11}} + \left(\frac{\xi_3}{r_{11}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2 \right] + 2\Gamma_{11} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

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$$2\varepsilon_{22} \left(\frac{H_2}{A_2} \right)^2 = \left(\frac{\mathcal{A}_2}{A_2} \right)^2 \left[1 + \frac{2\xi_3}{r_{22}} + \left(\frac{\xi_3}{r_{21}} \right)^2 + \left(\frac{\xi_3}{r_{22}} \right)^2 \right] - \left[1 + \frac{2\xi_3}{r_{22}} + \left(\frac{\xi_3}{r_{21}} \right)^2 + \left(\frac{\xi_3}{r_{22}} \right)^2 \right] + 2\Gamma_{22} + \mathcal{O}(\varepsilon^4)$$

and

$$\begin{aligned} 2\varepsilon_{12} \frac{H_1 H_2}{A_1 A_2} = & \frac{\mathcal{A}_1 \mathcal{A}_2}{A_1 A_2} \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cos \theta_{12} - \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cos \theta_{12} \\ & + \frac{\mathcal{A}_1 \mathcal{A}_2}{A_1 A_2} \left(\frac{\xi_3}{r_{21}} - \frac{\xi_3}{r_{12}} \right) \sin \theta_{12} - \left(\frac{\xi_3}{r_{21}} - \frac{\xi_3}{r_{12}} \right) \sin \theta_{12} \\ & + \frac{\mathcal{A}_1 \mathcal{A}_2}{A_1 A_2} \left(\frac{\xi_3 \xi_3}{r_{11} r_{21}} - \frac{\xi_3 \xi_3}{r_{12} r_{22}} \right) \sin \theta_{12} - \left(\frac{\xi_3 \xi_3}{r_{11} r_{21}} - \frac{\xi_3 \xi_3}{r_{12} r_{22}} \right) \sin \theta_{12} \\ & + \frac{\mathcal{A}_1 \mathcal{A}_2}{A_1 A_2} \left(\frac{\xi_3 \xi_3}{r_{12} r_{21}} + \frac{\xi_3 \xi_3}{r_{11} r_{22}} \right) \cos \theta_{12} - \left(\frac{\xi_3 \xi_3}{r_{12} r_{21}} + \frac{\xi_3 \xi_3}{r_{11} r_{22}} \right) \cos \theta_{12} + 2\Gamma_{12} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

PARALLEL-SURFACE TANGENTIAL STRAINS

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- Next,

$$\cot\theta_{12} = \cot\theta_{12}^{\circ} + 2\varepsilon_{12}^{\circ} \csc^3\theta_{12} - \left(\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}\right) \csc^2\theta_{12} \cot\theta_{12} + \mathcal{O}(\varepsilon^4)$$

and the identity $\frac{1}{r_{12}} + \frac{1}{r_{21}} = \cot\theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$ are used with

$$-\frac{1}{r_{12}} = K_{12}^{\circ} - \frac{\cot\theta_{12}}{2} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} + K_{11}^{\circ} - K_{22}^{\circ} \right) + \frac{1}{2} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \quad \text{and}$$

$$\frac{1}{r_{21}} = K_{12}^{\circ} + \frac{\cot\theta_{12}}{2} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} + K_{11}^{\circ} - K_{22}^{\circ} \right) + \frac{1}{2} \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \quad \text{to get}$$

PARALLEL-SURFACE TANGENTIAL STRAINS

CONTINUED

$$\frac{1}{r_{12}} = \frac{1}{r_{12}} - K_{12}^{\circ} + \frac{1}{2} \cot \theta_{12} (K_{11}^{\circ} - K_{22}^{\circ})$$

$$+ \frac{1}{2} \left[2\epsilon_{12}^{\circ} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc \theta_{12} - (\epsilon_{11}^{\circ} + \epsilon_{22}^{\circ}) \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \right] \csc^2 \theta_{12} + \mathcal{O}(\epsilon^4)$$

and

$$\frac{1}{r_{21}} = \frac{1}{r_{21}} + K_{12}^{\circ} + \frac{1}{2} \cot \theta_{12} (K_{11}^{\circ} - K_{22}^{\circ})$$

$$+ \frac{1}{2} \left[2\epsilon_{12}^{\circ} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc \theta_{12} - (\epsilon_{11}^{\circ} + \epsilon_{22}^{\circ}) \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \right] \csc^2 \theta_{12} + \mathcal{O}(\epsilon^4)$$

PARALLEL-SURFACE TANGENTIAL STRAINS

CONTINUED

- The strains are simplified by using these “small-strain” expressions for the torsions of the deformed reference surface along with

$$\mathcal{A}_1 = A_1 \left(1 + \varepsilon_{11}^o + \mathcal{O}(\varepsilon^4) \right) \quad \mathcal{A}_2 = A_2 \left(1 + \varepsilon_{22}^o + \mathcal{O}(\varepsilon^4) \right)$$

$$\frac{1}{r_{11}} = \frac{1}{r_{11}} + K_{11}^o$$

$$\cos \theta_{12} = 2\varepsilon_{12}^o + \cos \theta_{12} \left(1 - \varepsilon_{11}^o - \varepsilon_{22}^o \right) + \mathcal{O}(\varepsilon^4)$$

$$\frac{1}{r_{22}} = \frac{1}{r_{22}} + K_{22}^o$$

$$\sin \theta_{12} = \sin \theta_{12} - \cot \theta_{12} \left[2\varepsilon_{12}^o - \cos \theta_{12} \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \right] + \mathcal{O}(\varepsilon^4)$$

and the identity

$$\frac{1}{r_{12}} + \frac{1}{r_{21}} = \cot \theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$$

PARALLEL-SURFACE TANGENTIAL STRAINS

CONTINUED

- This process yields

$$\begin{aligned} \epsilon_{11} \left(\frac{H_1}{A_1} \right)^2 = & \epsilon_{11}^o \left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right] + \xi_3 K_{11}^o \left(1 + \frac{\xi_3}{r_{11}} \right) \\ & + \frac{1}{2} \left(\frac{\xi_3}{r_{12}} \right) \left[2\epsilon_{12}^o \left(\frac{\xi_3}{r_{11}} - \frac{\xi_3}{r_{22}} \right) \csc \theta_{12} - \left(\epsilon_{11}^o + \epsilon_{22}^o \right) \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right) \csc^2 \theta_{12} \right. \\ & \left. - \xi_3 \left(\frac{\xi_3}{r_{12}} \right) \left[K_{12}^o - \frac{1}{2} \cot \theta_{12} (K_{11}^o - K_{22}^o) \right] + \Gamma_{11} + \mathcal{O}(\epsilon^4) \right] \end{aligned}$$

where

$$\left(\frac{H_1}{A_1} \right)^2 = \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}} \right)^2 \quad \text{and}$$

$$\Gamma_{11} = \Gamma_{11} \left(\frac{\xi_3}{r_{11}}, \frac{\xi_3}{r_{12}}, \frac{\xi_3}{r_{21}}, \frac{\xi_3}{r_{22}}, F_1(\xi_3), F_2(\xi_3), 2\epsilon_{13}^o, 2\epsilon_{23}^o \right)$$

PARALLEL-SURFACE TANGENTIAL STRAINS

CONTINUED

- Likewise

$$\begin{aligned} \epsilon_{22} \left(\frac{H_2}{A_2} \right)^2 &= \epsilon_{22}^o \left[\left(1 + \frac{\xi_3}{r_{22}} \right)^2 + \left(\frac{\xi_3}{r_{21}} \right)^2 \right] + \xi_3 K_{22}^o \left(1 + \frac{\xi_3}{r_{22}} \right) \\ &+ \frac{1}{2} \left(\frac{\xi_3}{r_{21}} \right) \left[2\epsilon_{12}^o \left(\frac{\xi_3}{r_{11}} - \frac{\xi_3}{r_{22}} \right) \csc \theta_{12} - \left(\epsilon_{11}^o + \epsilon_{22}^o \right) \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right) \csc^2 \theta_{12} \right. \\ &\quad \left. + \xi_3 \left(\frac{\xi_3}{r_{21}} \right) \left[K_{12}^o + \frac{1}{2} \cot \theta_{12} (K_{11}^o - K_{22}^o) \right] + \Gamma_{22} + \mathcal{O}(\epsilon^4) \right] \end{aligned}$$

where

$$\left(\frac{H_2}{A_2} \right)^2 = \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}} \right)^2 \quad \text{and}$$

$$\Gamma_{22} = \Gamma_{22} \left(\frac{\xi_3}{r_{11}}, \frac{\xi_3}{r_{12}}, \frac{\xi_3}{r_{21}}, \frac{\xi_3}{r_{22}}, F_1(\xi_3), F_2(\xi_3), 2\epsilon_{13}^o, 2\epsilon_{23}^o \right)$$

PARALLEL-SURFACE TANGENTIAL STRAINS

CONTINUED

$$\begin{aligned}
 2\varepsilon_{12} \frac{H_1 H_2}{A_1 A_2} = & \varepsilon_{12}^o \left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 - \left(\frac{\xi_3}{r_{12}} \right)^2 + \left(1 + \frac{\xi_3}{r_{22}} \right)^2 - \left(\frac{\xi_3}{r_{21}} \right)^2 \right] \\
 & + \varepsilon_{12}^o \left[\left(\frac{\xi_3}{r_{12}} - \frac{\xi_3}{r_{21}} \right) \left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cot \theta_{12} + \left(\frac{\xi_3}{r_{11}} - \frac{\xi_3}{r_{22}} \right)^2 \left(\csc^4 \theta_{12} - 1 \right) \right] \\
 & - \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \left[\left(\frac{\xi_3}{r_{12}} - \frac{\xi_3}{r_{21}} \right) \left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) + \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right)^2 \cot \theta_{12} \right] \frac{\csc \theta_{12}}{2} \\
 & + \xi_3 \frac{K_{11}^o}{2} \left[\left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cot \theta_{12} + \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right) \csc^2 \theta_{12} - \frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right] \sin \theta_{12} \\
 & + \xi_3 \frac{K_{22}^o}{2} \left[\left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cot \theta_{12} - \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right) \csc^2 \theta_{12} - \frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right] \sin \theta_{12} \\
 & + \xi_3 K_{12}^o \left[2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} + \left(\frac{\xi_3}{r_{12}} - \frac{\xi_3}{r_{21}} \right) \cot \theta_{12} \right] \sin \theta_{12} + 2\Gamma_{12} + \mathcal{O}(\varepsilon^4)
 \end{aligned}$$

PARALLEL-SURFACE TANGENTIAL STRAINS

CONCLUDED

where

$$\frac{H_1 H_2}{A_1 A_2} = \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}} \right)^2 \right]^{\frac{1}{2}} \left[\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}} \right)^2 \right]^{\frac{1}{2}}$$

and

$$2\Gamma_{12} = 2\Gamma_{12} \left(\frac{\xi_3}{r_{11}}, \frac{\xi_3}{r_{12}}, \frac{\xi_3}{r_{21}}, \frac{\xi_3}{r_{22}}, F_1(\xi_3), F_2(\xi_3), 2\varepsilon_{13}^{\circ}, 2\varepsilon_{23}^{\circ} \right)$$

DISPLACEMENT-VECTOR FIELD

- The displacement fields are obtained by using $\vec{X} = \vec{x} + \xi_3 \hat{n}$, $\vec{\mathcal{X}} = \vec{X} + \vec{U}$,
and $\vec{x} = \vec{X} + \vec{u}$ with $\vec{\mathcal{X}} = \vec{x} + \xi_3 \hat{a} + F_1(\xi_3) \gamma_1^0 \hat{a}_1 + F_2(\xi_3) \gamma_2^0 \hat{a}_2 + \overrightarrow{\mathcal{O}(\varepsilon^4)}$ to get

$$\vec{U} = \vec{u} + \xi_3 (\hat{a} - \hat{n}) + F_1(\xi_3) \gamma_1^0 \hat{a}_1 + F_2(\xi_3) \gamma_2^0 \hat{a}_2 + \overrightarrow{\mathcal{O}(\varepsilon^4)}$$

- The component form of this equation, with respect to the basis $\{\hat{a}_1, \hat{a}_2, \hat{n}\}$, given by $\vec{U} = U_1^0 \hat{a}_1 + U_2^0 \hat{a}_2 + U_3^0 \hat{n}$, is obtained by using

$$\hat{a}_1 = \left[(1 + \Delta_{11}) \hat{a}_1 + \Delta_{12} \hat{a}_2 + \Delta_{13} \hat{n} \right] (1 - \varepsilon_{11}^0 + \mathcal{O}(\varepsilon^4))$$

$$\hat{a}_2 = \left[\Delta_{21} \hat{a}_1 + (1 + \Delta_{22}) \hat{a}_2 + \Delta_{23} \hat{n} \right] (1 - \varepsilon_{22}^0 + \mathcal{O}(\varepsilon^4))$$

$$\hat{a} = \frac{\sqrt{a}}{\sqrt{a}} (m_1 \hat{a}_1 + m_2 \hat{a}_2 + m_3 \hat{n})$$

DISPLACEMENT-VECTOR FIELD

CONTINUED

$$\frac{\sqrt{a}}{\sqrt{a}} = 1 + 2\varepsilon_{12}^{\circ} \cot\theta_{12} \csc\theta_{12} - \left(\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ} \right) \csc^2\theta_{12} + \mathcal{O}(\varepsilon^4) \quad \text{and}$$

$$\vec{u} = u_1 \hat{a}_1 + u_2 \hat{a}_2 + w \hat{n}$$

- The resulting expressions are

$$\begin{aligned} U_1^0 = u_1 + \xi_3 m_1 & \left(1 + 2\varepsilon_{12}^{\circ} \cot\theta_{12} \csc\theta_{12} - \left(\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ} \right) \csc^2\theta_{12} + \mathcal{O}(\varepsilon^4) \right) \\ & + \left(1 + \Delta_{11} \right) \left(F_1(\xi_3) \gamma_1^{\circ} + \mathcal{O}(\varepsilon^4) \right) + \Delta_{21} \left(F_2(\xi_3) \gamma_2^{\circ} + \mathcal{O}(\varepsilon^4) \right) \end{aligned}$$

$$\begin{aligned} U_2^0 = u_2 + \xi_3 m_2 & \left(1 + 2\varepsilon_{12}^{\circ} \cot\theta_{12} \csc\theta_{12} - \left(\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ} \right) \csc^2\theta_{12} + \mathcal{O}(\varepsilon^4) \right) \\ & + \Delta_{12} \left(F_1(\xi_3) \gamma_1^{\circ} + \mathcal{O}(\varepsilon^4) \right) + \left(1 + \Delta_{22} \right) \left(F_2(\xi_3) \gamma_2^{\circ} + \mathcal{O}(\varepsilon^4) \right) \end{aligned}$$

DISPLACEMENT-VECTOR FIELD CONTINUED

$$\mathbf{U}_3^0 = \mathbf{w} + \xi_3 \left[\mathbf{m}_3 \left(1 + 2\varepsilon_{12}^0 \cot\theta_{12} \csc\theta_{12} - \left(\varepsilon_{11}^0 + \varepsilon_{22}^0 \right) \csc^2\theta_{12} + \mathcal{O}(\varepsilon^4) \right) - 1 \right] \\ + \Delta_{13} \left(\mathbf{F}_1(\xi_3) \boldsymbol{\gamma}_1^0 + \mathcal{O}(\varepsilon^4) \right) + \Delta_{23} \left(\mathbf{F}_2(\xi_3) \boldsymbol{\gamma}_2^0 + \mathcal{O}(\varepsilon^4) \right)$$

- These equations for the displacements are “exact” within the precision associated with “small” strains and strain derivatives

DISPLACEMENT-VECTOR FIELD CONTINUED

- The component form of the displacement vector field at an arbitrary point of a parallel reference surface, with respect to the basis

$\{\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \hat{\mathbf{g}}_3\}$, given by

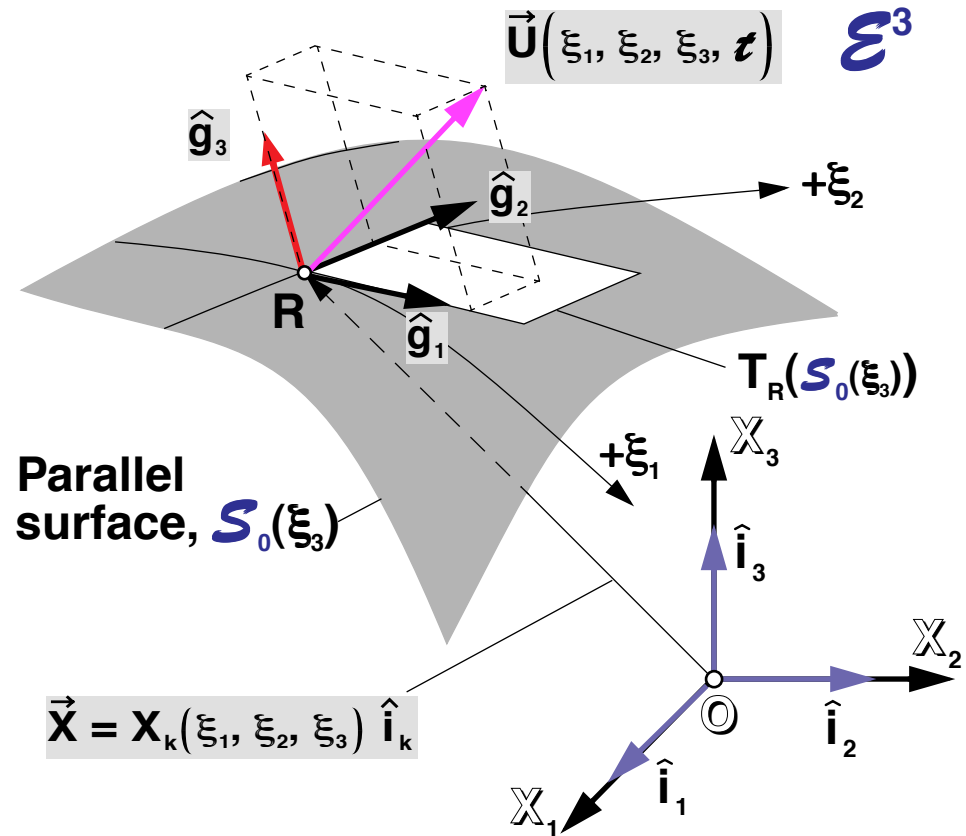
$$\vec{\mathbf{U}} = U_1 \hat{\mathbf{g}}_1 + U_2 \hat{\mathbf{g}}_2 + U_3 \hat{\mathbf{g}}_3$$

- Next, it is recalled that

$$\hat{\mathbf{g}}_1 = \mu_{11} \hat{\mathbf{a}}_1 + \mu_{12} \hat{\mathbf{a}}_2 \quad \text{and}$$

$$\hat{\mathbf{g}}_2 = \mu_{21} \hat{\mathbf{a}}_1 + \mu_{22} \hat{\mathbf{a}}_2, \quad \text{where}$$

$$\mu_{\alpha\beta} = \mu_{\alpha\beta}(\xi_1, \xi_2, \xi_3) \quad \text{are shifters that are defined as follows}$$



DISPLACEMENT-VECTOR FIELD

CONTINUED

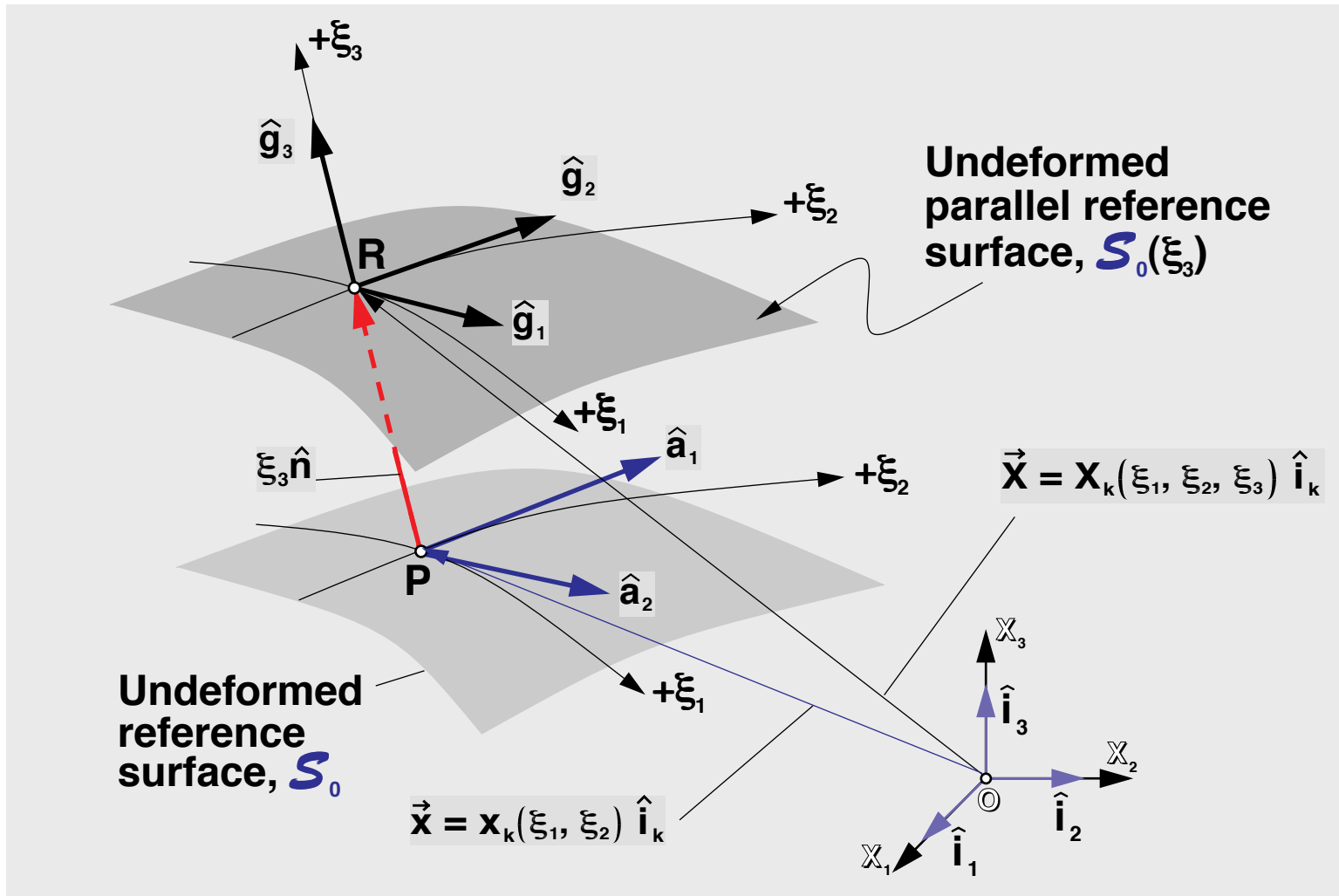
$$\mu_{11} = \frac{1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{12}}\right)^2}}$$

$$\mu_{12} = \frac{-\frac{\xi_3 \csc\theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{12}}\right)^2}}$$

$$\mu_{21} = \frac{\frac{\xi_3 \csc\theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}}$$

$$\mu_{22} = \frac{1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}}$$

DISPLACEMENT-VECTOR FIELD CONTINUED



DISPLACEMENT-VECTOR FIELD

CONCLUDED

- By using the shifters, it follows that

$$\mu_{11}U_1 + \mu_{21}U_2 = U_1^0 \quad \mu_{12}U_1 + \mu_{22}U_2 = U_2^0 \quad U_3 = U_3^0$$

- Thus, $U_1 = \frac{\mu_{22}U_1^0 - \mu_{21}U_2^0}{\mu_{11}\mu_{22} - \mu_{12}\mu_{21}}$ and $U_2 = \frac{\mu_{11}U_2^0 - \mu_{12}U_1^0}{\mu_{11}\mu_{22} - \mu_{12}\mu_{21}}$

- These expressions give the components of the displacement vector field at **an arbitrary point of a parallel reference surface**, with no simplifications made that are based on the thinness of the shell

**RESUME' OF EQUATIONS FOR
“SMALL” STRAINS AND FINITE ROTATIONS -
GENERAL GAUSSIAN COORDINATES**

EQUATIONS FOR GENERAL COORDINATES

- The Green-Lagrange shell strains $\left\{ \varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23} \right\}$ are given by

$$\begin{aligned} \varepsilon_{11} \left(\frac{H_1}{A_1} \right)^2 &= \varepsilon_{11}^o \left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right] + \xi_3 K_{11}^o \left(1 + \frac{\xi_3}{r_{11}} \right) \\ &+ \frac{1}{2} \left(\frac{\xi_3}{r_{12}} \right) \left[2\varepsilon_{12}^o \left(\frac{\xi_3}{r_{11}} - \frac{\xi_3}{r_{22}} \right) \csc \theta_{12} - \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right) \right] \csc^2 \theta_{12} \\ &- \xi_3 \left(\frac{\xi_3}{r_{12}} \right) \left[K_{12}^o - \frac{1}{2} \cot \theta_{12} \left(K_{11}^o - K_{22}^o \right) \right] + \Gamma_{11} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

where

$$\left(\frac{H_1}{A_1} \right)^2 = \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}} \right)^2$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned} \epsilon_{22} \left(\frac{H_2}{A_2} \right)^2 &= \epsilon_{22}^o \left[\left(1 + \frac{\xi_3}{r_{22}} \right)^2 + \left(\frac{\xi_3}{r_{21}} \right)^2 \right] + \xi_3 K_{22}^o \left(1 + \frac{\xi_3}{r_{22}} \right) \\ &+ \frac{1}{2} \left(\frac{\xi_3}{r_{21}} \right) \left[2\epsilon_{12}^o \left(\frac{\xi_3}{r_{11}} - \frac{\xi_3}{r_{22}} \right) \csc \theta_{12} - \left(\epsilon_{11}^o + \epsilon_{22}^o \right) \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right) \right] \csc^2 \theta_{12} \\ &+ \xi_3 \left(\frac{\xi_3}{r_{21}} \right) \left[K_{12}^o + \frac{1}{2} \cot \theta_{12} (K_{11}^o - K_{22}^o) \right] + \Gamma_{22} + \mathcal{O}(\epsilon^4) \end{aligned}$$

where

$$\left(\frac{H_2}{A_2} \right)^2 = \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}} \right)^2$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned}
 2\varepsilon_{12} \frac{H_1 H_2}{A_1 A_2} = & \varepsilon_{12}^o \left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 - \left(\frac{\xi_3}{r_{12}} \right)^2 + \left(1 + \frac{\xi_3}{r_{22}} \right)^2 - \left(\frac{\xi_3}{r_{21}} \right)^2 \right] \\
 & + \varepsilon_{12}^o \left[\left(\frac{\xi_3}{r_{12}} - \frac{\xi_3}{r_{21}} \right) \left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cot \theta_{12} + \left(\frac{\xi_3}{r_{11}} - \frac{\xi_3}{r_{22}} \right)^2 \left(\csc^4 \theta_{12} - 1 \right) \right] \\
 & - \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \left[\left(\frac{\xi_3}{r_{12}} - \frac{\xi_3}{r_{21}} \right) \left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) + \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right)^2 \cot \theta_{12} \right] \frac{\csc \theta_{12}}{2} \\
 & + \xi_3 \frac{K_{11}^o}{2} \left[\left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cot \theta_{12} + \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right) \csc^2 \theta_{12} - \frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right] \sin \theta_{12} \\
 & + \xi_3 \frac{K_{22}^o}{2} \left[\left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cot \theta_{12} - \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right) \csc^2 \theta_{12} - \frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right] \sin \theta_{12} \\
 & + \xi_3 K_{12}^o \left[2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} + \left(\frac{\xi_3}{r_{12}} - \frac{\xi_3}{r_{21}} \right) \cot \theta_{12} \right] \sin \theta_{12} + 2\Gamma_{12} + \mathcal{O}(\varepsilon^4)
 \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

where

$$\frac{H_1 H_2}{A_1 A_2} = \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}} \right)^2 \right]^{\frac{1}{2}} \left[\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}} \right)^2 \right]^{\frac{1}{2}}$$

$$\begin{aligned} \Gamma_{11} = & F_2(\xi_3) \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(\frac{1}{A_1} \frac{\partial}{\partial \xi_1} (\gamma_2^\circ \cos \theta_{12}) - \frac{\gamma_2^\circ \sin \theta_{12}}{\rho_{11}} \right) \\ & - F_2(\xi_3) \frac{\xi_3}{r_{12}} \left(\frac{\csc \theta_{12}}{A_1} \frac{\partial \gamma_2^\circ}{\partial \xi_1} \right) + F_1(\xi_3) \left[\left(1 + \frac{\xi_3}{r_{11}} \right) \frac{1}{A_1} \frac{\partial \gamma_1^\circ}{\partial \xi_1} - \frac{\xi_3}{r_{12}} \frac{\gamma_1^\circ}{\rho_{11}} \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\begin{aligned} \Gamma_{22} = & F_1(\xi_3) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \left(\frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\gamma_1^\circ \cos \theta_{12}) + \frac{\gamma_1^\circ \sin \theta_{12}}{\rho_{22}} \right) \\ & + F_1(\xi_3) \frac{\xi_3}{r_{21}} \left(\frac{\csc \theta_{12}}{A_2} \frac{\partial \gamma_1^\circ}{\partial \xi_2} \right) + F_2(\xi_3) \left[\left(1 + \frac{\xi_3}{r_{22}} \right) \frac{1}{A_2} \frac{\partial \gamma_2^\circ}{\partial \xi_2} - \frac{\xi_3}{r_{21}} \frac{\gamma_2^\circ}{\rho_{22}} \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned}
 2\Gamma_{12} = & F_1(\xi_3) \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \frac{1}{A_2} \frac{\partial \gamma_1^o}{\partial \xi_2} + F_2(\xi_3) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \frac{1}{A_1} \frac{\partial \gamma_2^o}{\partial \xi_1} \\
 & + F_1(\xi_3) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \left(\frac{\cos \theta_{12}}{A_1} \frac{\partial \gamma_1^o}{\partial \xi_1} + \frac{\gamma_1^o \sin \theta_{12}}{\rho_{11}} \right) \\
 & + F_2(\xi_3) \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(\frac{\cos \theta_{12}}{A_2} \frac{\partial \gamma_2^o}{\partial \xi_2} - \frac{\gamma_2^o \sin \theta_{12}}{\rho_{22}} \right) \\
 & + F_1(\xi_3) \left[\frac{\xi_3}{r_{21}} \left(\frac{\csc \theta_{12}}{A_1} \frac{\partial \gamma_1^o}{\partial \xi} \right) - \frac{\xi_3}{r_{12}} \left(\frac{\csc \theta_{12}}{A_2} \frac{\partial}{\partial \xi_2} (\gamma_1^o \cos \theta_{12}) + \frac{\gamma_1^o}{\rho_{22}} \right) \right] \\
 & + F_2(\xi_3) \left[\frac{\xi_3}{r_{21}} \left(\frac{\csc \theta_{12}}{A_1} \frac{\partial}{\partial \xi_1} (\gamma_2^o \cos \theta_{12}) - \frac{\gamma_2^o}{\rho_{11}} \right) - \frac{\xi_3}{r_{12}} \left(\frac{\csc \theta_{12}}{A_2} \frac{\partial \gamma_2^o}{\partial \xi_2} \right) \right] + \mathcal{O}(\varepsilon^4)
 \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\gamma_1^o = \csc^2 \theta_{12} \left(2\varepsilon_{13}^o - 2\varepsilon_{23}^o \cos \theta_{12} \right)$$

$$\gamma_2^o = \csc^2 \theta_{12} \left(2\varepsilon_{23}^o - 2\varepsilon_{13}^o \cos \theta_{12} \right)$$

and where it is noted that $2\varepsilon_{13}^o$ and $2\varepsilon_{23}^o$ are fundamental unknowns

- The transverse shearing strains are given by

$$2\varepsilon_{13} \frac{H_1}{A_1} = \left[F_1'(\xi_3) + \frac{P_1(\xi_3)}{r_{11}} \right] \gamma_1^o + \left[\left(F_2'(\xi_3) + \frac{P_2(\xi_3)}{r_{11}} \right) \cos \theta_{12} - \frac{P_2(\xi_3) \sin \theta_{12}}{r_{12}} \right] \gamma_2^o + \mathcal{O}(\varepsilon^4)$$

$$2\varepsilon_{23} \frac{H_2}{A_2} = \left[F_2'(\xi_3) + \frac{P_2(\xi_3)}{r_{22}} \right] \gamma_2^o + \left[\left(F_1'(\xi_3) + \frac{P_1(\xi_3)}{r_{22}} \right) \cos \theta_{12} + \frac{P_1(\xi_3) \sin \theta_{12}}{r_{21}} \right] \gamma_1^o + \mathcal{O}(\varepsilon^4)$$

where $P_1(\xi_3) \equiv \xi_3 F_1'(\xi_3) - F_1(\xi_3)$ and $P_2(\xi_3) \equiv \xi_3 F_2'(\xi_3) - F_2(\xi_3)$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

- The reference-surface membrane strains are given by

$$\begin{aligned} \boldsymbol{\varepsilon}_{11}^{\circ} = \mathbf{e}_{11}^{\circ} + \frac{1}{2} \left[\left(\mathbf{e}_{11}^{\circ} \csc \theta_{12} \right)^2 + \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} + \varphi \right)^2 \right. \\ \left. - 2 \mathbf{e}_{11}^{\circ} \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} + \varphi \right) \cot \theta_{12} + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)^2 \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\varepsilon}_{22}^{\circ} = \mathbf{e}_{22}^{\circ} + \frac{1}{2} \left[\left(\mathbf{e}_{22}^{\circ} \csc \theta_{12} \right)^2 + \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \varphi \right)^2 \right. \\ \left. - 2 \mathbf{e}_{22}^{\circ} \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \varphi \right) \cot \theta_{12} + \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right)^2 \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\begin{aligned} 2\boldsymbol{\varepsilon}_{12}^{\circ} = 2\mathbf{e}_{12}^{\circ} + \left[\mathbf{e}_{11}^{\circ} \left(\mathbf{e}_{12}^{\circ} - \varphi \sin \theta_{12} \right) + \mathbf{e}_{22}^{\circ} \left(\mathbf{e}_{12}^{\circ} + \varphi \sin \theta_{12} \right) \right] \csc^2 \theta_{12} \\ - \left[\mathbf{e}_{11}^{\circ} \mathbf{e}_{22}^{\circ} + \left(\mathbf{e}_{12}^{\circ} + \varphi \sin \theta_{12} \right) \left(\mathbf{e}_{12}^{\circ} - \varphi \sin \theta_{12} \right) \right] \csc \theta_{12} \cot \theta_{12} \\ + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) + \mathcal{O}(\varepsilon^4) \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

- The reference-surface bending strains are given by

$$K_{11}^{\circ} = \frac{1}{\bar{r}_{11}} - \frac{1 + 3\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}}{r_{11}} + \mathcal{O}(\varepsilon^4)$$

$$K_{22}^{\circ} = \frac{1}{\bar{r}_{22}} - \frac{1 + \varepsilon_{11}^{\circ} + 3\varepsilon_{22}^{\circ}}{r_{22}} + \mathcal{O}(\varepsilon^4)$$

$$2K_{12}^{\circ} = - \left[\frac{1}{\bar{r}_{12}} - \frac{1}{\bar{r}_{21}} - \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) \right] + 3 \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) (\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}) \\ + \left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) (\varepsilon_{22}^{\circ} - \varepsilon_{11}^{\circ}) \cot \theta_{12} + \mathcal{O}(\varepsilon^4)$$

where

$$\frac{1}{\bar{r}_{11}} = (1 + e_{11}^{\circ}) \mathbf{m}^{(1)} \Big|_{(1)} + (\cos \theta_{12} + e_{12}^{\circ} + \varphi \sin \theta_{12}) \mathbf{m}^{(2)} \Big|_{(1)} - (\varphi_1 + \varphi_2 \cos \theta_{12}) \mathbf{m}^{(3)} \Big|_{(1)}$$

$$\frac{1}{\bar{r}_{22}} = (\cos \theta_{12} + e_{12}^{\circ} - \varphi \sin \theta_{12}) \mathbf{m}^{(1)} \Big|_{(2)} + (1 + e_{22}^{\circ}) \mathbf{m}^{(2)} \Big|_{(2)} - (\varphi_1 \cos \theta_{12} + \varphi_2) \mathbf{m}^{(3)} \Big|_{(2)}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned} \frac{1}{\bar{r}_{12}} = & \left(1 + e_{11}^{\circ} \csc^2 \theta_{12} - (e_{12}^{\circ} \csc \theta_{12} + \varphi) \cot \theta_{12} \right) \left[m_3 m^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} - m_2 m^{\langle 3 \rangle} \Big|_{\langle 1 \rangle} \right] \sin \theta_{12} \\ & + \left(e_{12}^{\circ} \csc \theta_{12} + \varphi - e_{11}^{\circ} \cot \theta_{12} \right) \left[m_1 m^{\langle 3 \rangle} \Big|_{\langle 1 \rangle} - m_3 m^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} \right] \\ & + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \left[m_1 m^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} - m_2 m^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} \right] \sin \theta_{12} \end{aligned}$$

$$\begin{aligned} \frac{1}{\bar{r}_{21}} = & \left(1 + e_{22}^{\circ} \csc^2 \theta_{12} - (e_{12}^{\circ} \csc \theta_{12} - \varphi) \cot \theta_{12} \right) \left[m_3 m^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} - m_1 m^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} \right] \sin \theta_{12} \\ & + \left(e_{12}^{\circ} \csc \theta_{12} - \varphi - e_{22}^{\circ} \cot \theta_{12} \right) \left[m_2 m^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} - m_3 m^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} \right] \\ & + \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) \left[m_2 m^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} - m_1 m^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} \right] \sin \theta_{12} \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$m^{(1)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_1}{\partial \xi_1} - \frac{m_2 \csc \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\csc \theta_{12}}{\rho_{11}} \left(m_1 \cos \theta_{12} + m_2 \right) + m_3 \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right)$$

$$m^{(2)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_2}{\partial \xi_1} + \frac{\csc \theta_{12}}{\rho_{11}} \left(m_1 + m_2 \cos \theta_{12} \right) + \frac{m_2 \cot \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - m_3 \frac{\csc \theta_{12}}{r_{12}}$$

$$m^{(3)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_3}{\partial \xi_1} - \frac{m_1}{r_{11}} + m_2 \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right)$$

$$m^{(1)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_1}{\partial \xi_2} + \frac{m_1 \cot \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} - \frac{\csc \theta_{12}}{\rho_{22}} \left(m_1 \cos \theta_{12} + m_2 \right) + m_3 \frac{\csc \theta_{12}}{r_{21}}$$

$$m^{(2)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_2}{\partial \xi_2} - \frac{m_1 \csc \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} + \frac{\csc \theta_{12}}{\rho_{22}} \left(m_1 + m_2 \cos \theta_{12} \right) + m_3 \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right)$$

$$m^{(3)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_3}{\partial \xi_2} - m_1 \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) - \frac{m_2}{r_{22}}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$m_1 = \varphi_1 - \left(e_{12}^{\circ} \csc \theta_{12} + \varphi \right) \left(\varphi_1 \cot \theta_{12} + \varphi_2 \csc \theta_{12} \right) \\ + e_{22}^{\circ} \csc \theta_{12} \left(\varphi_1 \csc \theta_{12} + \varphi_2 \cot \theta_{12} \right)$$

$$m_2 = \varphi_2 - \left(e_{12}^{\circ} \csc \theta_{12} - \varphi \right) \left(\varphi_1 \csc \theta_{12} + \varphi_2 \cot \theta_{12} \right) \\ + e_{11}^{\circ} \csc \theta_{12} \left(\varphi_1 \cot \theta_{12} + \varphi_2 \csc \theta_{12} \right)$$

$$m_3 = 1 + \varphi^2 + \left(e_{11}^{\circ} + e_{22}^{\circ} + e_{11}^{\circ} e_{22}^{\circ} - \left(e_{12}^{\circ} \right)^2 \right) \csc^2 \theta_{12} \\ - 2 e_{12}^{\circ} \cot \theta_{12} \csc \theta_{12}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\Delta_{11} = \mathbf{e}_{11}^{\circ} \csc^2 \theta_{12} - \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} + \varphi \right) \cot \theta_{12}$$

$$\Delta_{12} = \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} + \varphi - \mathbf{e}_{11}^{\circ} \cot \theta_{12} \right) \csc \theta_{12}$$

$$\Delta_{13} = - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)$$

$$\Delta_{21} = \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \varphi - \mathbf{e}_{22}^{\circ} \cot \theta_{12} \right) \csc \theta_{12}$$

$$\Delta_{22} = \mathbf{e}_{22}^{\circ} \csc^2 \theta_{12} - \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \varphi \right) \cot \theta_{12}$$

$$\Delta_{23} = - \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right)$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\varphi_1 = \left(\frac{u_1}{r_{11}} - \frac{1}{A_1} \frac{\partial w}{\partial \xi_1} \right) \csc^2 \theta_{12} - u_1 \left(\frac{1}{r_{21}} + \frac{\cot \theta_{12}}{r_{22}} \right) \cot \theta_{12} + \left(\frac{\cot \theta_{12}}{A_2} \frac{\partial w}{\partial \xi_2} + \frac{u_2}{r_{21}} \right) \csc \theta_{12}$$

$$\varphi_2 = \left(\frac{u_2}{r_{22}} - \frac{1}{A_2} \frac{\partial w}{\partial \xi_2} \right) \csc^2 \theta_{12} + u_2 \left(\frac{1}{r_{12}} - \frac{\cot \theta_{12}}{r_{11}} \right) \cot \theta_{12} + \left(\frac{\cot \theta_{12}}{A_1} \frac{\partial w}{\partial \xi_1} - \frac{u_1}{r_{12}} \right) \csc \theta_{12}$$

$$2\varphi = \left(\frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} \right) \cot \theta_{12} + \left(\frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} \right) \csc \theta_{12} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}}$$

$$\mathbf{e}_{11}^{\circ} = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (u_1 + u_2 \cos \theta_{12}) - \frac{u_2 \sin \theta_{12}}{\rho_{11}} + \frac{w}{r_{11}}$$

$$\mathbf{e}_{22}^{\circ} = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (u_2 + u_1 \cos \theta_{12}) + \frac{u_1 \sin \theta_{12}}{\rho_{22}} + \frac{w}{r_{22}}$$

$$2\mathbf{e}_{12}^{\circ} = \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} + \left(\frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} + \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} \right) \cos \theta_{12} \\ + \left(\frac{u_1}{\rho_{11}} - \frac{u_2}{\rho_{22}} \right) \sin \theta_{12} + w \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) \cos \theta_{12} + w \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \sin \theta_{12}$$

EQUATIONS FOR GENERAL COORDINATES CONTINUED

$$\frac{1}{\rho_{11}} = \frac{\csc\theta_{12}}{A_1 A_2} \left(\frac{\partial}{\partial \xi_1} [A_2 \cos\theta_{12}] - \frac{\partial A_1}{\partial \xi_2} \right)$$

$$\frac{1}{\rho_{22}} = -\frac{\csc\theta_{12}}{A_1 A_2} \left(\frac{\partial}{\partial \xi_2} [A_1 \cos\theta_{12}] - \frac{\partial A_2}{\partial \xi_1} \right)$$

- The displacement fields for points of the shell are given by

$$\vec{U} = U_1 \hat{g}_1 + U_2 \hat{g}_2 + U_3 \hat{g}_3 \quad \text{with}$$

$$U_1 = \frac{\mu_{22} U_1^0 - \mu_{21} U_2^0}{\mu_{11} \mu_{22} - \mu_{12} \mu_{21}}, \quad U_2 = \frac{\mu_{11} U_2^0 - \mu_{12} U_1^0}{\mu_{11} \mu_{22} - \mu_{12} \mu_{21}}, \quad \text{and} \quad U_3 = U_3^0; \quad \text{and where}$$

$$\mu_{11} = \frac{1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{12}}\right)^2}}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\mu_{12} = \frac{-\frac{\xi_3 \csc\theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{12}}\right)^2}}$$

$$\mu_{21} = \frac{\frac{\xi_3 \csc\theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}}$$

$$\mu_{22} = \frac{1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned} \mathbf{U}_1^0 = \mathbf{u}_1 + \xi_3 \mathbf{m}_1 & \left(1 + 2\varepsilon_{12}^0 \cot \theta_{12} \csc \theta_{12} - \left(\varepsilon_{11}^0 + \varepsilon_{22}^0 \right) \csc^2 \theta_{12} + \mathcal{O}(\varepsilon^4) \right) \\ & + \left(1 + \Delta_{11} \right) \left(\mathbf{F}_1(\xi_3) \boldsymbol{\gamma}_1^0 + \mathcal{O}(\varepsilon^4) \right) + \Delta_{21} \left(\mathbf{F}_2(\xi_3) \boldsymbol{\gamma}_2^0 + \mathcal{O}(\varepsilon^4) \right) \end{aligned}$$

$$\begin{aligned} \mathbf{U}_2^0 = \mathbf{u}_2 + \xi_3 \mathbf{m}_2 & \left(1 + 2\varepsilon_{12}^0 \cot \theta_{12} \csc \theta_{12} - \left(\varepsilon_{11}^0 + \varepsilon_{22}^0 \right) \csc^2 \theta_{12} + \mathcal{O}(\varepsilon^4) \right) \\ & + \Delta_{12} \left(\mathbf{F}_1(\xi_3) \boldsymbol{\gamma}_1^0 + \mathcal{O}(\varepsilon^4) \right) + \left(1 + \Delta_{22} \right) \left(\mathbf{F}_2(\xi_3) \boldsymbol{\gamma}_2^0 + \mathcal{O}(\varepsilon^4) \right) \end{aligned}$$

$$\begin{aligned} \mathbf{U}_3^0 = \mathbf{w} + \xi_3 & \left[\mathbf{m}_3 \left(1 + 2\varepsilon_{12}^0 \cot \theta_{12} \csc \theta_{12} - \left(\varepsilon_{11}^0 + \varepsilon_{22}^0 \right) \csc^2 \theta_{12} + \mathcal{O}(\varepsilon^4) \right) - \mathbf{1} \right] \\ & + \Delta_{13} \left(\mathbf{F}_1(\xi_3) \boldsymbol{\gamma}_1^0 + \mathcal{O}(\varepsilon^4) \right) + \Delta_{23} \left(\mathbf{F}_2(\xi_3) \boldsymbol{\gamma}_2^0 + \mathcal{O}(\varepsilon^4) \right) \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

- The compatibility equations are given by

$$\mathcal{C}_{11}(\epsilon_{11}^{\circ}) + \mathcal{C}_{12}(\epsilon_{22}^{\circ}) + \mathcal{C}_{13}(\epsilon_{12}^{\circ}) + \mathcal{C}_{14}(K_{11}^{\circ}) + \mathcal{C}_{15}(K_{22}^{\circ}) + \mathcal{C}_{16}(K_{12}^{\circ}) = 0$$

$$\mathcal{C}_{21}(\epsilon_{11}^{\circ}) + \mathcal{C}_{22}(\epsilon_{22}^{\circ}) + \mathcal{C}_{23}(\epsilon_{12}^{\circ}) + \mathcal{C}_{24}(K_{11}^{\circ}) + \mathcal{C}_{25}(K_{22}^{\circ}) + \mathcal{C}_{26}(K_{12}^{\circ}) = 0$$

$$\mathcal{C}_{31}(\epsilon_{11}^{\circ}) + \mathcal{C}_{32}(\epsilon_{22}^{\circ}) + \mathcal{C}_{33}(\epsilon_{12}^{\circ}) + \mathcal{C}_{34}(K_{11}^{\circ}) + \mathcal{C}_{35}(K_{22}^{\circ}) + \mathcal{C}_{36}(K_{12}^{\circ}) = 0$$

where

$$\begin{aligned} \mathcal{C}_{11}(\epsilon_{11}^{\circ}) = & \frac{\partial}{\partial \xi_1} \left(\frac{A_2}{\rho_{22}} \csc^2 \theta_{12} \epsilon_{11}^{\circ} \right) - \frac{\partial}{\partial \xi_2} \left(\frac{A_1}{\rho_{11}} \cot^2 \theta_{12} \epsilon_{11}^{\circ} \right) \\ & - \frac{\partial}{\partial \xi_2} \left(\frac{\csc \theta_{12}}{A_2} \left[\frac{\partial}{\partial \xi_1} (A_2 \cos \theta_{12} \epsilon_{11}^{\circ}) + \frac{\partial}{\partial \xi_2} (A_1 \epsilon_{11}^{\circ}) \right] \right) + \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} (\cot \theta_{12} \epsilon_{11}^{\circ}) \\ & - \left(\frac{1}{r_{11} r_{22}} + \frac{1}{r_{12} r_{21}} \right) A_1 A_2 \csc \theta_{12} \epsilon_{11}^{\circ} + \mathcal{O}(\epsilon^4) \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned}
 \mathcal{C}_{12}(\epsilon_{22}^{\circ}) &= \frac{\partial}{\partial \xi_1} \left(\frac{A_2}{\rho_{22}} \cot^2 \theta_{12} \epsilon_{22}^{\circ} \right) - \frac{\partial}{\partial \xi_2} \left(\frac{A_1}{\rho_{11}} \csc^2 \theta_{12} \epsilon_{22}^{\circ} \right) \\
 &\quad - \frac{\partial}{\partial \xi_1} \left(\frac{\csc \theta_{12}}{A_1} \left[\frac{\partial}{\partial \xi_2} (A_1 \cos \theta_{12} \epsilon_{22}^{\circ}) + \frac{\partial}{\partial \xi_1} (A_2 \epsilon_{22}^{\circ}) \right] \right) + \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} (\cot \theta_{12} \epsilon_{22}^{\circ}) \\
 &\quad - \left(\frac{1}{r_{11} r_{22}} + \frac{1}{r_{12} r_{21}} \right) A_1 A_2 \csc \theta_{12} \epsilon_{22}^{\circ} + \mathcal{O}(\epsilon^4)
 \end{aligned}$$

$$\mathcal{C}_{14}(K_{11}^{\circ}) = -A_1 A_2 \left(\frac{\sin \theta_{12}}{r_{22}} - \frac{\cos \theta_{12}}{r_{21}} \right) K_{11}^{\circ} + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{15}(K_{22}^{\circ}) = -A_1 A_2 \left(\frac{\sin \theta_{12}}{r_{11}} + \frac{\cos \theta_{12}}{r_{12}} \right) K_{22}^{\circ} + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{16}(K_{12}^{\circ}) = -A_1 A_2 \left[\left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) \sin \theta_{12} - \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) \cos \theta_{12} \right] K_{12}^{\circ} + \mathcal{O}(\epsilon^4)$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned}
 \mathcal{C}_{13}(\boldsymbol{\varepsilon}_{12}^{\circ}) = & \frac{\partial}{\partial \xi_1} \left(\frac{\csc \theta_{12}}{\mathbf{A}_1} \frac{\partial}{\partial \xi_2} (\mathbf{A}_1 \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ}) \right) - \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\rho_{22}} \cot \theta_{12} \csc \theta_{12} \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ} \right) \\
 & + \frac{\partial}{\partial \xi_2} \left(\frac{\csc \theta_{12}}{\mathbf{A}_2} \frac{\partial}{\partial \xi_1} (\mathbf{A}_2 \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ}) \right) + \frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\rho_{11}} \cot \theta_{12} \csc \theta_{12} \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ} \right) \\
 & - \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} (\csc \theta_{12} \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ}) + \mathbf{A}_1 \mathbf{A}_2 (1 - \cos \theta_{12}) \cot^2 \theta_{12} \left(\frac{1}{\mathbf{r}_{12} \mathbf{r}_{22}} - \frac{1}{\mathbf{r}_{21} \mathbf{r}_{11}} \right) \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ} \\
 & + \mathbf{A}_1 \mathbf{A}_2 \cot \theta_{12} \left(\frac{1}{\mathbf{r}_{11} \mathbf{r}_{22}} + \frac{1}{\mathbf{r}_{12} \mathbf{r}_{21}} \right) \mathbf{2}\boldsymbol{\varepsilon}_{12}^{\circ} + \mathcal{O}(\boldsymbol{\varepsilon}^4)
 \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned}
 \mathcal{C}_{21}(\epsilon_{11}^{\circ}) = & - \left[\frac{\csc\theta_{12}}{2} \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \right] \frac{1}{A_1} \frac{\partial \epsilon_{11}^{\circ}}{\partial \xi_1} - \left[\left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc^2\theta_{12} - \frac{2\cot\theta_{12}}{r_{12}} \right] \frac{1}{A_2} \frac{\partial \epsilon_{11}^{\circ}}{\partial \xi_2} \\
 & - \csc\theta_{12} \left[\frac{1}{2A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) + \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{22}} \right) \sin 2\theta_{12} - \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{21}} \right) \cos 2\theta_{12} \right] \epsilon_{11}^{\circ} \\
 & + \csc\theta_{12} \left[\left(-\frac{1}{\rho_{22}} + \frac{1}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \csc\theta_{12} + \frac{1}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \cot\theta_{12} \right] \epsilon_{11}^{\circ} + \mathcal{O}(\epsilon^4)
 \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned}
 \mathcal{C}_{22}(\epsilon_{22}^{\circ}) = & \frac{\csc\theta_{12}}{2} \left[\frac{1}{r_{21}} - \frac{3}{r_{12}} \right] \frac{1}{A_1} \frac{\partial \epsilon_{22}^{\circ}}{\partial \xi_1} \\
 & + \frac{\csc\theta_{12}}{2} \left[\frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{21}} \right) \cos 2\theta_{12} - \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{12}} \right) \right] \epsilon_{22}^{\circ} \\
 & + \left[\left(\frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{11}} \right) - \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{22}} \right) \cos \theta_{12} \right) - \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \frac{\cot \theta_{12} \csc \theta_{12}}{\rho_{11}} \right] \epsilon_{22}^{\circ} \\
 & - \csc \theta_{12} \left(\frac{1}{\rho_{11}} + \frac{1}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) \left[\left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc^2 \theta_{12} - \frac{4 \cot \theta_{12}}{r_{12}} \right] \epsilon_{22}^{\circ} \\
 & - \cot \theta_{12} \left(-\frac{1}{\rho_{22}} + \frac{1}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \left[\left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc^2 \theta_{12} - \frac{2 \cot \theta_{12}}{r_{12}} \right] \epsilon_{22}^{\circ} + \mathcal{O}(\epsilon^4)
 \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned}
 \mathcal{C}_{23}(\epsilon_{12}^{\circ}) = & \left[\frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc^2 \theta_{12} + \frac{2}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{22}} \right) - 2 \cot \theta_{12} \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{21}} \right) \right] \epsilon_{12}^{\circ} \\
 & + \csc^2 \theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \frac{1}{A_1} \frac{\partial \epsilon_{12}^{\circ}}{\partial \xi_1} \\
 & - \csc^2 \theta_{12} \left[\left(\frac{1}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) \left(\frac{1}{r_{21}} + \frac{5}{r_{12}} \right) + \frac{2}{\rho_{11}} \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) \right] \epsilon_{12}^{\circ} + \mathcal{O}(\epsilon^4)
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{C}_{24}(K_{11}^{\circ}) = & \frac{\cos \theta_{12}}{2} \frac{1}{A_1} \frac{\partial K_{11}^{\circ}}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial K_{11}^{\circ}}{\partial \xi_2} \\
 & + \left[\csc \theta_{12} \left(\frac{1}{\rho_{11}} + \frac{1}{2} \frac{1}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) + \cot \theta_{12} \left(-\frac{1}{\rho_{22}} + \frac{1}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \right] K_{11}^{\circ} + \mathcal{O}(\epsilon^4)
 \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned} \mathcal{C}_{25}(\mathbf{K}_{22}^{\circ}) &= \frac{\cos\theta_{12}}{2} \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{K}_{22}^{\circ}}{\partial \xi_1} \\ &\quad - \left[\csc\theta_{12} \left(\frac{1}{\rho_{11}} + \frac{1}{2} \frac{1}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) + \cot\theta_{12} \left(-\frac{1}{\rho_{22}} + \frac{1}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \right] \mathbf{K}_{22}^{\circ} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\mathcal{C}_{26}(\mathbf{K}_{12}^{\circ}) = \sin\theta_{12} \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{K}_{12}^{\circ}}{\partial \xi_1} - 2 \left(-\frac{1}{\rho_{22}} + \frac{1}{\mathbf{A}_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \mathbf{K}_{12}^{\circ} + \mathcal{O}(\varepsilon^4)$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned}
 \mathcal{C}_{31}(\epsilon_{11}^{\circ}) = & \frac{\csc\theta_{12}}{2} \left[\frac{3}{r_{21}} - \frac{1}{r_{12}} \right] \frac{1}{A_2} \frac{\partial \epsilon_{11}^{\circ}}{\partial \xi_2} \\
 & + \frac{\csc\theta_{12}}{2} \left[\frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{21}} \right) - \frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{12}} \right) \cos 2\theta_{12} \right] \epsilon_{11}^{\circ} \\
 & + \left[\frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{22}} \right) - \frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{11}} \right) \cos \theta_{12} - \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \frac{\cot \theta_{12} \csc \theta_{12}}{\rho_{22}} \right] \epsilon_{11}^{\circ} \\
 & - \cot \theta_{12} \left(\frac{1}{\rho_{11}} + \frac{1}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) \left[\left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) \csc^2 \theta_{12} + \frac{2 \cot \theta_{12}}{r_{21}} \right] \epsilon_{11}^{\circ} \\
 & - \csc \theta_{12} \left(-\frac{1}{\rho_{22}} + \frac{1}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \left[\left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) \csc^2 \theta_{12} + \frac{4 \cot \theta_{12}}{r_{21}} \right] \epsilon_{11}^{\circ} + \mathcal{O}(\epsilon^4)
 \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned}
 \mathcal{C}_{32}(\epsilon_{22}^{\circ}) = & - \left[\left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) \csc^2 \theta_{12} + \frac{2 \cot \theta_{12}}{r_{21}} \right] \frac{1}{A_1} \frac{\partial \epsilon_{22}^{\circ}}{\partial \xi_1} + \left[\frac{\csc \theta_{12}}{2} \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \right] \frac{1}{A_2} \frac{\partial \epsilon_{22}^{\circ}}{\partial \xi_2} \\
 & - \csc \theta_{12} \left[\frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{11}} \right) \sin 2\theta_{12} + \frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{12}} \right) \cos 2\theta_{12} - \frac{1}{2A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \right] \epsilon_{22}^{\circ} \\
 & + \csc \theta_{12} \left[\left(\frac{1}{\rho_{11}} + \frac{1}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \csc \theta_{12} - \frac{1}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} \left(\frac{1}{r_{12}} + \frac{1}{r_{21}} \right) \cot \theta_{12} \right] \epsilon_{22}^{\circ} + \mathcal{O}(\epsilon^4)
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{C}_{34}(\kappa_{11}^{\circ}) = & \frac{\cos \theta_{12}}{2} \frac{1}{A_2} \frac{\partial \kappa_{11}^{\circ}}{\partial \xi_2} \\
 & - \left[\cot \theta_{12} \left(\frac{1}{\rho_{11}} + \frac{1}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) + \csc \theta_{12} \left(-\frac{1}{\rho_{22}} + \frac{1}{2} \frac{1}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \right] \kappa_{11}^{\circ} + \mathcal{O}(\epsilon^4)
 \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned}
 \mathcal{C}_{33}(\epsilon_{12}^{\circ}) = & \left[\frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) \csc^2 \theta_{12} + \frac{2}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{11}} \right) + 2 \cot \theta_{12} \frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{12}} \right) \right] \epsilon_{12}^{\circ} \\
 & + \csc^2 \theta_{12} \left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) \frac{1}{A_2} \frac{\partial \epsilon_{12}^{\circ}}{\partial \xi_2} \\
 & + \csc^2 \theta_{12} \left[\left(\frac{1}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \left(\frac{1}{r_{12}} + \frac{5}{r_{12}} \right) + \frac{2}{\rho_{22}} \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) \right] \epsilon_{12}^{\circ} + \mathcal{O}(\epsilon^4)
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{C}_{35}(\mathbf{K}_{22}^{\circ}) = & - \frac{1}{A_1} \frac{\partial \mathbf{K}_{22}^{\circ}}{\partial \xi_1} + \frac{\cos \theta_{12}}{2} \frac{1}{A_2} \frac{\partial \mathbf{K}_{22}^{\circ}}{\partial \xi_2} \\
 & + \left[\cot \theta_{12} \left(\frac{1}{\rho_{11}} + \frac{1}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) + \csc \theta_{12} \left(- \frac{1}{\rho_{22}} + \frac{1}{2} \frac{1}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} \right) \right] \mathbf{K}_{22}^{\circ} + \mathcal{O}(\epsilon^4)
 \end{aligned}$$

EQUATIONS FOR GENERAL COORDINATES

CONCLUDED

$$\mathcal{C}_{36}(\mathbf{K}_{12}^{\circ}) = \sin\theta_{12} \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{K}_{12}^{\circ}}{\partial \xi_2} - 2 \left(\frac{1}{\rho_{11}} + \frac{1}{\mathbf{A}_1} \frac{\partial \theta_{12}}{\partial \xi_1} \right) \mathbf{K}_{12}^{\circ} + \mathcal{O}(\varepsilon^4)$$

**RESUME' OF EQUATIONS FOR
“SMALL” STRAINS AND FINITE ROTATIONS -
ORTHOGONAL GAUSSIAN COORDINATES**

EQUATIONS FOR ORTHOGONAL COORDINATES

- For **orthogonal reference-surface Gaussian coordinates**, the Green-Lagrange shell strains $\{\epsilon_{11}, \epsilon_{22}, 2\epsilon_{12}, 2\epsilon_{13}, 2\epsilon_{23}\}$ are given by

$$\epsilon_{11} \left(\frac{\mathbf{H}_1}{\mathbf{A}_1} \right)^2 = \epsilon_{11}^o \left[\left(1 + \frac{\xi_3}{\mathbf{r}_{11}} \right)^2 + \left(\frac{\xi_3}{\mathbf{r}_{12}} \right)^2 \right] + \xi_3 K_{11}^o \left(1 + \frac{\xi_3}{\mathbf{r}_{11}} \right) + \epsilon_{12}^o \left(\frac{\xi_3}{\mathbf{r}_{12}} \right) \left(\frac{\xi_3}{\mathbf{r}_{11}} - \frac{\xi_3}{\mathbf{r}_{22}} \right) - \xi_3 K_{12}^o \left(\frac{\xi_3}{\mathbf{r}_{12}} \right) + \Gamma_{11} + \mathcal{O}(\epsilon^4)$$

$$\epsilon_{22} \left(\frac{\mathbf{H}_2}{\mathbf{A}_2} \right)^2 = \epsilon_{22}^o \left[\left(1 + \frac{\xi_3}{\mathbf{r}_{22}} \right)^2 + \left(\frac{\xi_3}{\mathbf{r}_{12}} \right)^2 \right] + \xi_3 K_{22}^o \left(1 + \frac{\xi_3}{\mathbf{r}_{22}} \right) - \epsilon_{12}^o \left(\frac{\xi_3}{\mathbf{r}_{12}} \right) \left(\frac{\xi_3}{\mathbf{r}_{11}} - \frac{\xi_3}{\mathbf{r}_{22}} \right) - \xi_3 K_{12}^o \left(\frac{\xi_3}{\mathbf{r}_{12}} \right) + \Gamma_{22} + \mathcal{O}(\epsilon^4)$$

EQUATIONS FOR ORTHOGONAL COORDINATES

CONTINUED

$$2\varepsilon_{12} \frac{H_1 H_2}{A_1 A_2} = \varepsilon_{12}^o \left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(1 + \frac{\xi_3}{r_{22}} \right)^2 - 2 \left(\frac{\xi_3}{r_{12}} \right)^2 \right] - \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \left(\frac{\xi_3}{r_{12}} \right) \left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) - \xi_3 \left(K_{11}^o + K_{22}^o \right) \left(\frac{\xi_3}{r_{12}} \right) + \xi_3 K_{12}^o \left[2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right] + 2\Gamma_{12} + \mathcal{O}(\varepsilon^4)$$

$$2\varepsilon_{13} \frac{H_1}{A_1} = \left[F_1'(\xi_3) \left(1 + \frac{\xi_3}{r_{11}} \right) - \frac{F_1(\xi_3)}{r_{11}} \right] 2\varepsilon_{13}^o - \left[\frac{\xi_3 F_2'(\xi_3) - F_2(\xi_3)}{r_{12}} \right] 2\varepsilon_{23}^o + \mathcal{O}(\varepsilon^4)$$

$$2\varepsilon_{23} \frac{H_2}{A_2} = \left[F_2'(\xi_3) \left(1 + \frac{\xi_3}{r_{22}} \right) - \frac{F_2(\xi_3)}{r_{22}} \right] 2\varepsilon_{23}^o - \left[\frac{\xi_3 F_1'(\xi_3) - F_1(\xi_3)}{r_{12}} \right] 2\varepsilon_{13}^o + \mathcal{O}(\varepsilon^4) \quad \text{with}$$

$$\left(\frac{H_1}{A_1} \right)^2 = \left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2$$

$$\left(\frac{H_2}{A_2} \right)^2 = \left(1 + \frac{\xi_3}{r_{22}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2$$

$$\frac{H_1 H_2}{A_1 A_2} = \left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right]^{\frac{1}{2}} \left[\left(1 + \frac{\xi_3}{r_{22}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right]^{\frac{1}{2}}$$

EQUATIONS FOR ORTHOGONAL COORDINATES

CONTINUED

where

$$\Gamma_{11} = \left(1 + \frac{\xi_3}{r_{11}}\right) \left[F_1(\xi_3) \frac{1}{A_1} \frac{\partial 2\epsilon_{13}^o}{\partial \xi_1} - F_2(\xi_3) \frac{2\epsilon_{23}^o}{\rho_{11}} \right] - \frac{\xi_3}{r_{12}} \left[F_1(\xi_3) \frac{2\epsilon_{13}^o}{\rho_{11}} + F_2(\xi_3) \frac{1}{A_1} \frac{\partial 2\epsilon_{23}^o}{\partial \xi_1} \right] + \mathcal{O}(\epsilon^4)$$

$$\Gamma_{22} = \left(1 + \frac{\xi_3}{r_{22}}\right) \left[F_1(\xi_3) \frac{2\epsilon_{13}^o}{\rho_{22}} + F_2(\xi_3) \frac{1}{A_2} \frac{\partial 2\epsilon_{23}^o}{\partial \xi_2} \right] - \frac{\xi_3}{r_{12}} \left[F_1(\xi_3) \frac{1}{A_2} \frac{\partial 2\epsilon_{13}^o}{\partial \xi_2} - F_2(\xi_3) \frac{2\epsilon_{23}^o}{\rho_{22}} \right] + \mathcal{O}(\epsilon^4)$$

EQUATIONS FOR ORTHOGONAL COORDINATES

CONTINUED

$$\begin{aligned}
 2\Gamma_{12} = & \left(1 + \frac{\xi_3}{r_{11}}\right) \left[F_1(\xi_3) \frac{1}{A_2} \frac{\partial 2\varepsilon_{13}^o}{\partial \xi_2} - F_2(\xi_3) \frac{2\varepsilon_{23}^o}{\rho_{22}} \right] \\
 & + \left(1 + \frac{\xi_3}{r_{22}}\right) \left[F_2(\xi_3) \frac{1}{A_1} \frac{\partial 2\varepsilon_{23}^o}{\partial \xi_1} + F_1(\xi_3) \frac{2\varepsilon_{13}^o}{\rho_{11}} \right] \\
 & - \frac{\xi_3}{r_{12}} \left[F_1(\xi_3) \left(\frac{1}{A_1} \frac{\partial 2\varepsilon_{13}^o}{\partial \xi_1} + \frac{2\varepsilon_{13}^o}{\rho_{22}} \right) F_2(\xi_3) \left(\frac{1}{A_2} \frac{\partial 2\varepsilon_{23}^o}{\partial \xi_2} - \frac{2\varepsilon_{23}^o}{\rho_{11}} \right) \right] + \mathcal{O}(\varepsilon^4)
 \end{aligned}$$

and where it is noted that $2\varepsilon_{13}^o$ and $2\varepsilon_{23}^o$ are fundamental unknowns

EQUATIONS FOR ORTHOGONAL COORDINATES

CONTINUED

- The reference-surface membrane strains are given by

$$\epsilon_{11}^o = e_{11}^o + \frac{1}{2} \left[(e_{11}^o)^2 + (e_{12}^o + \varphi)^2 + \varphi_1^2 \right] + \mathcal{O}(\epsilon^4)$$

$$\epsilon_{22}^o = e_{22}^o + \frac{1}{2} \left[(e_{12}^o - \varphi)^2 + (e_{22}^o)^2 + \varphi_2^2 \right] + \mathcal{O}(\epsilon^4)$$

$$2\epsilon_{12}^o = 2e_{12}^o + e_{11}^o(e_{12}^o - \varphi) + e_{22}^o(e_{12}^o + \varphi) + \varphi_1\varphi_2 + \mathcal{O}(\epsilon^4)$$

- The reference-surface bending strains are given by

$$K_{11}^o = \frac{1}{\bar{r}_{11}} - \frac{1 + 3\epsilon_{11}^o + \epsilon_{22}^o}{r_{11}} + \mathcal{O}(\epsilon^4)$$

$$K_{22}^o = \frac{1}{\bar{r}_{22}} - \frac{1 + \epsilon_{11}^o + 3\epsilon_{22}^o}{r_{22}} + \mathcal{O}(\epsilon^4)$$

$$2K_{12}^o = \frac{1}{\bar{r}_{21}} - \frac{1}{\bar{r}_{12}} + 2 \frac{1 + 3(\epsilon_{11}^o + \epsilon_{22}^o)}{r_{12}} + \mathcal{O}(\epsilon^4)$$

EQUATIONS FOR ORTHOGONAL COORDINATES

CONTINUED

where

$$\frac{1}{\bar{\eta}_{11}} = \left(1 + \mathbf{e}_{11}^{\circ}\right) m^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} + \left(\mathbf{e}_{12}^{\circ} + \varphi\right) m^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} - \varphi_1 m^{\langle 3 \rangle} \Big|_{\langle 1 \rangle}$$

$$\frac{1}{\bar{\eta}_{22}} = \left(\mathbf{e}_{12}^{\circ} - \varphi\right) m^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} + \left(1 + \mathbf{e}_{22}^{\circ}\right) m^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} - \varphi_2 m^{\langle 3 \rangle} \Big|_{\langle 2 \rangle}$$

$$\begin{aligned} \frac{1}{\bar{\eta}_{12}} = & \left(1 + \mathbf{e}_{11}^{\circ}\right) \left[m_3 m^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} - m_2 m^{\langle 3 \rangle} \Big|_{\langle 1 \rangle} \right] \\ & + \left(\mathbf{e}_{12}^{\circ} + \varphi\right) \left[m_1 m^{\langle 3 \rangle} \Big|_{\langle 1 \rangle} - m_3 m^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} \right] + \varphi_1 \left[m_1 m^{\langle 2 \rangle} \Big|_{\langle 1 \rangle} - m_2 m^{\langle 1 \rangle} \Big|_{\langle 1 \rangle} \right] \end{aligned}$$

$$\begin{aligned} \frac{1}{\bar{\eta}_{21}} = & \left(1 + \mathbf{e}_{22}^{\circ}\right) \left[m_3 m^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} - m_1 m^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} \right] \\ & + \left(\mathbf{e}_{12}^{\circ} - \varphi\right) \left[m_2 m^{\langle 3 \rangle} \Big|_{\langle 2 \rangle} - m_3 m^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} \right] + \varphi_2 \left[m_2 m^{\langle 1 \rangle} \Big|_{\langle 2 \rangle} - m_1 m^{\langle 2 \rangle} \Big|_{\langle 2 \rangle} \right] \end{aligned}$$

EQUATIONS FOR ORTHOGONAL COORDINATES

CONTINUED

and where

$$m_1 = \varphi_1(1 + \mathbf{e}_{22}^o) - \varphi_2(\mathbf{e}_{12}^o + \varphi)$$

$$m_2 = \varphi_2(1 + \mathbf{e}_{11}^o) - \varphi_1(\mathbf{e}_{12}^o - \varphi)$$

$$m_3 = (1 + \mathbf{e}_{11}^o)(1 + \mathbf{e}_{22}^o) - (\mathbf{e}_{12}^o + \varphi)(\mathbf{e}_{12}^o - \varphi)$$

$$m^{(1)} \Big|_{\langle 1 \rangle} = \frac{1}{A_1} \frac{\partial m_1}{\partial \xi_1} - \frac{m_2}{\rho_{11}} + \frac{m_3}{r_{11}}$$

$$m^{(2)} \Big|_{\langle 1 \rangle} = \frac{1}{A_1} \frac{\partial m_2}{\partial \xi_1} + \frac{m_1}{\rho_{11}} - \frac{m_3}{r_{12}}$$

$$m^{(3)} \Big|_{\langle 1 \rangle} = \frac{1}{A_1} \frac{\partial m_3}{\partial \xi_1} - \frac{m_1}{r_{11}} + \frac{m_2}{r_{12}}$$

$$m^{(1)} \Big|_{\langle 2 \rangle} = \frac{1}{A_2} \frac{\partial m_1}{\partial \xi_2} - \frac{m_2}{\rho_{22}} - \frac{m_3}{r_{12}}$$

$$m^{(2)} \Big|_{\langle 2 \rangle} = \frac{1}{A_2} \frac{\partial m_2}{\partial \xi_2} + \frac{m_1}{\rho_{22}} + \frac{m_3}{r_{22}}$$

$$m^{(3)} \Big|_{\langle 2 \rangle} = \frac{1}{A_2} \frac{\partial m_3}{\partial \xi_2} + \frac{m_1}{r_{12}} - \frac{m_2}{r_{22}}$$

EQUATIONS FOR ORTHOGONAL COORDINATES

CONTINUED

- In addition, the linear deformation measures are given by

$$e_{11}^{\circ} = \frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} - \frac{u_2}{\rho_{11}} + \frac{w}{r_{11}} \quad e_{22}^{\circ} = \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{\rho_{22}} + \frac{w}{r_{22}}$$

$$2e_{12}^{\circ} = \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} + \frac{u_1}{\rho_{11}} - \frac{u_2}{\rho_{22}} - \frac{2w}{r_{12}}$$

$$\varphi_1 = \frac{u_1}{r_{11}} - \frac{u_2}{r_{12}} - \frac{1}{A_1} \frac{\partial w}{\partial \xi_1} \quad \varphi_2 = \frac{u_2}{r_{22}} - \frac{u_1}{r_{12}} - \frac{1}{A_2} \frac{\partial w}{\partial \xi_2}$$

$$\varphi = \frac{1}{2} \left(\frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}} \right)$$

where

$$\frac{1}{\rho_{11}} = - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2} \quad \text{and} \quad \frac{1}{\rho_{22}} = \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \xi_1}$$

EQUATIONS FOR ORTHOGONAL COORDINATES

CONTINUED

- The displacement fields for points of the shell are given by

$$\vec{U} = U_1 \hat{g}_1 + U_2 \hat{g}_2 + U_3 \hat{g}_3 \quad \text{with}$$

$$U_1 = \frac{\mu_{22} U_1^0 - \mu_{21} U_2^0}{\mu_{11} \mu_{22} - \mu_{12} \mu_{21}}, \quad U_2 = \frac{\mu_{11} U_2^0 - \mu_{12} U_1^0}{\mu_{11} \mu_{22} - \mu_{12} \mu_{21}}, \quad \text{and} \quad U_3 = U_3^0; \quad \text{and where}$$

$$\mu_{11} = \frac{1 + \frac{\xi_3}{r_{11}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

$$\mu_{12} = \frac{-\frac{\xi_3}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

$$\mu_{21} = \frac{-\frac{\xi_3}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

$$\mu_{22} = \frac{1 + \frac{\xi_3}{r_{22}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

EQUATIONS FOR ORTHOGONAL COORDINATES

CONTINUED

$$\begin{aligned} \mathbf{U}_1^0 = \mathbf{u}_1 + \xi_3 \mathbf{m}_1 \left(1 - \varepsilon_{11}^o - \varepsilon_{22}^o + \mathcal{O}(\varepsilon^4) \right) + \mathbf{F}(\xi_3) \left(1 + \mathbf{e}_{11}^o \right) \left(2\varepsilon_{13}^o + \mathcal{O}(\varepsilon^4) \right) \\ + \mathbf{F}(\xi_3) \left(\mathbf{e}_{12}^o - \varphi \right) \left(2\varepsilon_{23}^o + \mathcal{O}(\varepsilon^4) \right) \end{aligned}$$

$$\begin{aligned} \mathbf{U}_2^0 = \mathbf{u}_2 + \xi_3 \mathbf{m}_2 \left(1 - \varepsilon_{11}^o - \varepsilon_{22}^o + \mathcal{O}(\varepsilon^4) \right) + \mathbf{F}(\xi_3) \left(\mathbf{e}_{12}^o + \varphi \right) \left(2\varepsilon_{13}^o + \mathcal{O}(\varepsilon^4) \right) \\ + \mathbf{F}(\xi_3) \left(1 + \mathbf{e}_{22}^o \right) \left(2\varepsilon_{23}^o + \mathcal{O}(\varepsilon^4) \right) \end{aligned}$$

$$\begin{aligned} \mathbf{U}_3^0 = \mathbf{w} + \xi_3 \left[\mathbf{m}_3 \left(1 - \varepsilon_{11}^o - \varepsilon_{22}^o + \mathcal{O}(\varepsilon^4) \right) - \mathbf{1} \right] - \mathbf{F}(\xi_3) \varphi_1 \left(2\varepsilon_{13}^o + \mathcal{O}(\varepsilon^4) \right) \\ - \mathbf{F}(\xi_3) \varphi_2 \left(2\varepsilon_{23}^o + \mathcal{O}(\varepsilon^4) \right) \end{aligned}$$

EQUATIONS FOR ORTHOGONAL COORDINATES

CONTINUED

- The compatibility equations are given by

$$\mathcal{C}_{11}(\epsilon_{11}^{\circ}) + \mathcal{C}_{12}(\epsilon_{22}^{\circ}) + \mathcal{C}_{13}(\epsilon_{12}^{\circ}) + \mathcal{C}_{14}(K_{11}^{\circ}) + \mathcal{C}_{15}(K_{22}^{\circ}) + \mathcal{C}_{16}(K_{12}^{\circ}) = 0$$

$$\mathcal{C}_{21}(\epsilon_{11}^{\circ}) + \mathcal{C}_{22}(\epsilon_{22}^{\circ}) + \mathcal{C}_{23}(\epsilon_{12}^{\circ}) + \mathcal{C}_{24}(K_{11}^{\circ}) + \mathcal{C}_{25}(K_{22}^{\circ}) + \mathcal{C}_{26}(K_{12}^{\circ}) = 0$$

$$\mathcal{C}_{31}(\epsilon_{11}^{\circ}) + \mathcal{C}_{32}(\epsilon_{22}^{\circ}) + \mathcal{C}_{33}(\epsilon_{12}^{\circ}) + \mathcal{C}_{34}(K_{11}^{\circ}) + \mathcal{C}_{35}(K_{22}^{\circ}) + \mathcal{C}_{36}(K_{12}^{\circ}) = 0$$

where

$$\begin{aligned} \mathcal{C}_{11}(\epsilon_{11}^{\circ}) = & \frac{\partial}{\partial \xi_1} \left(\frac{\mathbf{A}_2}{\rho_{22}} \epsilon_{11}^{\circ} \right) - \frac{\partial}{\partial \xi_2} \left(\frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} (\mathbf{A}_1 \epsilon_{11}^{\circ}) \right) \\ & - \left(\frac{1}{r_{11} r_{22}} - \frac{1}{(r_{12})^2} \right) \mathbf{A}_1 \mathbf{A}_2 \epsilon_{11}^{\circ} + \mathcal{O}(\epsilon^4) \end{aligned}$$

EQUATIONS FOR ORTHOGONAL COORDINATES

CONTINUED

$$\begin{aligned} \mathcal{C}_{12}(\epsilon_{22}^{\circ}) = & -\frac{\partial}{\partial \xi_2} \left(\frac{\mathbf{A}_1}{\rho_{11}} \epsilon_{22}^{\circ} \right) - \frac{\partial}{\partial \xi_1} \left(\frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} (\mathbf{A}_2 \epsilon_{22}^{\circ}) \right) \\ & - \left(\frac{1}{\mathbf{r}_{11} \mathbf{r}_{22}} - \frac{1}{(\mathbf{r}_{12})^2} \right) \mathbf{A}_1 \mathbf{A}_2 \epsilon_{22}^{\circ} + \mathcal{O}(\epsilon^4) \end{aligned}$$

$$\begin{aligned} \mathcal{C}_{13}(\epsilon_{12}^{\circ}) = & \frac{\partial}{\partial \xi_1} \left(\frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_2} (\mathbf{A}_1 2\epsilon_{12}^{\circ}) \right) + \frac{\partial}{\partial \xi_2} \left(\frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_1} (\mathbf{A}_2 2\epsilon_{12}^{\circ}) \right) \\ & - \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} (2\epsilon_{12}^{\circ}) + \mathcal{O}(\epsilon^4) \end{aligned}$$

$$\mathcal{C}_{14}(\kappa_{11}^{\circ}) = -\frac{\mathbf{A}_1 \mathbf{A}_2}{\mathbf{r}_{22}} \kappa_{11}^{\circ} + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{15}(\kappa_{22}^{\circ}) = -\frac{\mathbf{A}_1 \mathbf{A}_2}{\mathbf{r}_{11}} \kappa_{22}^{\circ} + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{16}(\kappa_{12}^{\circ}) = -\frac{2\mathbf{A}_1 \mathbf{A}_2}{\mathbf{r}_{12}} \kappa_{12}^{\circ} + \mathcal{O}(\epsilon^4)$$

EQUATIONS FOR ORTHOGONAL COORDINATES

CONTINUED

$$\mathcal{C}_{21}(\epsilon_{11}^{\circ}) = - \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \frac{1}{A_2} \frac{\partial \epsilon_{11}^{\circ}}{\partial \xi_2} + \frac{\epsilon_{11}^{\circ}}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{12}} \right) + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \xi_1} \left(\frac{2}{r_{12}} \right) \epsilon_{11}^{\circ} + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{22}(\epsilon_{22}^{\circ}) = \frac{\epsilon_{22}^{\circ}}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{11}} \right) - \frac{2}{r_{12}} \frac{1}{A_1} \frac{\partial \epsilon_{22}^{\circ}}{\partial \xi_1} + \frac{\epsilon_{22}^{\circ}}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{23}(\epsilon_{12}^{\circ}) = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) \epsilon_{12}^{\circ} + \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \frac{1}{A_1} \frac{\partial \epsilon_{12}^{\circ}}{\partial \xi_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2} \frac{4 \epsilon_{12}^{\circ}}{r_{12}} + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{24}(\kappa_{11}^{\circ}) = - \frac{1}{A_2} \frac{\partial \kappa_{11}^{\circ}}{\partial \xi_2} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2} \kappa_{11}^{\circ} + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{25}(\kappa_{22}^{\circ}) = \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2} \kappa_{22}^{\circ} + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{26}(\kappa_{12}^{\circ}) = \frac{1}{A_1} \frac{\partial \kappa_{12}^{\circ}}{\partial \xi_1} + \frac{2}{A_1 A_2} \frac{\partial A_2}{\partial \xi_1} \kappa_{12}^{\circ} + \mathcal{O}(\epsilon^4)$$

EQUATIONS FOR ORTHOGONAL COORDINATES

CONCLUDED

$$\mathcal{C}_{31}(\epsilon_{11}^{\circ}) = \left[\frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{1}{r_{22}} \right) \right] \epsilon_{11}^{\circ} - \frac{2}{r_{12}} \frac{1}{A_2} \frac{\partial \epsilon_{11}^{\circ}}{\partial \xi_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \xi_1} \left[\left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) \right] \epsilon_{11}^{\circ} + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{32}(\epsilon_{22}^{\circ}) = - \left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) \frac{1}{A_1} \frac{\partial \epsilon_{22}^{\circ}}{\partial \xi_1} + \frac{\epsilon_{22}^{\circ}}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{12}} \right) + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2} \frac{2 \epsilon_{22}^{\circ}}{r_{12}} + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{33}(\epsilon_{12}^{\circ}) = \left[\frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{1}{r_{22}} + \frac{1}{r_{11}} \right) \right] \epsilon_{12}^{\circ} + \left(\frac{1}{r_{22}} - \frac{1}{r_{11}} \right) \frac{1}{A_2} \frac{\partial \epsilon_{12}^{\circ}}{\partial \xi_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \xi_1} \frac{4 \epsilon_{12}^{\circ}}{r_{12}} + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{34}(\kappa_{11}^{\circ}) = \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \xi_1} \kappa_{11}^{\circ} + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{35}(\kappa_{22}^{\circ}) = - \frac{1}{A_1} \frac{\partial \kappa_{22}^{\circ}}{\partial \xi_1} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \xi_1} \kappa_{22}^{\circ} + \mathcal{O}(\epsilon^4)$$

$$\mathcal{C}_{36}(\kappa_{12}^{\circ}) = \frac{1}{A_2} \frac{\partial \kappa_{12}^{\circ}}{\partial \xi_2} + \frac{2}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2} \kappa_{12}^{\circ} + \mathcal{O}(\epsilon^4)$$

SIMPLIFICATION OF THE NONLINEAR SHELL STRAINS

BASIS FOR SIMPLIFICATION OF THE NONLINEAR SHELL STRAINS

- The nonlinear shell strains presented herein for “small” strains consist of complicated expressions that are difficult, at best, to apply to practical situations
 - Thus, a great deal of effort has been expended over nearly 70 years to systematically simplify these equations
- To simplify the membrane strains, changes in the curvatures, and changes in the torsions for the shell reference surface, the magnitude of the linear-deformation parameters appearing in the expressions must be established on a **physically meaningful basis**
- Because shell deformation generally consists of **strain and rotation**, and a restriction on the order of the strains has been given, the order of magnitude of the rotation of infinitesimal material line elements of a shell are needed to quantify the deformation

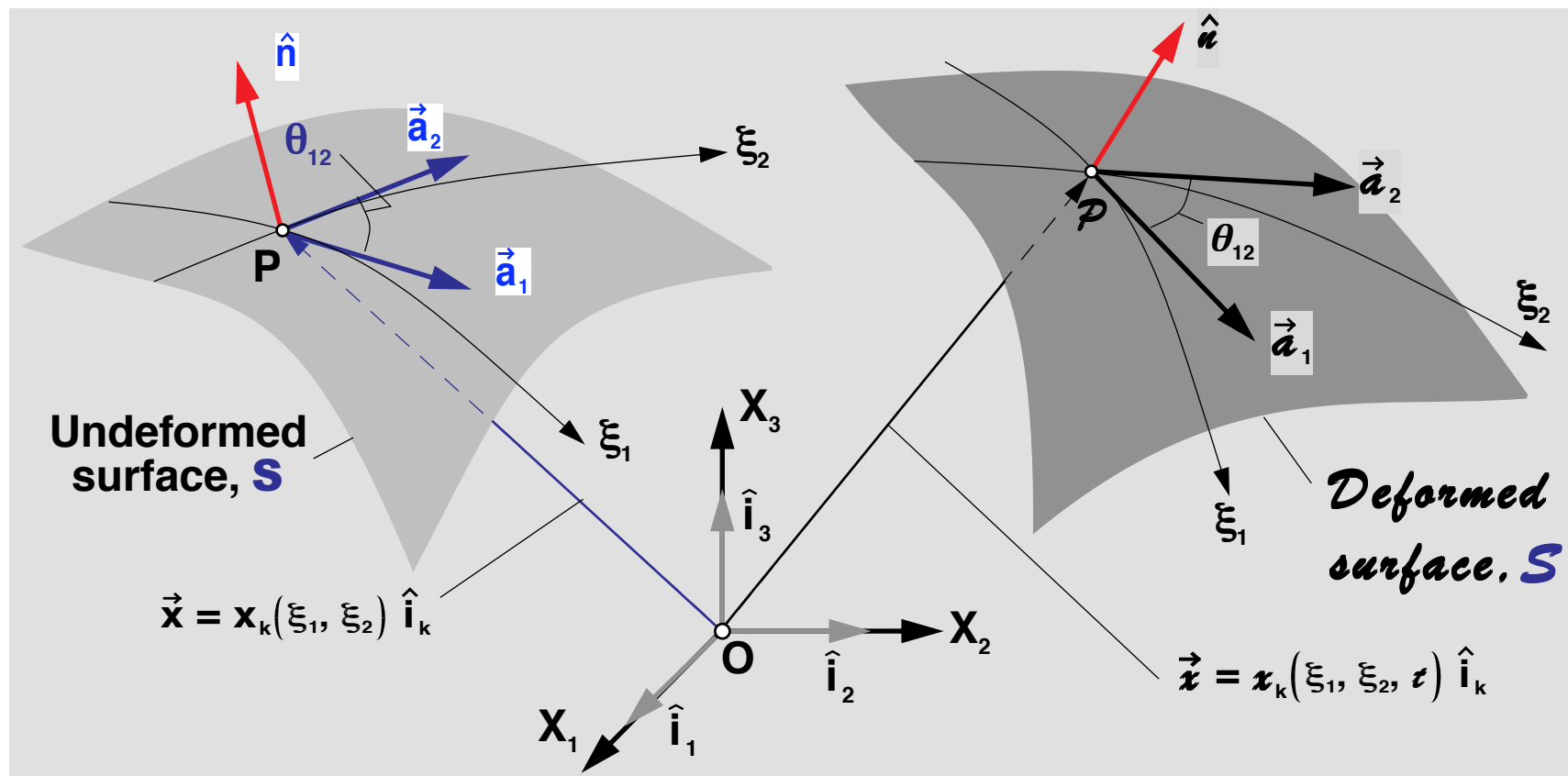
BASIS FOR SIMPLIFICATION OF THE NONLINEAR SHELL STRAINS - CONCLUDED

- **In addition, for many practical applications, the maximum thickness of a shell is small compared to its other areal dimensions**
- **Thus, two sets of criteria for simplifying the nonlinear “small-strain” shell equations have emerged that are based on placing restrictions on the magnitude of the rotational part of deformation and on placing restrictions on the relative thickness**
- **When these two sets are combined, significant simplification of the nonlinear shell equations is obtained**

SIMPLIFICATIONS BASED ON ROTATION SIZE

- The rotation of infinitesimal material line elements of a shell can be characterized fully by the **rotations of the convected base-vector fields**

$\{\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{n}}\}$ during deformation, as illustrated in the figure



SIMPLIFICATIONS BASED ON ROTATION SIZE

CONTINUED

- In general, **finite rotations** of infinitesimal material line elements can be represented by the magnitude and direction of a **finite-rotation pseudo vector** given by $\vec{\Omega} = \Omega_1 \hat{a}_1 + \Omega_2 \hat{a}_2 + \Omega_3 \hat{n} \equiv \hat{\Omega} \sin \omega$; where $\hat{\Omega}$ is a unit vector, $\sin \omega \equiv |\vec{\Omega}|$, and ω is the angle of rotation about $\hat{\Omega}$
- It has been shown in the references:
 - Pietraszkiewicz, W.: Langrangian Description and Incremental Formulation in the Non-Linear Theory of Thin Shells. International Journal of Non-Linear Mechanics, vol. 19, no. 2, 1983, pp. 115-140.
 - Pietraszkiewicz, W.: Finite Rotations in the Nonlinear Theory of Thin Shells. Thin Shell Theory, New Trends and Applications, W. Olszak, ed., International Centre for Mechanical Sciences, Course and Lectures no. 240, Springer-Verlag, 1980, pp. 153-208.

that for “small” strains, the components of $\vec{\Omega}(\xi_1, \xi_2, t)$, are given in terms of the linear deformation parameters by

SIMPLIFICATIONS BASED ON ROTATION SIZE CONTINUED

$$\Omega_1 = \left[\varphi_1 \left(\cot \theta_{12} + \frac{1}{2} \varphi - \frac{1}{2} \mathbf{e}_{12}^{\circ} \sin \theta_{12} \right) + \varphi_2 \left(\csc \theta_{12} + \frac{1}{2} \mathbf{e}_{11}^{\circ} \sin \theta_{12} \right) \right] \left[1 + \mathcal{O}(\varepsilon^2) \right]$$

$$\Omega_2 = \left[-\varphi_1 \left(\csc \theta_{12} + \frac{1}{2} \mathbf{e}_{22}^{\circ} \sin \theta_{12} \right) + \varphi_2 \left(\frac{1}{2} \mathbf{e}_{12}^{\circ} \sin \theta_{12} + \frac{1}{2} \varphi - \cot \theta_{12} \right) \right] \left[1 + \mathcal{O}(\varepsilon^2) \right]$$

$$\Omega_3 = \varphi \left[1 + \mathcal{O}(\varepsilon^2) \right]$$

- Also, in these references:
 - "Small" rotations correspond to $\omega \leq \mathcal{O}(\theta^2)$, where $0 < \theta < 1$
 - "Moderate" rotations correspond to $\omega = \mathcal{O}(\theta)$
 - "Large" rotations correspond to $\omega = \mathcal{O}(\sqrt{\theta})$
 - Unrestricted, finite rotations correspond to $\omega \geq \mathcal{O}(1)$

SIMPLIFICATIONS BASED ON ROTATION SIZE

CONTINUED

- Here, θ is a “small” positive quantity associated with rotations
- A practical alternative to using $|\omega|$ for characterizing rotation size, given in the references by Pietraszkiewicz, is to place restrictions directly on the $|\Omega_k|$
- This approach is physically appealing because Ω_1 and Ω_2 represent the rotation of the tangent plane at a point of the shell reference surface during deformation, and Ω_3 represent the rotation about the normal vector at that point
 - Thus, insight can be gained by comparing the flexibility of a shell with respect to “in-surface” and “out-of-surface” deformations
- To illustrate this approach, consider the class of rotations defined by $|\Omega_k| \leq \mathcal{O}(\theta^2)$, such that θ is approximately 100 times smaller than unity

SIMPLIFICATIONS BASED ON ROTATION SIZE CONTINUED

- Then, from

$$\Omega_1 = \left[\varphi_1 \left(\cot \theta_{12} + \frac{1}{2} \varphi - \frac{1}{2} \mathbf{e}_{12}^{\circ} \sin \theta_{12} \right) + \varphi_2 \left(\csc \theta_{12} + \frac{1}{2} \mathbf{e}_{11}^{\circ} \sin \theta_{12} \right) \right] \left[1 + \mathcal{O}(\varepsilon^2) \right]$$

$$\Omega_2 = \left[-\varphi_1 \left(\csc \theta_{12} + \frac{1}{2} \mathbf{e}_{22}^{\circ} \sin \theta_{12} \right) + \varphi_2 \left(\frac{1}{2} \mathbf{e}_{12}^{\circ} \sin \theta_{12} + \frac{1}{2} \varphi - \cot \theta_{12} \right) \right] \left[1 + \mathcal{O}(\varepsilon^2) \right]$$

$$\text{and } \Omega_3 = \varphi \left[1 + \mathcal{O}(\varepsilon^2) \right]$$

it follows that

$$|\varphi_{\alpha}| \leq \mathcal{O}(\theta^2), \quad |\varphi| \leq \mathcal{O}(\theta^2), \quad \text{and} \quad |\varphi_{\alpha} \mathbf{e}_{\beta\gamma}^{\circ}| \leq \mathcal{O}(\theta^2)$$

SIMPLIFICATIONS BASED ON ROTATION SIZE CONTINUED

- Next, from examination of the strains

$$\begin{aligned}\epsilon_{11}^{\circ} = \epsilon_{11}^{\circ} + \frac{1}{2} & \left[\left(\epsilon_{11}^{\circ} \csc \theta_{12} \right)^2 + \left(\epsilon_{12}^{\circ} \csc \theta_{12} + \varphi \right)^2 \right. \\ & \left. - 2\epsilon_{11}^{\circ} \left(\epsilon_{12}^{\circ} \csc \theta_{12} + \varphi \right) \cot \theta_{12} + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)^2 \right] + \mathcal{O}(\epsilon^4)\end{aligned}$$

$$\begin{aligned}\epsilon_{22}^{\circ} = \epsilon_{22}^{\circ} + \frac{1}{2} & \left[\left(\epsilon_{22}^{\circ} \csc \theta_{12} \right)^2 + \left(\epsilon_{12}^{\circ} \csc \theta_{12} - \varphi \right)^2 \right. \\ & \left. - 2\epsilon_{22}^{\circ} \left(\epsilon_{12}^{\circ} \csc \theta_{12} - \varphi \right) \cot \theta_{12} + \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right)^2 \right] + \mathcal{O}(\epsilon^4)\end{aligned}$$

$$\begin{aligned}2\epsilon_{12}^{\circ} = 2\epsilon_{12}^{\circ} + & \left[\epsilon_{11}^{\circ} \left(\epsilon_{12}^{\circ} - \varphi \sin \theta_{12} \right) + \epsilon_{22}^{\circ} \left(\epsilon_{12}^{\circ} + \varphi \sin \theta_{12} \right) \right] \csc^2 \theta_{12} \\ & - \left[\epsilon_{11}^{\circ} \epsilon_{22}^{\circ} + \left(\epsilon_{12}^{\circ} + \varphi \sin \theta_{12} \right) \left(\epsilon_{12}^{\circ} - \varphi \sin \theta_{12} \right) \right] \csc \theta_{12} \cot \theta_{12} \\ & + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) + \mathcal{O}(\epsilon^4)\end{aligned}$$

SIMPLIFICATIONS BASED ON ROTATION SIZE CONTINUED

it is seen that $\left| \mathbf{e}_{\beta\gamma}^o \right|$ are the largest quantities appearing in strain expressions

- Thus, for “small” strains characterized by $\left| \boldsymbol{\varepsilon}_{\beta\gamma}^o \right| \leq \mathcal{O}(\varepsilon^2)$, the magnitude of $\mathbf{e}_{\beta\gamma}^o$ must be $\left| \mathbf{e}_{\beta\gamma}^o \right| \leq \mathcal{O}(\theta^2)$ to yield “small” strains that are consistent with $\boldsymbol{\varepsilon}_{\alpha\beta}^o$ being of $\mathcal{O}(\varepsilon^2)$
- A similar procedure is followed to obtain magnitude estimates for the linear deformation parameters that correspond to different size restrictions placed on the magnitudes of $\boldsymbol{\Omega}_k$
- The results of this approach are given in references by Pietraszkiewicz for several classes of rotations and are stated in the following Table

SIMPLIFICATIONS BASED ON ROTATION SIZE

CONTINUED

Order of magnitude estimates for the linear deformation parameters are given for the case of “small” strains as follows, where θ is a small positive-valued parameter and $\theta^2 \ll 1$

$ \Omega_1 , \Omega_2 $	$ \Omega_3 $	$ \varphi_1 , \varphi_2 $	$ \varphi $	$ \mathbf{e}_{\alpha\beta}^o $
small	small	θ^2	θ^2	θ^2
moderate	small	θ	θ^2	θ^2
moderate	moderate	θ	θ	θ^2
large	small	$\theta^{1/2}$	θ^2	θ
large	moderate	$\theta^{1/2}$	θ	θ
large	large	$\theta^{1/2}$	$\theta^{1/2}$	θ
finite	small	1	θ^2	1
finite	moderate	1	θ	1
finite	large	1	$\theta^{1/2}$	1
finite	finite	1	1	1

SIMPLIFICATIONS BASED ON ROTATION SIZE

CONCLUDED

- In this table, the magnitudes of Ω_1 , Ω_2 , and Ω_3 in the shaded columns are considered as input and the information in the unshaded columns are the output
- The term “**small**” used in the table corresponds to magnitudes smaller than or equal to $\mathcal{O}(\theta^2)$, where $\theta^2 \ll 1$
- The term “**moderate**” corresponds to magnitudes smaller than or equal to $\mathcal{O}(\theta)$
- The term “**large**” corresponds to magnitudes smaller than or equal to $\leq \mathcal{O}(\theta^{1/2})$
- The term “**finite**” means that the magnitudes are ≥ 1

SIMPLIFICATIONS BASED ON SHELL THINNESS

- Let h denote the maximum value of the shell thickness, $h(\xi_1, \xi_2)$, and let $\mathcal{R}$ denote the smallest magnitude of the reference-surface curvatures and torsions given by the reciprocals of $r_{\alpha\beta}(\xi_1, \xi_2)$
- **Simplifications based on the shell thickness** involve placing limitations on the size of $\frac{h}{\mathcal{R}}$
- To take advantage of shell thinness, it is noted that $\left| \frac{\xi_3}{r_{\alpha\beta}} \right| \leq \frac{h}{\mathcal{R}}$ and that $\frac{h}{\mathcal{R}} \ll 1$ in many practical cases
- For this case, binomial expansions of quantities involving powers and products of $\frac{\xi_3}{r_{11}}$, $\frac{\xi_3}{r_{12}}$, $\frac{\xi_3}{r_{22}}$, and $\frac{\xi_3}{r_{21}}$ are used to simplify the expressions for the shell strains

SIMPLIFICATIONS BASED ON SHELL THINNESS

CONTINUED

- For example, consider the Green-Lagrange shell strain

$$\begin{aligned} \varepsilon_{11} \left(\frac{H_1}{A_1} \right)^2 &= \varepsilon_{11}^o \left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right] + \xi_3 K_{11}^o \left(1 + \frac{\xi_3}{r_{11}} \right) \\ &+ \frac{1}{2} \left(\frac{\xi_3}{r_{12}} \right) \left[2\varepsilon_{12}^o \left(\frac{\xi_3}{r_{11}} - \frac{\xi_3}{r_{22}} \right) \csc \theta_{12} - \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right) \right] \csc^2 \theta_{12} \\ &- \xi_3 \left(\frac{\xi_3}{r_{12}} \right) \left[K_{12}^o - \frac{1}{2} \cot \theta_{12} (K_{11}^o - K_{22}^o) \right] + \Gamma_{11} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

where

$$\left(\frac{H_1}{A_1} \right)^2 = \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}} \right)^2 \quad \text{and}$$

SIMPLIFICATIONS BASED ON SHELL THINNESS

CONTINUED

$$\begin{aligned} \Gamma_{11} = & F_2(\xi_3) \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(\frac{1}{A_1} \frac{\partial}{\partial \xi_1} (\gamma_2^\circ \cos \theta_{12}) - \frac{\gamma_2^\circ \sin \theta_{12}}{\rho_{11}} \right) \\ & - F_2(\xi_3) \frac{\xi_3}{r_{12}} \left(\frac{\csc \theta_{12}}{A_1} \frac{\partial \gamma_2^\circ}{\partial \xi_1} \right) + F_1(\xi_3) \left[\left(1 + \frac{\xi_3}{r_{11}} \right) \frac{1}{A_1} \frac{\partial \gamma_1^\circ}{\partial \xi_1} - \frac{\xi_3}{r_{12}} \frac{\gamma_1^\circ}{\rho_{11}} \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

- Applying the binomial theorem to $\left(\frac{H_1}{A_1} \right)^{-2}$ gives

$$\begin{aligned} \left(\frac{H_1}{A_1} \right)^{-2} = & 1 - \frac{2\xi_3}{r_{11}} - \frac{2\xi_3}{r_{12}} \cot \theta_{12} - \left(\frac{\xi_3}{r_{11}} \right)^2 - \frac{2\xi_3}{r_{11}} \frac{\xi_3}{r_{12}} \cot \theta_{12} \\ & - \left(\frac{\xi_3}{r_{12}} \right)^2 (\cot^2 \theta_{12} + \csc^2 \theta_{12}) + \mathcal{O} \left(\left(\frac{\ell}{\mathcal{R}} \right)^3 \right) \end{aligned}$$

SIMPLIFICATIONS BASED ON SHELL THINNESS

CONTINUED

- Thus, it follows that

$$\left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right] \left(\frac{H_1}{A_1} \right)^{-2} = 1 - \frac{2\xi_3}{r_{12}} \cot\theta_{12} - \frac{6\xi_3}{r_{11}} \frac{\xi_3}{r_{12}} \cot\theta_{12} \\ - 2 \left(\frac{\xi_3}{r_{12}} \right)^2 \cot^2\theta_{12} - 4 \left(\frac{\xi_3}{r_{11}} \right)^2 + \mathcal{O}\left(\left(\frac{t}{R}\right)^3\right)$$

$$\left(1 + \frac{\xi_3}{r_{11}} \right) \left(\frac{H_1}{A_1} \right)^{-2} = 1 - \frac{\xi_3}{r_{11}} - \frac{2\xi_3}{r_{12}} \cot\theta_{12} - 3 \left(\frac{\xi_3}{r_{11}} \right)^2 - \frac{4\xi_3}{r_{11}} \frac{\xi_3}{r_{12}} \cot\theta_{12} \\ - \left(\frac{\xi_3}{r_{12}} \right)^2 \left(\cot^2\theta_{12} + \csc^2\theta_{12} \right) + \mathcal{O}\left(\left(\frac{t}{R}\right)^3\right)$$

$$\left(\frac{\xi_3}{r_{12}} \right) \left(\frac{H_1}{A_1} \right)^{-2} = \frac{\xi_3}{r_{12}} - \frac{2\xi_3}{r_{11}} \frac{\xi_3}{r_{12}} - 2 \left(\frac{\xi_3}{r_{12}} \right)^2 \cot\theta_{12} + \mathcal{O}\left(\left(\frac{t}{R}\right)^3\right)$$

SIMPLIFICATIONS BASED ON SHELL THINNESS

CONTINUED

- Likewise,

$$\left(\frac{\xi_3}{\mathbf{r}_{12}}\right)\left(\frac{\xi_3}{\mathbf{r}_{11}} - \frac{\xi_3}{\mathbf{r}_{22}}\right)\left(\frac{\mathbf{H}_1}{\mathbf{A}_1}\right)^{-2} = \left(\frac{\xi_3}{\mathbf{r}_{12}}\right)\left(\frac{\xi_3}{\mathbf{r}_{11}} - \frac{\xi_3}{\mathbf{r}_{22}}\right) + \mathcal{O}\left(\left(\frac{t}{R}\right)^3\right)$$

$$\left(\frac{\xi_3}{\mathbf{r}_{12}}\right)\left(\frac{\xi_3}{\mathbf{r}_{12}} + \frac{\xi_3}{\mathbf{r}_{21}}\right)\left(\frac{\mathbf{H}_1}{\mathbf{A}_1}\right)^{-2} = \left(\frac{\xi_3}{\mathbf{r}_{12}}\right)\left(\frac{\xi_3}{\mathbf{r}_{12}} + \frac{\xi_3}{\mathbf{r}_{21}}\right) + \mathcal{O}\left(\left(\frac{t}{R}\right)^3\right)$$

- Applying these simplifications yields

$$\begin{aligned} \epsilon_{11} = & \epsilon_{11}^{\circ} + \xi_3 K_{11}^{\circ} + \mathbf{F}_1(\xi_3) \frac{1}{\mathbf{A}_1} \frac{\partial \gamma_1^{\circ}}{\partial \xi_1} \\ & + \mathbf{F}_2(\xi_3) \left(\frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\gamma_2^{\circ} \cos \theta_{12} \right) - \frac{\gamma_2^{\circ} \sin \theta_{12}}{\rho_{11}} \right) + \mathcal{O}(\epsilon^2) \mathcal{O}\left(\frac{t}{R}\right) + \mathcal{O}(\epsilon^4) \end{aligned}$$

SIMPLIFICATIONS BASED ON SHELL THINNESS

CONTINUED

- Likewise,

$$\begin{aligned}\varepsilon_{22} = & \varepsilon_{22}^{\circ} + \xi_3 K_{22}^{\circ} + F_2(\xi_3) \frac{1}{A_2} \frac{\partial \gamma_2^{\circ}}{\partial \xi_2} \\ & + F_1(\xi_3) \left(\frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\gamma_1^{\circ} \cos \theta_{12}) + \frac{\gamma_1^{\circ} \sin \theta_{12}}{\rho_{22}} \right) + \mathcal{O}(\varepsilon^2) \mathcal{O}\left(\frac{t}{R}\right) + \mathcal{O}(\varepsilon^4)\end{aligned}$$

$$\begin{aligned}2\varepsilon_{12} = & 2\varepsilon_{12}^{\circ} + \xi_3 \left[2K_{12}^{\circ} \sin \theta_{12} + (K_{11}^{\circ} + K_{22}^{\circ}) \cos \theta_{12} \right] \\ & + F_1(\xi_3) \left(\frac{\cos \theta_{12}}{A_1} \frac{\partial \gamma_1^{\circ}}{\partial \xi_1} + \frac{1}{A_2} \frac{\partial \gamma_1^{\circ}}{\partial \xi_2} + \frac{\gamma_1^{\circ} \sin \theta_{12}}{\rho_{11}} \right) \\ & + F_2(\xi_3) \left(\frac{1}{A_1} \frac{\partial \gamma_2^{\circ}}{\partial \xi_1} + \frac{\cos \theta_{12}}{A_2} \frac{\partial \gamma_2^{\circ}}{\partial \xi_2} - \frac{\gamma_2^{\circ} \sin \theta_{12}}{\rho_{22}} \right) + \mathcal{O}(\varepsilon^2) \mathcal{O}\left(\frac{t}{R}\right) + \mathcal{O}(\varepsilon^4)\end{aligned}$$

SIMPLIFICATIONS BASED ON SHELL THINNESS

CONTINUED

$$2\epsilon_{13} = \left[F_1'(\xi_3) - \frac{F_1(\xi_3)}{r_{11}} \right] \gamma_1^o + \left[\left(F_2'(\xi_3) - \frac{F_2(\xi_3)}{r_{11}} \right) \cos\theta_{12} + \frac{F_2(\xi_3) \sin\theta_{12}}{r_{12}} \right] \gamma_2^o + \mathcal{O}(\epsilon^2) \mathcal{O}\left(\frac{L}{R}\right) + \mathcal{O}(\epsilon^4)$$

$$2\epsilon_{23} = \left[F_2'(\xi_3) - \frac{F_2(\xi_3)}{r_{22}} \right] \gamma_2^o + \left[\left(F_1'(\xi_3) - \frac{F_1(\xi_3)}{r_{22}} \right) \cos\theta_{12} - \frac{F_1(\xi_3) \sin\theta_{12}}{r_{21}} \right] \gamma_1^o + \mathcal{O}(\epsilon^2) \mathcal{O}\left(\frac{L}{R}\right) + \mathcal{O}(\epsilon^4)$$

- The shifters $\mu_{\alpha\beta}(\xi_1, \xi_2, \xi_3)$ that have been defined herein and used in $\hat{\mathbf{g}}_1 = \mu_{11} \hat{\mathbf{a}}_1 + \mu_{12} \hat{\mathbf{a}}_2$ and $\hat{\mathbf{g}}_2 = \mu_{21} \hat{\mathbf{a}}_1 + \mu_{22} \hat{\mathbf{a}}_2$ are also simplified by using power series

SIMPLIFICATIONS BASED ON SHELL THINNESS CONTINUED

- In particular,

$$\mu_{11} = \frac{1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}}\right)^2}} \quad \rightarrow$$

$$\mu_{11} = 1 - \frac{1}{2} \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}} \right)^2 + \mathcal{O} \left(\left(\frac{l}{R} \right)^3 \right)$$

$$\mu_{12} = \frac{-\frac{\xi_3 \csc \theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}}\right)^2}} \quad \rightarrow$$

$$\mu_{12} = -\frac{\xi_3 \csc \theta_{12}}{r_{12}} \left(1 - \frac{\xi_3}{r_{11}} - \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) + \mathcal{O} \left(\left(\frac{l}{R} \right)^3 \right)$$

SIMPLIFICATIONS BASED ON SHELL THINNESS CONCLUDED

$$\mu_{21} = \frac{\frac{\xi_3 \csc \theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}}\right)^2}} \quad \rightarrow$$

$$\mu_{21} = \frac{\xi_3 \csc \theta_{12}}{r_{21}} \left(1 - \frac{\xi_3}{r_{22}} + \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right) + \mathcal{O}\left(\left(\frac{l}{\mathcal{R}}\right)^3\right)$$

$$\mu_{22} = \frac{1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}}\right)^2}} \quad \rightarrow$$

$$\mu_{22} = 1 - \frac{1}{2} \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}}\right)^2 + \mathcal{O}\left(\left(\frac{l}{\mathcal{R}}\right)^3\right)$$

SPECIAL CASES OF THE NONLINEAR SHELL STRAINS

SPECIAL CASES OF THE NONLINEAR SHELL STRAINS

- A variety of expressions for the nonlinear shell strains have appeared in the literature that can be obtained from the baseline equations presented herein for “small” strains
- For example, in the paper:
 - Koiter, W. T.: General Equations of Elastic Stability for Thin Shells. Proceedings - Symposium on the Theory of Shells, University of Houston, 1967, pp. 187-227.

Koiter indicates that for thin-shell elastic stability problems, it is reasonable to use the exact form of the “small-strain” equations for the membrane strains and a set of linearized equations for the changes in reference-surface curvatures and torsions

- This statement is based on the presumption that the effects of the changes in reference-surface curvatures and torsions are relatively much smaller than the nonlinear effects of the membrane strains that generate interaction between membrane stresses and shell deformations

SPECIAL CASES OF THE NONLINEAR SHELL STRAINS

CONTINUED

- In particular, Koiter presents the tensor form of the following equations

$$\begin{aligned} \boldsymbol{\varepsilon}_{11}^{\circ} = & \boldsymbol{e}_{11}^{\circ} + \frac{1}{2} \left[\left(\boldsymbol{e}_{11}^{\circ} \csc \theta_{12} \right)^2 + \left(\boldsymbol{e}_{12}^{\circ} + \varphi \sin \theta_{12} \right)^2 \csc^2 \theta_{12} \right. \\ & \left. - 2 \boldsymbol{e}_{11}^{\circ} \left(\boldsymbol{e}_{12}^{\circ} + \varphi \sin \theta_{12} \right) \csc \theta_{12} \cot \theta_{12} + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)^2 \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\varepsilon}_{22}^{\circ} = & \boldsymbol{e}_{22}^{\circ} + \frac{1}{2} \left[\left(\boldsymbol{e}_{22}^{\circ} \csc \theta_{12} \right)^2 + \left(\boldsymbol{e}_{12}^{\circ} - \varphi \sin \theta_{12} \right)^2 \csc^2 \theta_{12} \right. \\ & \left. - 2 \boldsymbol{e}_{22}^{\circ} \left(\boldsymbol{e}_{12}^{\circ} - \varphi \sin \theta_{12} \right) \csc \theta_{12} \cot \theta_{12} + \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right)^2 \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\begin{aligned} 2\boldsymbol{\varepsilon}_{12}^{\circ} = & 2\boldsymbol{e}_{12}^{\circ} + \left[\boldsymbol{e}_{11}^{\circ} \left(\boldsymbol{e}_{12}^{\circ} - \varphi \sin \theta_{12} \right) + \boldsymbol{e}_{22}^{\circ} \left(\boldsymbol{e}_{12}^{\circ} + \varphi \sin \theta_{12} \right) \right] \csc^2 \theta_{12} \\ & - \left[\boldsymbol{e}_{11}^{\circ} \boldsymbol{e}_{22}^{\circ} + \left(\boldsymbol{e}_{12}^{\circ} + \varphi \sin \theta_{12} \right) \left(\boldsymbol{e}_{12}^{\circ} - \varphi \sin \theta_{12} \right) \right] \csc \theta_{12} \cot \theta_{12} \\ & + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) + \mathcal{O}(\varepsilon^4) \end{aligned}$$

SPECIAL CASES OF THE NONLINEAR SHELL STRAINS

CONTINUED

$$\kappa_{11}^{\circ} = \chi_{11}^{\circ} = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (\varphi_1 + \varphi_2 \cos \theta_{12}) - \frac{\varphi_2 \sin \theta_{12}}{\rho_{11}} - \frac{\varphi}{r_{12}}$$

$$\kappa_{22}^{\circ} = \chi_{22}^{\circ} = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\varphi_2 + \varphi_1 \cos \theta_{12}) + \frac{\varphi_1 \sin \theta_{12}}{\rho_{22}} - \frac{\varphi}{r_{21}}$$

$$\begin{aligned} 2\kappa_{12}^{\circ} = 2\chi_{12}^{\circ} = & \frac{1}{A_2} \frac{\partial \varphi_1}{\partial \xi_2} + \frac{\cos \theta_{12}}{A_2} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\cos \theta_{12}}{A_1} \frac{\partial \varphi_1}{\partial \xi_1} + \frac{1}{A_1} \frac{\partial \varphi_2}{\partial \xi_1} \\ & + \left(\frac{\varphi_1}{\rho_{11}} - \frac{\varphi_2}{\rho_{22}} \right) \sin \theta_{12} - \varphi \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc \theta_{12} \end{aligned}$$

where

$$\varphi_1 = \left(\frac{u_1}{r_{11}} - \frac{1}{A_1} \frac{\partial w}{\partial \xi_1} \right) \csc^2 \theta_{12} - u_1 \left(\frac{1}{r_{21}} + \frac{\cot \theta_{12}}{r_{22}} \right) \cot \theta_{12} + \left(\frac{\cot \theta_{12}}{A_2} \frac{\partial w}{\partial \xi_2} + \frac{u_2}{r_{21}} \right) \csc \theta_{12}$$

$$\varphi_2 = \left(\frac{u_2}{r_{22}} - \frac{1}{A_2} \frac{\partial w}{\partial \xi_2} \right) \csc^2 \theta_{12} + u_2 \left(\frac{1}{r_{12}} - \frac{\cot \theta_{12}}{r_{11}} \right) \cot \theta_{12} + \left(\frac{\cot \theta_{12}}{A_1} \frac{\partial w}{\partial \xi_1} - \frac{u_1}{r_{12}} \right) \csc \theta_{12}$$

SPECIAL CASES OF THE NONLINEAR SHELL STRAINS

CONTINUED

$$2\varphi = \left(\frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} \right) \cot \theta_{12} + \left(\frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} \right) \csc \theta_{12} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}}$$

$$\mathbf{e}_{11}^o = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (u_1 + u_2 \cos \theta_{12}) - \frac{u_2 \sin \theta_{12}}{\rho_{11}} + \frac{w}{r_{11}}$$

$$\mathbf{e}_{22}^o = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (u_2 + u_1 \cos \theta_{12}) + \frac{u_1 \sin \theta_{12}}{\rho_{22}} + \frac{w}{r_{22}}$$

$$\begin{aligned} 2\mathbf{e}_{12}^o = & \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} + \left(\frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} + \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} \right) \cos \theta_{12} \\ & + \left(\frac{u_1}{\rho_{11}} - \frac{u_2}{\rho_{22}} \right) \sin \theta_{12} + w \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) \cos \theta_{12} + w \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \sin \theta_{12} \end{aligned}$$

- These membrane strains are exact within the realm of “small” strains, and the changes in reference-surface curvatures and torsions are identical to those used in the **Sanders-Budiansky-Koiter** linear shell theory, that is considered presently to be the best first-approximation shell theory

SPECIAL CASES OF THE NONLINEAR SHELL STRAINS

CONTINUED

- The expressions for the shell strains are given by

$$\varepsilon_{11} = \varepsilon_{11}^{\circ} + \xi_3 K_{11}^{\circ} + \mathcal{O}\left(\varepsilon^2 \frac{L}{R}, \varepsilon^4\right) \quad \varepsilon_{22} = \varepsilon_{22}^{\circ} + \xi_3 K_{22}^{\circ} + \mathcal{O}\left(\varepsilon^2 \frac{L}{R}, \varepsilon^4\right)$$

$$2\varepsilon_{12} = 2\varepsilon_{12}^{\circ} + \xi_3 \left[2K_{12}^{\circ} \sin\theta_{12} + (K_{11}^{\circ} + K_{22}^{\circ}) \cos\theta_{12} \right] + \mathcal{O}\left(\varepsilon^2 \frac{L}{R}, \varepsilon^4\right)$$

where the effects of transverse shearing deformations have been neglected

- It is noteworthy to point out that these equations are identical to those presented by Budiansky in the paper:
 - Budiansky, B.: Notes on Nonlinear Shell Theory. Journal of Applied Mechanics, vol. 35, no. 2, June, 1968, pp. 393-401.

SPECIAL CASES OF THE NONLINEAR SHELL STRAINS

CONTINUED

- For orthogonal reference-surface Gaussian coordinates, the equations presented by Koiter and by Budiansky reduce to

$$\varepsilon_{11}^{\circ} = \mathbf{e}_{11}^{\circ} + \frac{1}{2} \left[\left(\mathbf{e}_{11}^{\circ} \right)^2 + \left(\mathbf{e}_{12}^{\circ} + \varphi \right)^2 + \varphi_1^2 \right] + \mathcal{O}(\varepsilon^4)$$

$$\varepsilon_{22}^{\circ} = \mathbf{e}_{22}^{\circ} + \frac{1}{2} \left[\left(\mathbf{e}_{12}^{\circ} - \varphi \right)^2 + \left(\mathbf{e}_{22}^{\circ} \right)^2 + \varphi_2^2 \right] + \mathcal{O}(\varepsilon^4)$$

$$2\varepsilon_{12}^{\circ} = 2\mathbf{e}_{12}^{\circ} + \mathbf{e}_{11}^{\circ}(\mathbf{e}_{12}^{\circ} - \varphi) + \mathbf{e}_{22}^{\circ}(\mathbf{e}_{12}^{\circ} + \varphi) + \varphi_1\varphi_2 + \mathcal{O}(\varepsilon^4)$$

$$\mathbf{K}_{11}^{\circ} = \chi_{11}^{\circ}$$

$$\mathbf{K}_{22}^{\circ} = \chi_{22}^{\circ}$$

$$2\mathbf{K}_{12}^{\circ} = 2\chi_{12}^{\circ}$$

$$\varepsilon_{11} = \varepsilon_{11}^{\circ} + \xi_3 \mathbf{K}_{11}^{\circ} + \mathcal{O}\left(\varepsilon^2 \frac{h}{R}, \varepsilon^4\right)$$

$$\varepsilon_{22} = \varepsilon_{22}^{\circ} + \xi_3 \mathbf{K}_{22}^{\circ} + \mathcal{O}\left(\varepsilon^2 \frac{h}{R}, \varepsilon^4\right)$$

$$2\varepsilon_{12} = 2\varepsilon_{12}^{\circ} + 2\xi_3 \mathbf{K}_{12}^{\circ} + \mathcal{O}\left(\varepsilon^2 \frac{h}{R}, \varepsilon^4\right)$$

SPECIAL CASES OF THE NONLINEAR SHELL STRAINS

CONTINUED

where the linear deformation measures are given by

$$\mathbf{e}_{11}^{\circ} = \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_1}{\partial \xi_1} - \frac{\mathbf{u}_2}{\rho_{11}} + \frac{\mathbf{w}}{\mathbf{r}_{11}}$$

$$\mathbf{e}_{22}^{\circ} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_2}{\partial \xi_2} + \frac{\mathbf{u}_1}{\rho_{22}} + \frac{\mathbf{w}}{\mathbf{r}_{22}}$$

$$2\mathbf{e}_{12}^{\circ} = \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_1}{\partial \xi_2} + \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_2}{\partial \xi_1} + \frac{\mathbf{u}_1}{\rho_{11}} - \frac{\mathbf{u}_2}{\rho_{22}} - \frac{2\mathbf{w}}{\mathbf{r}_{12}}$$

$$\chi_{11}^{\circ} = \frac{1}{\mathbf{A}_1} \frac{\partial \varphi_1}{\partial \xi_1} - \frac{\varphi_2}{\rho_{11}} - \frac{\varphi}{\mathbf{r}_{12}}$$

$$\chi_{22}^{\circ} = \frac{1}{\mathbf{A}_2} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\varphi_1}{\rho_{22}} + \frac{\varphi}{\mathbf{r}_{12}}$$

$$2\chi_{12}^{\circ} = \frac{1}{\mathbf{A}_2} \frac{\partial \varphi_1}{\partial \xi_2} + \frac{1}{\mathbf{A}_1} \frac{\partial \varphi_2}{\partial \xi_1} + \left(\frac{\varphi_1}{\rho_{11}} - \frac{\varphi_2}{\rho_{22}} \right) - \varphi \left(\frac{1}{\mathbf{r}_{11}} - \frac{1}{\mathbf{r}_{22}} \right)$$

$$\varphi_1 = \frac{\mathbf{u}_1}{\mathbf{r}_{11}} - \frac{\mathbf{u}_2}{\mathbf{r}_{12}} - \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{w}}{\partial \xi_1}$$

$$\varphi_2 = \frac{\mathbf{u}_2}{\mathbf{r}_{22}} - \frac{\mathbf{u}_1}{\mathbf{r}_{12}} - \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{w}}{\partial \xi_2}$$

$$\varphi = \frac{1}{2} \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{u}_2}{\partial \xi_1} - \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{u}_1}{\partial \xi_2} + \frac{\mathbf{u}_1}{\rho_{11}} + \frac{\mathbf{u}_2}{\rho_{22}} \right)$$

$$\frac{1}{\rho_{11}} = - \frac{1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2}$$

$$\frac{1}{\rho_{22}} = \frac{1}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1}$$

SPECIAL CASES OF THE NONLINEAR SHELL STRAINS

CONTINUED

- Following a similar line of reasoning, in the paper:
 - Geier, B.: Energy-Based Task Formulations for Buckling Problems of Laminated Composite Shells. Zeitschrift für Flugwissenschaften und Weltraumforschung, vol. 10, no. 4, 1986, pp. 215-226.

Geier uses the same membrane strains as Koiter, but uses the following linear expressions for the changes in reference-surface curvatures and torsions

$$K_{11}^{\circ} = \chi_{11}^{\circ} + \frac{1}{r_{12}} \left(e_{11}^{\circ} \cot \theta_{12} - e_{12}^{\circ} \csc \theta_{12} \right) - \frac{e_{11}^{\circ}}{r_{11}}$$

$$K_{22}^{\circ} = \chi_{22}^{\circ} + \frac{1}{r_{21}} \left(e_{12}^{\circ} \csc \theta_{12} - e_{22}^{\circ} \cot \theta_{12} \right) - \frac{e_{22}^{\circ}}{r_{22}}$$

$$2K_{12}^{\circ} = 2\chi_{12}^{\circ} \csc \theta_{12} - \left(\chi_{11}^{\circ} + \chi_{22}^{\circ} \right) \cot \theta_{12} + \frac{e_{11}^{\circ}}{r_{12}} - \frac{e_{22}^{\circ}}{r_{21}} - \frac{e_{12}^{\circ} \csc \theta_{12} - e_{11}^{\circ} \cot \theta_{12}}{r_{11}} + \frac{e_{22}^{\circ} \cot \theta_{12} - e_{12}^{\circ} \csc \theta_{12}}{r_{22}}$$

SPECIAL CASES OF THE NONLINEAR SHELL STRAINS

CONCLUDED

- For orthogonal reference-surface Gaussian coordinates, the equations for the changes in reference-surface curvatures and torsions reduce to

$$K_{11}^{\circ} = \chi_{11}^{\circ} - \frac{e_{11}^{\circ}}{r_{11}} - \frac{e_{12}^{\circ}}{r_{12}} \quad K_{22}^{\circ} = \chi_{22}^{\circ} - \frac{e_{22}^{\circ}}{r_{22}} - \frac{e_{12}^{\circ}}{r_{12}}$$

$$2K_{12}^{\circ} = 2\chi_{12}^{\circ} + \frac{e_{11}^{\circ} + e_{22}^{\circ}}{r_{12}} - e_{12}^{\circ} \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right)$$

- Rigorous, systematic simplification of the nonlinear shell equations have been presented by **Sanders** (1963), **Koiter** (1966), **Budiansky** (1968), and **Pietraszkiewicz** (1977)
- For the most part, the equations given are very complicated
- In contrast, the simplified equations that are based on “small” strains and “moderate” rotations are manageable and described in the next section

“SMALL” STRAINS AND “MODERATE” ROTATIONS

- Consider the special case of the “small” strain and “moderate” rotation theory of Pietraszkiewicz (1980), where the magnitudes of the linear deformation parameters are restricted by

$$|\varphi_\alpha| \leq \mathcal{O}(\theta), \quad |\varphi| \leq \mathcal{O}(\theta), \quad \text{and} \quad |\mathbf{e}_{\beta\gamma}^\circ| \leq \mathcal{O}(\theta^2)$$

- For this case,

$$\begin{aligned} \boldsymbol{\varepsilon}_{11}^\circ = & \mathbf{e}_{11}^\circ + \frac{1}{2} \left[\left(\mathbf{e}_{11}^\circ \mathbf{csc} \theta_{12} \right)^2 + \left(\mathbf{e}_{12}^\circ + \varphi \sin \theta_{12} \right)^2 \mathbf{csc}^2 \theta_{12} \right. \\ & \left. - 2 \mathbf{e}_{11}^\circ \left(\mathbf{e}_{12}^\circ + \varphi \sin \theta_{12} \right) \mathbf{csc} \theta_{12} \cot \theta_{12} + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)^2 \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

reduces to

$$\begin{aligned} \boldsymbol{\varepsilon}_{11}^\circ = & \mathbf{e}_{11}^\circ + \frac{1}{2} \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)^2 + \frac{1}{2} \varphi^2 \\ & + \varphi \left(\mathbf{e}_{12}^\circ \mathbf{csc} \theta_{12} - \mathbf{e}_{11}^\circ \cot \theta_{12} \right) + \mathcal{O}(\theta^4, \varepsilon^4) \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\begin{aligned} \varepsilon_{22}^{\circ} = & \mathbf{e}_{22}^{\circ} + \frac{1}{2} \left[\left(\mathbf{e}_{22}^{\circ} \mathbf{csc} \theta_{12} \right)^2 + \left(\mathbf{e}_{12}^{\circ} - \varphi \sin \theta_{12} \right)^2 \mathbf{csc}^2 \theta_{12} \right. \\ & \left. - 2 \mathbf{e}_{22}^{\circ} \left(\mathbf{e}_{12}^{\circ} - \varphi \sin \theta_{12} \right) \mathbf{csc} \theta_{12} \cot \theta_{12} + \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right)^2 \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

reduces to

$$\begin{aligned} \varepsilon_{22}^{\circ} = & \mathbf{e}_{22}^{\circ} + \frac{1}{2} \left(\varphi_2 + \varphi_1 \cos \theta_{12} \right)^2 + \frac{1}{2} \varphi^2 \\ & + \varphi \left(\mathbf{e}_{22}^{\circ} \cot \theta_{12} - \mathbf{e}_{12}^{\circ} \mathbf{csc} \theta_{12} \right) + \mathcal{O}(\theta^4, \varepsilon^4) \end{aligned}$$

and

$$\begin{aligned} 2\varepsilon_{12}^{\circ} = & 2\mathbf{e}_{12}^{\circ} + \left[\mathbf{e}_{11}^{\circ} \left(\mathbf{e}_{12}^{\circ} - \varphi \sin \theta_{12} \right) + \mathbf{e}_{22}^{\circ} \left(\mathbf{e}_{12}^{\circ} + \varphi \sin \theta_{12} \right) \right] \mathbf{csc}^2 \theta_{12} \\ & - \left[\mathbf{e}_{11}^{\circ} \mathbf{e}_{22}^{\circ} + \left(\mathbf{e}_{12}^{\circ} + \varphi \sin \theta_{12} \right) \left(\mathbf{e}_{12}^{\circ} - \varphi \sin \theta_{12} \right) \right] \mathbf{csc} \theta_{12} \cot \theta_{12} \\ & + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) + \mathcal{O}(\varepsilon^4) \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

reduces to

$$2\boldsymbol{\varepsilon}_{12}^{\circ} = 2\mathbf{e}_{12}^{\circ} + \left(\varphi_1 + \varphi_2 \cos\theta_{12} \right) \left(\varphi_2 + \varphi_1 \cos\theta_{12} \right) \\ - \varphi^2 \cos\theta_{12} + \varphi \left(\mathbf{e}_{22}^{\circ} - \mathbf{e}_{11}^{\circ} \right) \csc\theta_{12} + \mathcal{O}\left(\theta^4, \varepsilon^4\right)$$

- In these strain expressions, $\mathcal{O}\left(\theta^4, \varepsilon^4\right)$ indicates that terms fourth order in the rotations and fourth order in the strains are neglected; that is,

$$\mathcal{O}\left(\theta^4, \varepsilon^4\right) \sim \mathcal{O}\left(\theta^4\right) + \mathcal{O}\left(\varepsilon^4\right)$$

- To obtain changes in reference-surface curvatures and torsions that have a consistent order-of-magnitude accuracy, it is useful to examine the Green-Lagrange shell strains

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- In particular, consider the following general expression for the Green-Lagrange strain $\boldsymbol{\varepsilon}_{11}$

$$\begin{aligned} \boldsymbol{\varepsilon}_{11} \left(\frac{\mathbf{H}_1}{\mathbf{A}_1} \right)^2 &= \boldsymbol{\varepsilon}_{11}^o \left[\left(\mathbf{1} + \frac{\xi_3}{\mathbf{r}_{11}} \right)^2 + \left(\frac{\xi_3}{\mathbf{r}_{12}} \right)^2 \right] + \xi_3 \mathbf{K}_{11}^o \left(\mathbf{1} + \frac{\xi_3}{\mathbf{r}_{11}} \right) \\ &+ \frac{1}{2} \left(\frac{\xi_3}{\mathbf{r}_{12}} \right) \left[2 \boldsymbol{\varepsilon}_{12}^o \left(\frac{\xi_3}{\mathbf{r}_{11}} - \frac{\xi_3}{\mathbf{r}_{22}} \right) \mathbf{csc} \theta_{12} - \left(\boldsymbol{\varepsilon}_{11}^o + \boldsymbol{\varepsilon}_{22}^o \right) \left(\frac{\xi_3}{\mathbf{r}_{12}} + \frac{\xi_3}{\mathbf{r}_{21}} \right) \right] \mathbf{csc}^2 \theta_{12} \\ &- \xi_3 \left(\frac{\xi_3}{\mathbf{r}_{12}} \right) \left[\mathbf{K}_{12}^o - \frac{1}{2} \cot \theta_{12} \left(\mathbf{K}_{11}^o - \mathbf{K}_{22}^o \right) \right] + \Gamma_{11} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

- Let ε^o denote that maximum magnitude of the reference-surface membrane strains that occur during deformation

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Examination of the expression for the Green-Lagrange strain ϵ_{11} indicates that the maximum bending strain contribution occurs on the parallel surface the farthest from the reference surface
- Thus, let ϵ^o_K denote that maximum bending strain of the shell with respect to the reference surface, where ϵ^o denote the maximum value of the shell thickness, $h(\xi_1, \xi_2)$
- Examination of the expression for the Green-Lagrange strain ϵ_{11} also indicates the relative proportions of membrane strain to bending strain is determined by the relative proportions of ϵ^o and ϵ^o_K
- The same observation applies to the other Green-Lagrange strains

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- If the magnitudes of the bending strains are on the same order as the magnitudes of the membrane strains, then it follows that

$$\mathcal{O}(\kappa^\circ) \equiv \mathcal{O}(\varepsilon^\circ)$$

- As a result, $\mathcal{O}(\kappa^\circ) \equiv \mathcal{O}\left(\frac{\varepsilon^\circ}{h}\right)$, which indicates that the changes in surface curvatures and torsions do not need to be represented to the same degree of accuracy as the membrane strains
- For “small” strains and “moderate” rotations, terms in the membrane strains given herein that are $\mathcal{O}(\theta^4, \varepsilon^4)$ are neglected

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Thus, in the presentation that follows, terms are neglected in quantities needed to obtain the changes in reference-surface curvatures and torsions that are $\mathcal{O}(\theta^3, \varepsilon^4)$ and smaller
- In addition, it is presumed that $\left| \frac{\partial \varphi_\alpha}{\partial \xi_\beta} \right| \leq \mathcal{O}(\theta)$ and $\left| \frac{\partial \varphi}{\partial \xi_\beta} \right| \leq \mathcal{O}(\theta)$
- The changes in the reference-surface curvatures and torsions, associated with “small” strains and “moderate” rotations, can be obtained by systematic simplification of the corresponding full nonlinear equations previously derived herein
- However, because of the practical significance of this special class of shell deformations, these quantities are re-derived subsequently from first principles, applying the order-of-magnitude simplifications at each step

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Hence, substituting the **simplified** membrane strain expressions for “small” strains and “moderate” rotations into

$$\mathcal{A}_1 = \mathbf{A}_1 \left[\mathbf{1} + \boldsymbol{\varepsilon}_{11}^o + \mathcal{O}(\varepsilon^4) \right]$$

$$\hat{\mathbf{a}}_1 = \frac{\mathbf{A}_1}{\mathcal{A}_1} \left[(1 + \Delta_{11}) \hat{\mathbf{a}}_1 + \Delta_{12} \hat{\mathbf{a}}_2 + \Delta_{13} \hat{\mathbf{n}} \right]$$

$$\mathcal{A}_2 = \mathbf{A}_2 \left[\mathbf{1} + \boldsymbol{\varepsilon}_{22}^o + \mathcal{O}(\varepsilon^4) \right]$$

$$\hat{\mathbf{a}}_2 = \frac{\mathbf{A}_2}{\mathcal{A}_2} \left[\Delta_{21} \hat{\mathbf{a}}_1 + (1 + \Delta_{22}) \hat{\mathbf{a}}_2 + \Delta_{23} \hat{\mathbf{n}} \right]$$

applying the **binomial theorem** and neglecting terms $\mathcal{O}(\theta^3, \varepsilon^4)$ and smaller gives

$$\begin{aligned} \hat{\mathbf{a}}_1 = & \left[1 - \frac{1}{2} (\varphi_1 + \varphi_2 \cos \theta_{12})^2 - \frac{1}{2} \varphi^2 + (\mathbf{e}_{11}^o \cot \theta_{12} - \mathbf{e}_{12}^o \csc \theta_{12} + \varphi) \cot \theta_{12} \right] \hat{\mathbf{a}}_1 \\ & + \left[(\mathbf{e}_{12}^o \csc \theta_{12} - \mathbf{e}_{11}^o \cot \theta_{12} + \varphi) \csc \theta_{12} \right] \hat{\mathbf{a}}_2 - \left[(\varphi_1 + \varphi_2 \cos \theta_{12}) \right] \hat{\mathbf{n}} + \overrightarrow{\mathcal{O}(\theta^3, \varepsilon^4)} \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

and

$$\begin{aligned}\hat{\mathbf{a}}_2 = & \left[1 - \frac{1}{2}(\varphi_2 + \varphi_1 \cos\theta_{12})^2 - \frac{1}{2}\varphi^2 + (\mathbf{e}_{22}^{\circ} \cot\theta_{12} - \mathbf{e}_{12}^{\circ} \csc\theta_{12} + \varphi) \cot\theta_{12} \right] \hat{\mathbf{a}}_2 \\ & + \left[(\mathbf{e}_{12}^{\circ} \csc\theta_{12} - \mathbf{e}_{22}^{\circ} \cot\theta_{12} - \varphi) \csc\theta_{12} \right] \hat{\mathbf{a}}_1 - \left[(\varphi_2 + \varphi_1 \cos\theta_{12}) \right] \hat{\mathbf{n}} + \overrightarrow{\mathcal{O}(\theta^3, \varepsilon^4)}\end{aligned}$$

where

$$\Delta_{11} = \mathbf{e}_{11}^{\circ} \csc^2\theta_{12} - (\mathbf{e}_{12}^{\circ} \csc\theta_{12} + \varphi) \cot\theta_{12}$$

$$\Delta_{12} = (\mathbf{e}_{12}^{\circ} \csc\theta_{12} + \varphi - \mathbf{e}_{11}^{\circ} \cot\theta_{12}) \csc\theta_{12}$$

$$\Delta_{13} = -(\varphi_1 + \varphi_2 \cos\theta_{12})$$

$$\Delta_{21} = (\mathbf{e}_{12}^{\circ} \csc\theta_{12} - \varphi - \mathbf{e}_{22}^{\circ} \cot\theta_{12}) \csc\theta_{12}$$

$$\Delta_{22} = \mathbf{e}_{22}^{\circ} \csc^2\theta_{12} - (\mathbf{e}_{12}^{\circ} \csc\theta_{12} - \varphi) \cot\theta_{12}$$

$$\Delta_{23} = -(\varphi_1 \cos\theta_{12} + \varphi_2)$$

have been used

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Similarly, $\frac{\sqrt{a}}{\sqrt{a}} = 1 + 2\varepsilon_{12}^{\circ} \cot\theta_{12} \csc\theta_{12} - (\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}) \csc^2\theta_{12} + \mathcal{O}(\varepsilon^4)$ reduces to

$$\frac{\sqrt{a}}{\sqrt{a}} = 1 - \frac{1}{2}(\varphi_1^2 + \varphi_2^2 + 2\varphi^2) - \varphi_1\varphi_2 \cos\theta_{12} + 2e_{12}^{\circ} \cot\theta_{12} \csc\theta_{12} - (e_{11}^{\circ} + e_{22}^{\circ}) \csc^2\theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

- Recalling that $\hat{\mathbf{a}} = \frac{\sqrt{a}}{\sqrt{a}} (\mathbf{m}_1 \hat{\mathbf{a}}_1 + \mathbf{m}_2 \hat{\mathbf{a}}_2 + \mathbf{m}_3 \hat{\mathbf{n}})$, where

$$\mathbf{m}_1 = \varphi_1 - (e_{12}^{\circ} \csc\theta_{12} + \varphi) (\varphi_1 \cot\theta_{12} + \varphi_2 \csc\theta_{12}) + e_{22}^{\circ} \csc\theta_{12} (\varphi_1 \csc\theta_{12} + \varphi_2 \cot\theta_{12})$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\begin{aligned} m_2 = \varphi_2 - \left(e_{12}^o \csc \theta_{12} - \varphi \right) \left(\varphi_1 \csc \theta_{12} + \varphi_2 \cot \theta_{12} \right) \\ + e_{11}^o \csc \theta_{12} \left(\varphi_1 \cot \theta_{12} + \varphi_2 \csc \theta_{12} \right) \end{aligned}$$

$$\begin{aligned} m_3 = 1 + \varphi^2 + \left(e_{11}^o + e_{22}^o + e_{11}^o e_{22}^o - (e_{12}^o)^2 \right) \csc^2 \theta_{12} \\ - 2e_{12}^o \cot \theta_{12} \csc \theta_{12} \end{aligned}$$

and simplifying for “small” strains and “moderate” rotations yields

$$m_1 = \varphi_1 - \varphi \left(\varphi_2 + \varphi_1 \cos \theta_{12} \right) \csc \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$m_2 = \varphi_2 + \varphi \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \csc \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$m_3 = 1 + \varphi^2 - 2e_{12}^o \cot \theta_{12} \csc \theta_{12} + (e_{11}^o + e_{22}^o) \csc^2 \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- In addition, the components of the vector field normal to the deformed reference surface, $\hat{\mathbf{n}} = \mathbf{n}_1 \hat{\mathbf{a}}_1 + \mathbf{n}_2 \hat{\mathbf{a}}_2 + \mathbf{n}_3 \hat{\mathbf{n}}$, reduce to

$$\mathbf{n}_1 = \varphi_1 - \varphi \left(\varphi_2 + \varphi_1 \cos \theta_{12} \right) \csc \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\mathbf{n}_2 = \varphi_2 + \varphi \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \csc \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\mathbf{n}_3 = 1 - \frac{1}{2}(\varphi_1^2 + \varphi_2^2) - \varphi_1 \varphi_2 \cos \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

- Moreover, the finite-rotation vector field reduces as follows:

$$\Omega_1 = \left[\varphi_1 \left(\cot \theta_{12} + \frac{1}{2} \varphi - \frac{1}{2} \mathbf{e}_{12}^o \sin \theta_{12} \right) + \varphi_2 \left(\csc \theta_{12} + \frac{1}{2} \mathbf{e}_{11}^o \sin \theta_{12} \right) \right] \left[1 + \mathcal{O}(\varepsilon^2) \right]$$

becomes

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\Omega_1 = \left[\varphi_1 \cot \theta_{12} + \varphi_2 \csc \theta_{12} + \frac{1}{2} \varphi_1 \varphi \right] \left[1 + \mathcal{O}(\theta^3, \varepsilon^2) \right]$$

$$\Omega_2 = \left[-\varphi_1 \left(\csc \theta_{12} + \frac{1}{2} \mathbf{e}_{22}^{\circ} \sin \theta_{12} \right) + \varphi_2 \left(\frac{1}{2} \mathbf{e}_{12}^{\circ} \sin \theta_{12} + \frac{1}{2} \varphi - \cot \theta_{12} \right) \right] \left[1 + \mathcal{O}(\varepsilon^2) \right]$$

becomes
$$\Omega_2 = \left[-\varphi_1 \csc \theta_{12} - \varphi_2 \cot \theta_{12} + \frac{1}{2} \varphi_2 \varphi \right] \left[1 + \mathcal{O}(\theta^3, \varepsilon^2) \right]$$

and $\Omega_3 = \varphi \left[1 + \mathcal{O}(\varepsilon^2) \right]$ becomes
$$\Omega_3 = \varphi \left[1 + \mathcal{O}(\theta^3, \varepsilon^2) \right]$$

- The next step in the derivation is to recall that the **deformed reference-surface curvatures** are given by

$$\frac{1}{r_{11}} = \hat{\mathbf{a}}_1 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_1} \quad \text{and} \quad \frac{1}{r_{22}} = \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_2}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- These expressions re-written as

$$\frac{1}{r_{11}} = \frac{A_1}{\mathcal{A}_1} \left(\hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \hat{u}}{\partial \xi_1} \right) = \left[1 - \varepsilon_{11}^o + \mathcal{O}(\varepsilon^4) \right] \left(\hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \hat{u}}{\partial \xi_1} \right) \quad \text{and}$$

$$\frac{1}{r_{22}} = \frac{A_2}{\mathcal{A}_2} \left(\hat{a}_2 \cdot \frac{1}{A_2} \frac{\partial \hat{u}}{\partial \xi_2} \right) = \left[1 - \varepsilon_{22}^o + \mathcal{O}(\varepsilon^4) \right] \left(\hat{a}_2 \cdot \frac{1}{A_2} \frac{\partial \hat{u}}{\partial \xi_2} \right)$$

and simplified by using the previous expressions for $\hat{a}_1$ and $\hat{a}_2$ to get

$$\frac{1}{r_{11}} = C_{11} \left(\hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \hat{u}}{\partial \xi_1} \right) + C_{12} \left(\hat{a}_2 \cdot \frac{1}{A_1} \frac{\partial \hat{u}}{\partial \xi_1} \right) + C_{13} \left(\hat{n} \cdot \frac{1}{A_1} \frac{\partial \hat{u}}{\partial \xi_1} \right) + \mathcal{O}(\theta^3, \varepsilon^4) \quad \text{and}$$

$$\frac{1}{r_{22}} = C_{21} \left(\hat{a}_1 \cdot \frac{1}{A_2} \frac{\partial \hat{u}}{\partial \xi_2} \right) + C_{22} \left(\hat{a}_2 \cdot \frac{1}{A_2} \frac{\partial \hat{u}}{\partial \xi_2} \right) + C_{23} \left(\hat{n} \cdot \frac{1}{A_2} \frac{\partial \hat{u}}{\partial \xi_2} \right) + \mathcal{O}(\theta^3, \varepsilon^4)$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

where

$$C_{11} = 1 - e_{11}^o - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)^2 - \varphi^2 - \left(e_{12}^o \csc \theta_{12} - e_{11}^o \cot \theta_{12} + \varphi \right) \cot \theta_{12}$$

$$C_{12} = \left(e_{12}^o \csc \theta_{12} - e_{11}^o \cot \theta_{12} + \varphi \right) \csc \theta_{12}$$

$$C_{13} = - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)$$

$$C_{22} = 1 - e_{22}^o - \left(\varphi_2 + \varphi_1 \cos \theta_{12} \right)^2 - \varphi^2 - \left(e_{12}^o \csc \theta_{12} - e_{22}^o \cot \theta_{12} - \varphi \right) \cot \theta_{12}$$

$$C_{21} = \left(e_{12}^o \csc \theta_{12} - e_{22}^o \cot \theta_{12} - \varphi \right) \csc \theta_{12}$$

$$C_{23} = - \left(\varphi_2 + \varphi_1 \cos \theta_{12} \right)$$

- Additionally, the **deformed reference-surface torsions** are given by

$$\frac{1}{r_{12}} = - \hat{\mathbf{a}}^2 \cdot \frac{1}{\mathcal{A}_1} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_1} \quad \text{and} \quad \frac{1}{r_{21}} = \hat{\mathbf{a}}^1 \cdot \frac{1}{\mathcal{A}_2} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_2}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- These expressions are re-written as

$$\frac{1}{r_{12}} = -\frac{A_1}{\mathcal{A}_1} \left(\hat{\mathbf{a}}^2 \cdot \frac{1}{A_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \right) = - \left[1 - \epsilon_{11}^o + \mathcal{O}(\epsilon^4) \right] \left(\hat{\mathbf{a}}^2 \cdot \frac{1}{A_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \right)$$

$$\frac{1}{r_{21}} = \frac{A_2}{\mathcal{A}_2} \left(\hat{\mathbf{a}}^1 \cdot \frac{1}{A_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \right) = \left[1 - \epsilon_{22}^o + \mathcal{O}(\epsilon^4) \right] \left(\hat{\mathbf{a}}^1 \cdot \frac{1}{A_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \right)$$

- From $\hat{\mathbf{a}}^1 \equiv \hat{\mathbf{a}}_2 \times \hat{\mathbf{n}}$ and $\hat{\mathbf{a}}^2 \equiv \hat{\mathbf{n}} \times \hat{\mathbf{a}}_1$, it follows that

$$\hat{\mathbf{a}}^1 = A_{11} \hat{\mathbf{a}}^1 + A_{12} \hat{\mathbf{a}}^2 + A_{13} \hat{\mathbf{n}} \quad \text{and} \quad \hat{\mathbf{a}}^2 = A_{21} \hat{\mathbf{a}}^1 + A_{22} \hat{\mathbf{a}}^2 + A_{23} \hat{\mathbf{n}}$$

where

$$A_{11} = 1 - \mathbf{e}_{22}^o - \frac{1}{2}(\varphi_1)^2 (1 + \cos^2 \theta_{12}) - \frac{1}{2} \varphi^2 - \varphi_1 \varphi_2 \cos \theta_{12} \\ + \left(\mathbf{e}_{22}^o \csc \theta_{12} - \mathbf{e}_{12}^o \cot \theta_{12} + \varphi \cos \theta_{12} \right) \csc \theta_{12} + \mathcal{O}(\theta^3, \epsilon^4)$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\mathbf{A}_{12} = -\varphi_1(\varphi_2 + \varphi_1 \cos\theta_{12}) + (\mathbf{e}_{22}^{\circ} \cot\theta_{12} - \mathbf{e}_{12}^{\circ} \csc\theta_{12} + \varphi) \csc\theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\mathbf{A}_{13} = -\varphi_1 \sin\theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\mathbf{A}_{21} = -\varphi_2(\varphi_1 + \varphi_2 \cos\theta_{12}) + (\mathbf{e}_{11}^{\circ} \cot\theta_{12} - \mathbf{e}_{12}^{\circ} \csc\theta_{12} - \varphi) \csc\theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\begin{aligned} \mathbf{A}_{22} = & 1 - \mathbf{e}_{11}^{\circ} - \frac{1}{2}(\varphi_2)^2(1 + \cos^2\theta_{12}) - \frac{1}{2}\varphi^2 - \varphi_1\varphi_2\cos\theta_{12} \\ & + (\mathbf{e}_{11}^{\circ} \csc\theta_{12} - \mathbf{e}_{12}^{\circ} \cot\theta_{12} - \varphi \cos\theta_{12}) \csc\theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

$$\mathbf{A}_{23} = -\varphi_2 \sin\theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

and

$$\hat{\mathbf{a}}^1 = \hat{\mathbf{a}}_1 \csc\theta_{12} - \hat{\mathbf{a}}_2 \cot\theta_{12}$$

$$\hat{\mathbf{a}}^2 = \hat{\mathbf{a}}_2 \csc\theta_{12} - \hat{\mathbf{a}}_1 \cot\theta_{12}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- The surface torsions are simplified to get

$$\frac{1}{\kappa_{12}} = D_{11} \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \right) + D_{12} \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \right) + D_{13} \left(\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \right) + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\frac{1}{\kappa_{21}} = D_{21} \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \right) + D_{22} \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \right) + D_{23} \left(\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \right) + \mathcal{O}(\theta^3, \varepsilon^4)$$

where

$$\begin{aligned} D_{11} = \varphi + \left(\mathbf{e}_{12}^0 - \varphi_1 \varphi_2 \cos 2\theta_{12} - \frac{1}{4} \varphi_2^2 \cos 3\theta_{12} \right) \csc \theta_{12} \\ + \left[1 - 2\mathbf{e}_{11}^0 - \frac{1}{4} (\varphi_2^2 + 2\varphi_1^2 + 4\varphi^2) \right] \cot \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\mathbf{D}_{12} = \varphi_1 \varphi_2 \cot \theta_{12} - \left[1 - \mathbf{e}_{11}^o - \frac{1}{2}(\varphi_1^2 + \varphi_2^2 + 2\varphi^2) \right] \csc \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\mathbf{D}_{13} = \varphi_2 \sin \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\mathbf{D}_{21} = -\varphi_1 \varphi_2 \cot \theta_{12} + \left[1 - \mathbf{e}_{22}^o - \frac{1}{2}(\varphi_1^2 + \varphi_2^2 + 2\varphi^2) \right] \csc \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\begin{aligned} \mathbf{D}_{22} = \varphi - & \left(\mathbf{e}_{12}^o - \varphi_1 \varphi_2 \cos 2\theta_{12} - \frac{1}{4} \varphi_1^2 \cos 3\theta_{12} \right) \csc \theta_{12} \\ & - \left[1 - 2\mathbf{e}_{22}^o - \frac{1}{4}(\varphi_1^2 + 2\varphi_2^2 + 4\varphi^2) \right] \cot \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

$$\mathbf{D}_{23} = -\varphi_1 \sin \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Examination of the components of $\hat{u} = u_1 \hat{a}_1 + u_2 \hat{a}_2 + u_3 \hat{n}$ suggests that it is beneficial to express this vector field as $\hat{u} = \hat{n} + \vec{\varphi} + \vec{\Psi}$, where $\vec{\varphi} = \varphi_1 \hat{a}_1 + \varphi_2 \hat{a}_2$ is the **linear-difference-vector field** defined previously herein and $\vec{\Psi} = \Psi_1 \hat{a}_1 + \Psi_2 \hat{a}_2 + \Psi_3 \hat{n}$ contains the nonlinear terms given by

$$\Psi_1 = -\varphi \left(\varphi_2 + \varphi_1 \cos \theta_{12} \right) \csc \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\Psi_2 = \varphi \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \csc \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\Psi_3 = -\frac{1}{2}(\varphi_1^2 + \varphi_2^2) - \varphi_1 \varphi_2 \cos \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Thus,

$$\frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} = \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} + \frac{1}{A_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} + \frac{1}{A_1} \frac{\partial \vec{\Psi}}{\partial \xi_1} \quad \text{and} \quad \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} = \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} + \frac{1}{A_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} + \frac{1}{A_2} \frac{\partial \vec{\Psi}}{\partial \xi_2}$$

- The components of these derivatives are simplified by using

$$\hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} = \frac{1}{r_{11}} \quad \hat{a}_2 \cdot \frac{1}{A_1} \frac{\partial \hat{n}}{\partial \xi_1} = \frac{\cos \theta_{12}}{r_{11}} - \frac{\sin \theta_{12}}{r_{12}}$$

$$\hat{a}_1 \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} = \frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \quad \hat{a}_2 \cdot \frac{1}{A_2} \frac{\partial \hat{n}}{\partial \xi_2} = \frac{1}{r_{22}}$$

and the previously defined linear deformation quantities

$$\chi_{11}^{\circ} \equiv \left(\hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) - \frac{\varphi}{r_{12}} \quad \chi_{22}^{\circ} \equiv \left(\hat{a}_2 \cdot \frac{1}{A_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) - \frac{\varphi}{r_{21}}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Moreover, it is convenient to introduce

$$\bar{\chi}_{12}^{\circ} \equiv \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) \quad \text{and} \quad \bar{\chi}_{21}^{\circ} \equiv \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) \quad \text{such that}$$

$$\bar{\chi}_{12}^{\circ} + \bar{\chi}_{21}^{\circ} = 2\chi_{12}^{\circ} - \left(\frac{\varphi}{r_{22}} - \frac{\varphi}{r_{11}} \right) \sin\theta_{12} + \left(\frac{\varphi}{r_{21}} + \frac{\varphi}{r_{12}} \right) \cos\theta_{12}$$

where $2\chi_{12}^{\circ}$ is the following previously defined linear deformation quantity given by

$$2\chi_{12}^{\circ} \equiv \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) + \left(\frac{\varphi}{r_{22}} - \frac{\varphi}{r_{11}} \right) \sin\theta_{12} - \left(\frac{\varphi}{r_{21}} + \frac{\varphi}{r_{12}} \right) \cos\theta_{12}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS

CONTINUED

- Using the previously derived general expressions for the components of the derivatives of a vector field given by

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{A_1} \frac{\partial \vec{V}}{\partial \xi_1} = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (V_1 + V_2 \cos \theta_{12}) - \frac{V_2 \sin \theta_{12}}{\rho_{11}} + \frac{V_3}{r_{11}}$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{A_1} \frac{\partial \vec{V}}{\partial \xi_1} = \left(\frac{1}{A_1} \frac{\partial V_1}{\partial \xi_1} + \frac{V_3}{r_{11}} \right) \cos \theta_{12} + \frac{1}{A_1} \frac{\partial V_2}{\partial \xi_1} + \left(\frac{V_1}{\rho_{11}} - \frac{V_3}{r_{12}} \right) \sin \theta_{12}$$

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{A_2} \frac{\partial \vec{V}}{\partial \xi_2} = \frac{1}{A_2} \frac{\partial V_1}{\partial \xi_2} + \left(\frac{1}{A_2} \frac{\partial V_2}{\partial \xi_2} + \frac{V_3}{r_{22}} \right) \cos \theta_{12} + \left(\frac{V_3}{r_{21}} - \frac{V_2}{\rho_{22}} \right) \sin \theta_{12}$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{A_2} \frac{\partial \vec{V}}{\partial \xi_2} = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (V_2 + V_1 \cos \theta_{12}) + \frac{V_1 \sin \theta_{12}}{\rho_{22}} + \frac{V_3}{r_{22}}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

yields

$$\chi_{11}^{\circ} \equiv \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) - \frac{\varphi}{\mathbf{r}_{12}} = \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) - \frac{\varphi_2 \sin \theta_{12}}{\rho_{11}} - \frac{\varphi}{\mathbf{r}_{12}}$$

$$\bar{\chi}_{12}^{\circ} \equiv \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) = \frac{\cos \theta_{12}}{\mathbf{A}_1} \frac{\partial \varphi_1}{\partial \xi_1} + \frac{1}{\mathbf{A}_1} \frac{\partial \varphi_2}{\partial \xi_1} + \frac{\varphi_1 \sin \theta_{12}}{\rho_{11}}$$

$$\bar{\chi}_{21}^{\circ} \equiv \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) = \frac{1}{\mathbf{A}_2} \frac{\partial \varphi_1}{\partial \xi_2} + \frac{\cos \theta_{12}}{\mathbf{A}_2} \frac{\partial \varphi_2}{\partial \xi_2} - \frac{\varphi_2 \sin \theta_{12}}{\rho_{22}}$$

$$\chi_{22}^{\circ} \equiv \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) - \frac{\varphi}{\mathbf{r}_{21}} = \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} \left(\varphi_2 + \varphi_1 \cos \theta_{12} \right) + \frac{\varphi_1 \sin \theta_{12}}{\rho_{22}} - \frac{\varphi}{\mathbf{r}_{21}}$$

and

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\begin{aligned}
 2\chi_{12}^{\circ} &\equiv \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\varphi}}{\partial \xi_2} \right) + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\varphi}}{\partial \xi_1} \right) + \left(\frac{\varphi}{r_{22}} - \frac{\varphi}{r_{11}} \right) \sin \theta_{12} - \left(\frac{\varphi}{r_{21}} + \frac{\varphi}{r_{12}} \right) \cos \theta_{12} \\
 &= \frac{1}{\mathbf{A}_2} \frac{\partial \varphi_1}{\partial \xi_2} + \frac{\cos \theta_{12}}{\mathbf{A}_2} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\cos \theta_{12}}{\mathbf{A}_1} \frac{\partial \varphi_1}{\partial \xi_1} + \frac{1}{\mathbf{A}_1} \frac{\partial \varphi_2}{\partial \xi_1} \\
 &\quad + \left(\frac{\varphi_1}{\rho_{11}} - \frac{\varphi_2}{\rho_{22}} \right) \sin \theta_{12} + \left(\frac{\varphi}{r_{22}} - \frac{\varphi}{r_{11}} \right) \sin \theta_{12} - \left(\frac{\varphi}{r_{21}} + \frac{\varphi}{r_{12}} \right) \cos \theta_{12}
 \end{aligned}$$

- In addition, $\vec{\varphi} \cdot \hat{\mathbf{n}} = 0$ yields the identity $\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \vec{\varphi}}{\partial \xi_{\alpha}} = - \vec{\varphi} \cdot \frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial \hat{\mathbf{n}}}{\partial \xi_{\alpha}}$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Using these expressions gives

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_1} = \frac{1}{r_{11}} + \chi_{11}^{\circ} + \frac{\varphi}{r_{12}} + \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\Psi}}{\partial \xi_1} \right)$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_1} = \bar{\chi}_{12}^{\circ} + \frac{\cos \theta_{12}}{r_{11}} - \frac{\sin \theta_{12}}{r_{12}} + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\Psi}}{\partial \xi_1} \right)$$

$$\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_1} = -\frac{\varphi_1}{r_{11}} + \varphi_2 \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right) + \left(\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\Psi}}{\partial \xi_1} \right)$$

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{a}}}{\partial \xi_2} = \bar{\chi}_{21}^{\circ} + \frac{\cos \theta_{12}}{r_{22}} + \frac{\sin \theta_{12}}{r_{21}} + \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\Psi}}{\partial \xi_2} \right)$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} = \frac{1}{r_{22}} + \chi_{22}^o + \frac{\varphi}{r_{21}} + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\Psi}}{\partial \xi_2} \right)$$

$$\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} = -\varphi_1 \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) - \frac{\varphi_2}{r_{22}} + \left(\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\Psi}}{\partial \xi_2} \right)$$

- The expressions

$$\frac{1}{r_{11}} = \mathbf{C}_{11} \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \right) + \mathbf{C}_{12} \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \right) + \mathbf{C}_{13} \left(\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_1} \right) + \mathcal{O}(\theta^3, \varepsilon^4) \quad \text{and}$$

$$\frac{1}{r_{22}} = \mathbf{C}_{21} \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \right) + \mathbf{C}_{22} \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \right) + \mathbf{C}_{23} \left(\hat{\mathbf{n}} \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \hat{\mathbf{u}}}{\partial \xi_2} \right) + \mathcal{O}(\theta^3, \varepsilon^4)$$

reduce to

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\begin{aligned} \frac{1}{r_{11}} = & \chi_{11}^{\circ} + \frac{1 - e_{11}^{\circ}}{r_{11}} + \frac{e_{11}^{\circ} \cot \theta_{12} - e_{12}^{\circ} \csc \theta_{12}}{r_{12}} - \left(\frac{\varphi_1 \varphi_2 + \varphi_2^2 \cos \theta_{12}}{r_{12}} \right) \sin \theta_{12} \\ & - \varphi^2 \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) + \varphi \left(\bar{\chi}_{12}^{\circ} \csc \theta_{12} - \chi_{11}^{\circ} \cot \theta_{12} \right) + \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\Psi}}{\partial \xi_1} \right) + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{r_{22}} = & \chi_{22}^{\circ} + \frac{1 - e_{22}^{\circ}}{r_{22}} + \frac{e_{12}^{\circ} \csc \theta_{12} - e_{22}^{\circ} \cot \theta_{12}}{r_{21}} + \left(\frac{\varphi_1 \varphi_2 + \varphi_1^2 \cos \theta_{12}}{r_{21}} \right) \sin \theta_{12} \\ & - \varphi^2 \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right) + \varphi \left(\chi_{22}^{\circ} \cot \theta_{12} - \bar{\chi}_{21}^{\circ} \csc \theta_{12} \right) + \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\Psi}}{\partial \xi_2} \right) + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Similarly,

$$\frac{1}{r_{12}} = D_{11} \left(\hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \hat{u}}{\partial \xi_1} \right) + D_{12} \left(\hat{a}_2 \cdot \frac{1}{A_1} \frac{\partial \hat{u}}{\partial \xi_1} \right) + D_{13} \left(\hat{n} \cdot \frac{1}{A_1} \frac{\partial \hat{u}}{\partial \xi_1} \right) + \mathcal{O}(\theta^3, \epsilon^4)$$

$$\frac{1}{r_{21}} = D_{21} \left(\hat{a}_1 \cdot \frac{1}{A_2} \frac{\partial \hat{u}}{\partial \xi_2} \right) + D_{22} \left(\hat{a}_2 \cdot \frac{1}{A_2} \frac{\partial \hat{u}}{\partial \xi_2} \right) + D_{23} \left(\hat{n} \cdot \frac{1}{A_2} \frac{\partial \hat{u}}{\partial \xi_2} \right) + \mathcal{O}(\theta^3, \epsilon^4)$$

reduce to

$$\begin{aligned} \frac{1}{r_{12}} = & \frac{1}{r_{12}} + \varphi \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) + \chi_{11}^{\circ} \cot \theta_{12} - \bar{\chi}_{12}^{\circ} \csc \theta_{12} + \frac{e_{12}^{\circ} \csc \theta_{12} - e_{11}^{\circ} \cot \theta_{12}}{r_{11}} \\ & - \frac{e_{11}^{\circ}}{r_{12}} - \frac{1}{2r_{12}} \left[\varphi_1^2 + \varphi_2^2 \cos 2\theta_{12} + \varphi_1 \varphi_2 \csc \theta_{12} \sin 2\theta_{12} \right] \\ & + \varphi \chi_{11}^{\circ} + \left(\hat{a}_1 \cdot \frac{1}{A_1} \frac{\partial \vec{\Psi}}{\partial \xi_1} \right) \cot \theta_{12} - \left(\hat{a}_2 \cdot \frac{1}{A_1} \frac{\partial \vec{\Psi}}{\partial \xi_1} \right) \csc \theta_{12} + \mathcal{O}(\theta^3, \epsilon^4) \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

and

$$\begin{aligned} \frac{1}{r_{21}} = & \frac{1}{r_{21}} + \varphi \left(\frac{1}{r_{22}} - \frac{\cot\theta_{12}}{r_{21}} \right) - \chi_{22}^{\circ} \cot\theta_{12} + \bar{\chi}_{21}^{\circ} \csc\theta_{12} + \frac{\mathbf{e}_{22}^{\circ} \cot\theta_{12} - \mathbf{e}_{12}^{\circ} \csc\theta_{12}}{r_{22}} \\ & - \frac{\mathbf{e}_{22}^{\circ}}{r_{21}} - \frac{1}{2r_{21}} \left[\varphi_2^2 + \varphi_1^2 \cos 2\theta_{12} + \varphi_1 \varphi_2 \csc\theta_{12} \sin 2\theta_{12} \right] \\ & + \varphi \chi_{22}^{\circ} + \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\Psi}}{\partial \xi_2} \right) \csc\theta_{12} - \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\Psi}}{\partial \xi_2} \right) \cot\theta_{12} + \mathcal{O}(\theta^3, \epsilon^4) \end{aligned}$$

● Next,

$$\Psi_1 = -\varphi \left(\varphi_2 + \varphi_1 \cos\theta_{12} \right) \csc\theta_{12} + \mathcal{O}(\theta^3, \epsilon^4)$$

$$\Psi_2 = \varphi \left(\varphi_1 + \varphi_2 \cos\theta_{12} \right) \csc\theta_{12} + \mathcal{O}(\theta^3, \epsilon^4)$$

$$\Psi_3 = -\frac{1}{2} \left(\varphi_1^2 + \varphi_2^2 \right) - \varphi_1 \varphi_2 \cos\theta_{12} + \mathcal{O}(\theta^3, \epsilon^4)$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

are substituted into

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\Psi}}{\partial \xi_1} = \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} (\Psi_1 + \Psi_2 \cos \theta_{12}) - \frac{\Psi_2 \sin \theta_{12}}{\rho_{11}} + \frac{\Psi_3}{r_{11}}$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\Psi}}{\partial \xi_1} = \left(\frac{1}{\mathbf{A}_1} \frac{\partial \Psi_1}{\partial \xi_1} + \frac{\Psi_3}{r_{11}} \right) \cos \theta_{12} + \frac{1}{\mathbf{A}_1} \frac{\partial \Psi_2}{\partial \xi_1} + \left(\frac{\Psi_1}{\rho_{11}} - \frac{\Psi_3}{r_{12}} \right) \sin \theta_{12}$$

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\Psi}}{\partial \xi_2} = \frac{1}{\mathbf{A}_2} \frac{\partial \Psi_1}{\partial \xi_2} + \left(\frac{1}{\mathbf{A}_2} \frac{\partial \Psi_2}{\partial \xi_2} + \frac{\Psi_3}{r_{22}} \right) \cos \theta_{12} + \left(\frac{\Psi_3}{r_{21}} - \frac{\Psi_2}{\rho_{22}} \right) \sin \theta_{12}$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\Psi}}{\partial \xi_2} = \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} (\Psi_2 + \Psi_1 \cos \theta_{12}) + \frac{\Psi_1 \sin \theta_{12}}{\rho_{22}} + \frac{\Psi_3}{r_{22}}$$

and simplified to get

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\Psi}}{\partial \xi_1} = -\frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} (\varphi \varphi_2 \sin \theta_{12}) - \frac{\varphi}{\rho_{11}} (\varphi_1 + \varphi_2 \cos \theta_{12})$$

$$- \frac{1}{2r_{11}} (\varphi_1^2 + \varphi_2^2 + 2\varphi_1 \varphi_2 \cos \theta_{12}) + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\Psi}}{\partial \xi_1} = \frac{\sin \theta_{12}}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} (\varphi \varphi_1) + \varphi \varphi_2 \csc \theta_{12} \frac{1}{\mathbf{A}_1} \frac{\partial \cos \theta_{12}}{\partial \xi_1} - \frac{\varphi}{\rho_{11}} (\varphi_2 + \varphi_1 \cos \theta_{12})$$

$$+ \frac{1}{2} \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right) (\varphi_1^2 + \varphi_2^2 + 2\varphi_1 \varphi_2 \cos \theta_{12}) + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\Psi}}{\partial \xi_2} = -\frac{\sin \theta_{12}}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} (\varphi \varphi_2) - \varphi \varphi_1 \csc \theta_{12} \frac{1}{\mathbf{A}_2} \frac{\partial \cos \theta_{12}}{\partial \xi_2} - \frac{\varphi}{\rho_{22}} (\varphi_1 + \varphi_2 \cos \theta_{12})$$

$$- \frac{1}{2} \left(\frac{\cos \theta_{12}}{r_{22}} + \frac{\sin \theta_{12}}{r_{21}} \right) (\varphi_1^2 + \varphi_2^2 + 2\varphi_1 \varphi_2 \cos \theta_{12}) + \mathcal{O}(\theta^3, \varepsilon^4)$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\begin{aligned} \hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\Psi}}{\partial \xi_2} = & \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} (\varphi \varphi_1 \sin \theta_{12}) - \frac{\varphi}{\rho_{22}} (\varphi_2 + \varphi_1 \cos \theta_{12}) \\ & - \frac{1}{2r_{22}} (\varphi_1^2 + \varphi_2^2 + 2\varphi_1 \varphi_2 \cos \theta_{12}) + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

- Moreover,

$$\begin{aligned} & \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\Psi}}{\partial \xi_1} \right) \cot \theta_{12} - \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_1} \frac{\partial \vec{\Psi}}{\partial \xi_1} \right) \csc \theta_{12} = \\ & - \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} (\varphi \varphi_2 \cos \theta_{12}) - \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} (\varphi \varphi_1) \\ & + \frac{\varphi \varphi_2}{\rho_{11}} \sin \theta_{12} - \frac{1}{2r_{12}} (\varphi_1^2 + \varphi_2^2 + 2\varphi_1 \varphi_2 \cos \theta_{12}) + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

and

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\begin{aligned} & \left(\hat{\mathbf{a}}_1 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\Psi}}{\partial \xi_2} \right) \csc \theta_{12} - \left(\hat{\mathbf{a}}_2 \cdot \frac{1}{\mathbf{A}_2} \frac{\partial \vec{\Psi}}{\partial \xi_2} \right) \cot \theta_{12} = \\ & - \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} (\varphi \varphi_1 \cos \theta_{12}) - \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} (\varphi \varphi_2) \\ & - \frac{\varphi \varphi_1}{\rho_{22}} \sin \theta_{12} - \frac{1}{2r_{21}} (\varphi_1^2 + \varphi_2^2 + 2\varphi_1 \varphi_2 \cos \theta_{12}) + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

- Furthermore,

$$\bar{\chi}_{12}^{\circ} \csc \theta_{12} - \chi_{11}^{\circ} \cot \theta_{12} = \frac{1}{\mathbf{A}_1} \frac{\partial}{\partial \xi_1} (\varphi_2 \sin \theta_{12}) + \frac{\varphi_1 + \varphi_2 \cos \theta_{12}}{\rho_{11}} + \frac{\varphi \cot \theta_{12}}{r_{12}} \quad \text{and}$$

$$\chi_{22}^{\circ} \cot \theta_{12} - \bar{\chi}_{21}^{\circ} \csc \theta_{12} = \frac{\varphi_1 \cos \theta_{12} + \varphi_2}{\rho_{22}} - \frac{\varphi \cot \theta_{12}}{r_{21}} - \frac{1}{\mathbf{A}_2} \frac{\partial}{\partial \xi_2} (\varphi_1 \sin \theta_{12})$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Substituting these simplified expressions, along with

$$\chi_{11}^{\circ} = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (\varphi_1 + \varphi_2 \cos \theta_{12}) - \frac{\varphi_2 \sin \theta_{12}}{\rho_{11}} - \frac{\varphi}{r_{12}} \quad \text{and}$$

$$\chi_{22}^{\circ} = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\varphi_2 + \varphi_1 \cos \theta_{12}) + \frac{\varphi_1 \sin \theta_{12}}{\rho_{22}} - \frac{\varphi}{r_{21}},$$

into the most recent expressions for the deformed reference-surface curvatures and torsions yields

$$\begin{aligned} \frac{1}{r_{11}} = \frac{1}{r_{11}} + \chi_{11}^{\circ} + \frac{1}{r_{12}} & \left[e_{11}^{\circ} \cot \theta_{12} - e_{12}^{\circ} \csc \theta_{12} - (\varphi_1 \varphi_2 + \varphi_2^2 \cos \theta_{12}) \sin \theta_{12} \right] \\ & - \frac{1}{2r_{11}} \left(2e_{11}^{\circ} + \varphi_1^2 + \varphi_2^2 + 2\varphi^2 + 2\varphi_1 \varphi_2 \cos \theta_{12} \right) - \frac{\varphi_2 \sin \theta_{12}}{A_1} \frac{\partial \varphi}{\partial \xi_1} + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\frac{1}{r_{22}} = \frac{1}{r_{22}} + \chi_{22}^{\circ} + \frac{1}{r_{21}} \left[\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \mathbf{e}_{22}^{\circ} \cot \theta_{12} + (\varphi_1 \varphi_2 + \varphi_1^2 \cos \theta_{12}) \sin \theta_{12} \right] \\ - \frac{1}{2r_{22}} \left[2\mathbf{e}_{22}^{\circ} + \varphi_1^2 + \varphi_2^2 + 2\varphi^2 + 2\varphi_1 \varphi_2 \cos \theta_{12} \right] + \frac{\varphi_1 \sin \theta_{12}}{\mathbf{A}_2} \frac{\partial \varphi}{\partial \xi_2} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\frac{1}{r_{12}} = \frac{1}{r_{12}} + \varphi \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) + \chi_{11}^{\circ} \cot \theta_{12} - \bar{\chi}_{12}^{\circ} \csc \theta_{12} + \frac{\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \mathbf{e}_{11}^{\circ} \cot \theta_{12}}{r_{11}} \\ - \frac{\mathbf{e}_{11}^{\circ}}{r_{12}} - \frac{1}{r_{12}} \left[(\varphi_1 + \varphi_2 \cos \theta_{12})^2 + \varphi^2 \right] - (\varphi_1 + \varphi_2 \cos \theta_{12}) \frac{1}{\mathbf{A}_1} \frac{\partial \varphi}{\partial \xi_1} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\frac{1}{r_{21}} = \frac{1}{r_{21}} + \varphi \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right) - \chi_{22}^{\circ} \cot \theta_{12} + \bar{\chi}_{21}^{\circ} \csc \theta_{12} + \frac{\mathbf{e}_{22}^{\circ} \cot \theta_{12} - \mathbf{e}_{12}^{\circ} \csc \theta_{12}}{r_{22}} \\ - \frac{\mathbf{e}_{22}^{\circ}}{r_{21}} - \frac{1}{r_{21}} \left[(\varphi_2 + \varphi_1 \cos \theta_{12})^2 + \varphi^2 \right] - (\varphi_2 + \varphi_1 \cos \theta_{12}) \frac{1}{\mathbf{A}_2} \frac{\partial \varphi}{\partial \xi_2} + \mathcal{O}(\theta^3, \varepsilon^4)$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- The changes in reference-surface curvatures, $\mathcal{K}_{11}^{\circ}(\xi_1, \xi_2, \boldsymbol{\epsilon})$ and $\mathcal{K}_{22}^{\circ}(\xi_1, \xi_2, \boldsymbol{\epsilon})$, and the change in reference-surface torsion, $\mathcal{K}_{12}^{\circ}(\xi_1, \xi_2, \boldsymbol{\epsilon})$, caused by deformation have been defined herein by

$$\mathcal{K}_{11}^{\circ} \equiv \frac{1}{\boldsymbol{r}_{11}} - \frac{1}{\boldsymbol{r}_{11}}$$

$$\mathcal{K}_{22}^{\circ} \equiv \frac{1}{\boldsymbol{r}_{22}} - \frac{1}{\boldsymbol{r}_{22}}$$

$$\mathcal{K}_{12}^{\circ} \equiv - \left[\frac{1}{2} \left(\frac{1}{\boldsymbol{r}_{12}} - \frac{1}{\boldsymbol{r}_{21}} \right) - \frac{1}{2} \left(\frac{1}{\boldsymbol{r}_{12}} - \frac{1}{\boldsymbol{r}_{21}} \right) \right]$$

- From these definitions, it follows that

$$\begin{aligned} \mathcal{K}_{11}^{\circ} = & \chi_{11}^{\circ} + \frac{1}{\boldsymbol{r}_{12}} \left[\boldsymbol{e}_{11}^{\circ} \cot \theta_{12} - \boldsymbol{e}_{12}^{\circ} \csc \theta_{12} - \left(\varphi_1 \varphi_2 + \varphi_2^2 \cos \theta_{12} \right) \sin \theta_{12} \right] \\ & - \frac{1}{2\boldsymbol{r}_{11}} \left(2\boldsymbol{e}_{11}^{\circ} + \varphi_1^2 + \varphi_2^2 + 2\varphi^2 + 2\varphi_1 \varphi_2 \cos \theta_{12} \right) - \frac{\varphi_2 \sin \theta_{12}}{\boldsymbol{A}_1} \frac{\partial \varphi}{\partial \xi_1} + \mathcal{O}(\theta^3, \epsilon^4) \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\begin{aligned} \mathbf{K}_{22}^{\circ} = & \boldsymbol{\chi}_{22}^{\circ} + \frac{1}{\mathbf{r}_{21}} \left[\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \mathbf{e}_{22}^{\circ} \cot \theta_{12} + (\varphi_1 \varphi_2 + \varphi_1^2 \cos \theta_{12}) \sin \theta_{12} \right] \\ & - \frac{1}{2\mathbf{r}_{22}} \left[2\mathbf{e}_{22}^{\circ} + \varphi_1^2 + \varphi_2^2 + 2\varphi^2 + 2\varphi_1 \varphi_2 \cos \theta_{12} \right] + \frac{\varphi_1 \sin \theta_{12}}{\mathbf{A}_2} \frac{\partial \varphi}{\partial \xi_2} + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

and

$$\begin{aligned} 2\mathbf{K}_{12}^{\circ} = & \left(\bar{\boldsymbol{\chi}}_{12}^{\circ} + \bar{\boldsymbol{\chi}}_{21}^{\circ} \right) \csc \theta_{12} - \left(\boldsymbol{\chi}_{11}^{\circ} + \boldsymbol{\chi}_{22}^{\circ} \right) \cot \theta_{12} + \varphi \left(\frac{1}{\mathbf{r}_{22}} - \frac{1}{\mathbf{r}_{11}} \right) \csc^2 \theta_{12} \\ & + \frac{\mathbf{e}_{11}^{\circ}}{\mathbf{r}_{12}} - \frac{\mathbf{e}_{22}^{\circ}}{\mathbf{r}_{21}} - \frac{\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \mathbf{e}_{11}^{\circ} \cot \theta_{12}}{\mathbf{r}_{11}} + \frac{\mathbf{e}_{22}^{\circ} \cot \theta_{12} - \mathbf{e}_{12}^{\circ} \csc \theta_{12}}{\mathbf{r}_{22}} \\ & + \frac{1}{\mathbf{r}_{12}} \left[\left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)^2 + \varphi^2 \right] - \frac{1}{\mathbf{r}_{21}} \left[\left(\varphi_2 + \varphi_1 \cos \theta_{12} \right)^2 + \varphi^2 \right] \\ & + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \frac{1}{\mathbf{A}_1} \frac{\partial \varphi}{\partial \xi_1} - \left(\varphi_2 + \varphi_1 \cos \theta_{12} \right) \frac{1}{\mathbf{A}_2} \frac{\partial \varphi}{\partial \xi_2} + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

where the identity $\frac{1}{r_{12}} + \frac{1}{r_{21}} = \cot\theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$ has been used

- Next,

$$\bar{\chi}_{12}^{\circ} + \bar{\chi}_{21}^{\circ} = 2\chi_{12}^{\circ} - \left(\frac{\varphi}{r_{22}} - \frac{\varphi}{r_{11}} \right) \sin\theta_{12} + \left(\frac{\varphi}{r_{21}} + \frac{\varphi}{r_{12}} \right) \cos\theta_{12}$$

is used with the previous identity to get

$$\left(\bar{\chi}_{12}^{\circ} + \bar{\chi}_{21}^{\circ} \right) \csc\theta_{12} = 2\chi_{12}^{\circ} \csc\theta_{12} + \varphi \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc^2\theta_{12}$$

- Substituting this expression into the previous expression for the change in reference-surface torsion yields

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\begin{aligned}
 2K_{12}^{\circ} = & 2\chi_{12}^{\circ} \csc \theta_{12} - (\chi_{11}^{\circ} + \chi_{22}^{\circ}) \cot \theta_{12} + \frac{\mathbf{e}_{11}^{\circ}}{\mathbf{r}_{12}} - \frac{\mathbf{e}_{22}^{\circ}}{\mathbf{r}_{21}} \\
 & - \frac{\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \mathbf{e}_{11}^{\circ} \cot \theta_{12}}{\mathbf{r}_{11}} + \frac{\mathbf{e}_{22}^{\circ} \cot \theta_{12} - \mathbf{e}_{12}^{\circ} \csc \theta_{12}}{\mathbf{r}_{22}} \\
 & + \frac{1}{\mathbf{r}_{12}} \left[(\varphi_1 + \varphi_2 \cos \theta_{12})^2 + \varphi^2 \right] - \frac{1}{\mathbf{r}_{21}} \left[(\varphi_2 + \varphi_1 \cos \theta_{12})^2 + \varphi^2 \right] \\
 & + (\varphi_1 + \varphi_2 \cos \theta_{12}) \frac{1}{\mathbf{A}_1} \frac{\partial \varphi}{\partial \xi_1} - (\varphi_2 + \varphi_1 \cos \theta_{12}) \frac{1}{\mathbf{A}_2} \frac{\partial \varphi}{\partial \xi_2} + \mathcal{O}(\theta^3, \varepsilon^4)
 \end{aligned}$$

- These equations for the changes in reference-surface curvatures and torsions correspond to a “small” strain and “moderate” rotation theory, like that of Pietraszkiewicz, that includes nonlinear bending action

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- For orthogonal reference-surface Gaussian coordinates,

$$\mathbf{K}_{11}^{\circ} = \mathbf{X}_{11}^{\circ} - \frac{\mathbf{e}_{11}^{\circ}}{r_{11}} - \frac{\mathbf{e}_{12}^{\circ}}{r_{12}} - \left\{ \frac{1}{2r_{11}} (\varphi_1^2 + \varphi_2^2 + 2\varphi^2) + \frac{\varphi_1\varphi_2}{r_{12}} + \frac{\varphi_2}{\mathbf{A}_1} \frac{\partial\varphi}{\partial\xi_1} \right\} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\mathbf{K}_{22}^{\circ} = \mathbf{X}_{22}^{\circ} - \frac{\mathbf{e}_{22}^{\circ}}{r_{22}} - \frac{\mathbf{e}_{12}^{\circ}}{r_{12}} - \left\{ \frac{1}{2r_{22}} (\varphi_1^2 + \varphi_2^2 + 2\varphi^2) + \frac{\varphi_1\varphi_2}{r_{12}} - \frac{\varphi_1}{\mathbf{A}_2} \frac{\partial\varphi}{\partial\xi_2} \right\} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\mathbf{K}_{12}^{\circ} = \mathbf{X}_{12}^{\circ} + \frac{\mathbf{e}_{11}^{\circ} + \mathbf{e}_{22}^{\circ}}{2r_{12}} - \frac{\mathbf{e}_{12}^{\circ}}{2} \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) + \frac{1}{2} \left\{ \frac{1}{r_{12}} (\varphi_1^2 + \varphi_2^2 + 2\varphi^2) + \frac{\varphi_1}{\mathbf{A}_1} \frac{\partial\varphi}{\partial\xi_1} - \frac{\varphi_2}{\mathbf{A}_2} \frac{\partial\varphi}{\partial\xi_2} \right\} + \mathcal{O}(\theta^3, \varepsilon^4)$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Further simplifications to the changes in reference-surface curvatures and torsion are obtained by examining the terms:

- $$\frac{\varphi_2 \sin \theta_{12}}{A_1} \frac{\partial \varphi}{\partial \xi_1}$$
 appearing in K_{11}^o

- $$\frac{\varphi_1 \sin \theta_{12}}{A_2} \frac{\partial \varphi}{\partial \xi_2}$$
 appearing in K_{22}^o

- $$\left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \frac{1}{A_1} \frac{\partial \varphi}{\partial \xi_1} - \left(\varphi_2 + \varphi_1 \cos \theta_{12} \right) \frac{1}{A_2} \frac{\partial \varphi}{\partial \xi_2}$$
 appearing in $2K_{12}^o$

- In particular, these simplifications are facilitated by considering the requirement for a single-valued reference-surface displacement vector field; that is

$$\frac{\partial}{\partial \xi_2} \left(\frac{\partial \vec{u}}{\partial \xi_1} \right) = \frac{\partial}{\partial \xi_1} \left(\frac{\partial \vec{u}}{\partial \xi_2} \right)$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- By using

$$\frac{1}{\mathbf{A}_1} \frac{\partial \vec{u}}{\partial \xi_1} = \Delta_{11} \hat{\mathbf{a}}_1 + \Delta_{12} \hat{\mathbf{a}}_2 + \Delta_{13} \hat{\mathbf{n}} \quad \text{and} \quad \frac{1}{\mathbf{A}_2} \frac{\partial \vec{u}}{\partial \xi_2} = \Delta_{21} \hat{\mathbf{a}}_1 + \Delta_{22} \hat{\mathbf{a}}_2 + \Delta_{23} \hat{\mathbf{n}}$$

and the general expression for the derivatives of the unit-magnitude base vector fields, $\frac{\partial}{\partial \xi_2} \left(\frac{\partial \vec{u}}{\partial \xi_1} \right) = \frac{\partial}{\partial \xi_1} \left(\frac{\partial \vec{u}}{\partial \xi_2} \right)$ gives the following three scalar equations

$$\begin{aligned} \frac{1}{\mathbf{A}_2} \frac{\partial \Delta_{11}}{\partial \xi_2} - \frac{1}{\mathbf{A}_1} \frac{\partial \Delta_{21}}{\partial \xi_1} &= \frac{\Delta_{21}}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1} - \frac{\Delta_{11}}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} + \frac{1}{\rho_{22}} \left(\Delta_{12} \csc \theta_{12} + \Delta_{11} \cot \theta_{12} \right) \\ &+ \Delta_{23} \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right) - \frac{\Delta_{13} \csc \theta_{12}}{r_{21}} - \frac{1}{\rho_{11}} \left(\Delta_{21} \cot \theta_{12} + \Delta_{22} \csc \theta_{12} \right) \\ &+ \Delta_{11} \left(\frac{\csc \theta_{12} \cot \theta_{12}}{\mathbf{A}_2} \frac{\partial \cos \theta_{12}}{\partial \xi_2} \right) + \Delta_{22} \left(\frac{\csc^2 \theta_{12}}{\mathbf{A}_1} \frac{\partial \cos \theta_{12}}{\partial \xi_1} \right) \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\begin{aligned} \frac{1}{\mathbf{A}_1} \frac{\partial \Delta_{22}}{\partial \xi_1} - \frac{1}{\mathbf{A}_2} \frac{\partial \Delta_{12}}{\partial \xi_2} &= \frac{\Delta_{12}}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} - \frac{\Delta_{22}}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1} - \frac{1}{\rho_{11}} \left(\Delta_{21} \mathbf{csc} \theta_{12} + \Delta_{22} \mathbf{cot} \theta_{12} \right) \\ &\quad + \Delta_{13} \left(\frac{1}{r_{22}} - \frac{\mathbf{cot} \theta_{12}}{r_{21}} \right) + \Delta_{23} \frac{\mathbf{csc} \theta_{12}}{r_{12}} + \frac{1}{\rho_{22}} \left(\Delta_{11} \mathbf{csc} \theta_{12} + \Delta_{12} \mathbf{cot} \theta_{12} \right) \\ &\quad + \Delta_{22} \left(\frac{\mathbf{csc} \theta_{12} \mathbf{cot} \theta_{12}}{\mathbf{A}_1} \frac{\partial \mathbf{cos} \theta_{12}}{\partial \xi_1} \right) + \Delta_{11} \left(\frac{\mathbf{csc}^2 \theta_{12}}{\mathbf{A}_2} \frac{\partial \mathbf{cos} \theta_{12}}{\partial \xi_2} \right) \end{aligned}$$

$$\begin{aligned} \frac{1}{\mathbf{A}_2} \frac{\partial \Delta_{13}}{\partial \xi_2} - \frac{1}{\mathbf{A}_1} \frac{\partial \Delta_{23}}{\partial \xi_1} &= \frac{\Delta_{23}}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_2}{\partial \xi_1} - \frac{\Delta_{13}}{\mathbf{A}_1 \mathbf{A}_2} \frac{\partial \mathbf{A}_1}{\partial \xi_2} \\ &\quad + \frac{\Delta_{12}}{r_{22}} - \frac{\Delta_{21}}{r_{11}} + \Delta_{11} \left(\frac{\mathbf{sin} \theta_{12}}{r_{21}} + \frac{\mathbf{cos} \theta_{12}}{r_{22}} \right) + \Delta_{22} \left(\frac{\mathbf{sin} \theta_{12}}{r_{12}} - \frac{\mathbf{cos} \theta_{12}}{r_{11}} \right) \end{aligned}$$

- Recall that in these expressions,

$$\Delta_{11} = \mathbf{e}_{11}^{\circ} \mathbf{csc}^2 \theta_{12} - \left(\mathbf{e}_{12}^{\circ} \mathbf{csc} \theta_{12} + \varphi \right) \mathbf{cot} \theta_{12}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\Delta_{12} = \left(\mathbf{e}_{12}^{\circ} \csc\theta_{12} + \varphi - \mathbf{e}_{11}^{\circ} \cot\theta_{12} \right) \csc\theta_{12}$$

$$\Delta_{13} = - \left(\varphi_1 + \varphi_2 \cos\theta_{12} \right)$$

$$\Delta_{21} = \left(\mathbf{e}_{12}^{\circ} \csc\theta_{12} - \varphi - \mathbf{e}_{22}^{\circ} \cot\theta_{12} \right) \csc\theta_{12}$$

$$\Delta_{22} = \mathbf{e}_{22}^{\circ} \csc^2\theta_{12} - \left(\mathbf{e}_{12}^{\circ} \csc\theta_{12} - \varphi \right) \cot\theta_{12}$$

$$\Delta_{23} = - \left(\varphi_1 \cos\theta_{12} + \varphi_2 \right)$$

- Thus, derivatives of φ appear only in the left-hand-side of the first two scalar equations
- Specifically,

“SMALL” STRAINS AND “MODERATE” ROTATIONS **CONTINUED**

$$\begin{aligned} \frac{1}{A_2} \frac{\partial \Delta_{11}}{\partial \xi_2} - \frac{1}{A_1} \frac{\partial \Delta_{21}}{\partial \xi_1} &= \frac{\csc \theta_{12}}{A_1} \frac{\partial \varphi}{\partial \xi_1} - \frac{\cot \theta_{12}}{A_2} \frac{\partial \varphi}{\partial \xi_2} \\ &+ \frac{\varphi}{A_1} \frac{\partial \csc \theta_{12}}{\partial \xi_1} + \frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left[\left(\mathbf{e}_{11}^{\circ} \csc \theta_{12} - \mathbf{e}_{12}^{\circ} \cot \theta_{12} \right) \csc \theta_{12} \right] \\ &- \frac{\varphi}{A_2} \frac{\partial \cot \theta_{12}}{\partial \xi_2} - \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left[\left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \mathbf{e}_{22}^{\circ} \cot \theta_{12} \right) \csc \theta_{12} \right] \end{aligned}$$

$$\begin{aligned} \frac{1}{A_1} \frac{\partial \Delta_{22}}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial \Delta_{12}}{\partial \xi_2} &= \frac{\cot \theta_{12}}{A_1} \frac{\partial \varphi}{\partial \xi_1} - \frac{\csc \theta_{12}}{A_2} \frac{\partial \varphi}{\partial \xi_2} \\ &+ \frac{\varphi}{A_1} \frac{\partial \cot \theta_{12}}{\partial \xi_1} + \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left[\left(\mathbf{e}_{22}^{\circ} \csc \theta_{12} - \mathbf{e}_{12}^{\circ} \cot \theta_{12} \right) \csc \theta_{12} \right] \\ &- \frac{\varphi}{A_2} \frac{\partial \csc \theta_{12}}{\partial \xi_2} - \frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left[\left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \mathbf{e}_{11}^{\circ} \cot \theta_{12} \right) \csc \theta_{12} \right] \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Thus, it follows that multiplying the first two scalar equations by φ_α and using the general expressions for the geodesic curvatures

$$\frac{1}{\rho_{11}} = \frac{\csc\theta_{12}}{\mathbf{A}_1\mathbf{A}_2} \left(\frac{\partial}{\partial\xi_1} [\mathbf{A}_2 \cos\theta_{12}] - \frac{\partial\mathbf{A}_1}{\partial\xi_2} \right) \text{ and}$$

$$\frac{1}{\rho_{22}} = - \frac{\csc\theta_{12}}{\mathbf{A}_1\mathbf{A}_2} \left(\frac{\partial}{\partial\xi_2} [\mathbf{A}_1 \cos\theta_{12}] - \frac{\partial\mathbf{A}_2}{\partial\xi_1} \right) \text{ yields}$$

$$\begin{aligned} \varphi_\alpha \left[\frac{\csc\theta_{12}}{\mathbf{A}_1} \frac{\partial\varphi}{\partial\xi_1} - \frac{\cot\theta_{12}}{\mathbf{A}_2} \frac{\partial\varphi}{\partial\xi_2} \right] = \\ - \varphi_\alpha (\varphi_1 \cos\theta_{12} + \varphi_2) \left(\frac{1}{r_{11}} + \frac{\cot\theta_{12}}{r_{12}} \right) + \frac{\varphi_\alpha}{\mathbf{A}_1} \left(\frac{\partial \mathbf{e}_{12}^\circ}{\partial\xi_1} \csc\theta_{12} - \frac{\partial \mathbf{e}_{22}^\circ}{\partial\xi_1} \cot\theta_{12} \right) \csc\theta_{12} \\ + \varphi_\alpha (\varphi_1 + \varphi_2 \cos\theta_{12}) \frac{\csc\theta_{12}}{r_{21}} - \frac{\varphi_\alpha}{\mathbf{A}_2} \left(\frac{\partial \mathbf{e}_{11}^\circ}{\partial\xi_2} \csc\theta_{12} - \frac{\partial \mathbf{e}_{12}^\circ}{\partial\xi_2} \cot\theta_{12} \right) \csc\theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

and

$$\begin{aligned} \varphi_\alpha \left[\frac{\cot\theta_{12}}{\mathbf{A}_1} \frac{\partial\varphi}{\partial\xi_1} - \frac{\csc\theta_{12}}{\mathbf{A}_2} \frac{\partial\varphi}{\partial\xi_2} \right] = \\ - \varphi_\alpha \left(\varphi_1 + \varphi_2 \cos\theta_{12} \right) \left(\frac{1}{r_{22}} - \frac{\cot\theta_{12}}{r_{21}} \right) - \frac{\varphi_\alpha}{\mathbf{A}_1} \left(\frac{\partial \mathbf{e}_{22}^\circ}{\partial\xi_1} \csc\theta_{12} - \frac{\partial \mathbf{e}_{12}^\circ}{\partial\xi_1} \cot\theta_{12} \right) \csc\theta_{12} \\ - \varphi_\alpha \left(\varphi_1 \cos\theta_{12} + \varphi_2 \right) \frac{\csc\theta_{12}}{r_{12}} + \frac{\varphi_\alpha}{\mathbf{A}_2} \left(\frac{\partial \mathbf{e}_{12}^\circ}{\partial\xi_2} \csc\theta_{12} - \frac{\partial \mathbf{e}_{11}^\circ}{\partial\xi_2} \cot\theta_{12} \right) \csc\theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

- Next, these two equations are solved for $\frac{\varphi_\alpha}{\mathbf{A}_1} \frac{\partial\varphi}{\partial\xi_1}$ and $\frac{\varphi_\alpha}{\mathbf{A}_2} \frac{\partial\varphi}{\partial\xi_2}$ to get

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\begin{aligned} \frac{\varphi_\alpha}{\mathbf{A}_1} \frac{\partial \varphi}{\partial \xi_1} = & \varphi_\alpha \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \left(\frac{1}{r_{21}} + \frac{\cot \theta_{12}}{r_{22}} \right) - \varphi_\alpha \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) \frac{\csc \theta_{12}}{r_{11}} \\ & + \varphi_\alpha \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{e}_{12}^o}{\partial \xi_1} - \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{e}_{11}^o}{\partial \xi_2} \right) \csc \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

$$\begin{aligned} \frac{\varphi_\alpha}{\mathbf{A}_2} \frac{\partial \varphi}{\partial \xi_2} = & \varphi_\alpha \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) \left(\frac{1}{r_{12}} - \frac{\cot \theta_{12}}{r_{11}} \right) + \varphi_\alpha \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \frac{\csc \theta_{12}}{r_{22}} \\ & + \varphi_\alpha \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{e}_{22}^o}{\partial \xi_1} - \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{e}_{12}^o}{\partial \xi_2} \right) \csc \theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Using these expressions and the identity
gives

$$\frac{1}{r_{12}} + \frac{1}{r_{21}} = \cot\theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$$

$$\begin{aligned} K_{11}^{\circ} = \chi_{11}^{\circ} + \frac{1}{r_{12}} \left(\mathbf{e}_{11}^{\circ} \cot\theta_{12} - \mathbf{e}_{12}^{\circ} \csc\theta_{12} \right) - \frac{\mathbf{e}_{11}^{\circ}}{r_{11}} - \varphi_2 \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{e}_{12}^{\circ}}{\partial \xi_1} - \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{e}_{11}^{\circ}}{\partial \xi_2} \right) \\ - \frac{1}{2r_{11}} \left(\varphi_1^2 + \varphi_2^2 \cos 2\theta_{12} + 2\varphi^2 + 2\varphi_1 \varphi_2 \cos\theta_{12} \right) + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

$$\begin{aligned} K_{22}^{\circ} = \chi_{22}^{\circ} + \frac{1}{r_{21}} \left(\mathbf{e}_{12}^{\circ} \csc\theta_{12} - \mathbf{e}_{22}^{\circ} \cot\theta_{12} \right) - \frac{\mathbf{e}_{22}^{\circ}}{r_{22}} - \varphi_1 \left(\frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{e}_{12}^{\circ}}{\partial \xi_2} - \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{e}_{22}^{\circ}}{\partial \xi_1} \right) \\ - \frac{1}{2r_{22}} \left[\varphi_1^2 \cos 2\theta_{12} + \varphi_2^2 + 2\varphi^2 + 2\varphi_1 \varphi_2 \cos\theta_{12} \right] + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$\begin{aligned}
 2\kappa_{12}^{\circ} = & 2\chi_{12}^{\circ} \csc\theta_{12} - (\chi_{11}^{\circ} + \chi_{22}^{\circ}) \cot\theta_{12} - \frac{\mathbf{e}_{12}^{\circ} \csc\theta_{12} - \mathbf{e}_{11}^{\circ} \cot\theta_{12}}{r_{11}} \\
 & + \frac{\mathbf{e}_{11}^{\circ}}{r_{12}} - \frac{\mathbf{e}_{22}^{\circ}}{r_{21}} + \frac{\mathbf{e}_{22}^{\circ} \cot\theta_{12} - \mathbf{e}_{12}^{\circ} \csc\theta_{12}}{r_{22}} + \varphi^2 \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) \\
 & - (\varphi_1 \cos\theta_{12} + \varphi_2) \left[\frac{\varphi_1 \sin\theta_{12}}{r_{22}} + \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{e}_{22}^{\circ}}{\partial \xi_1} - \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{e}_{12}^{\circ}}{\partial \xi_2} \right) \csc\theta_{12} \right] \\
 & - (\varphi_1 + \varphi_2 \cos\theta_{12}) \left[\frac{\varphi_2 \sin\theta_{12}}{r_{11}} - \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{e}_{12}^{\circ}}{\partial \xi_1} - \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{e}_{11}^{\circ}}{\partial \xi_2} \right) \csc\theta_{12} \right] + \mathcal{O}(\theta^3, \epsilon^4)
 \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- For **orthogonal reference-surface Gaussian coordinates**, these expressions reduce to

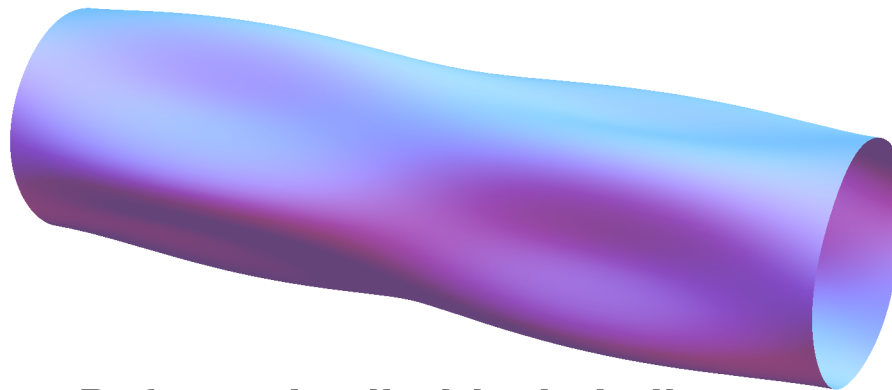
$$\mathbf{K}_{11}^{\circ} = \boldsymbol{\chi}_{11}^{\circ} - \frac{\mathbf{e}_{12}^{\circ}}{\mathbf{r}_{12}} - \frac{\mathbf{e}_{11}^{\circ}}{\mathbf{r}_{11}} - \varphi_2 \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{e}_{12}^{\circ}}{\partial \xi_1} - \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{e}_{11}^{\circ}}{\partial \xi_2} \right) - \frac{1}{2\mathbf{r}_{11}} (\varphi_1^2 - \varphi_2^2 + 2\varphi^2) + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$\mathbf{K}_{22}^{\circ} = \boldsymbol{\chi}_{22}^{\circ} - \frac{\mathbf{e}_{12}^{\circ}}{\mathbf{r}_{12}} - \frac{\mathbf{e}_{22}^{\circ}}{\mathbf{r}_{22}} - \varphi_1 \left(\frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{e}_{12}^{\circ}}{\partial \xi_2} - \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{e}_{22}^{\circ}}{\partial \xi_1} \right) - \frac{1}{2\mathbf{r}_{22}} (\varphi_2^2 - \varphi_1^2 + 2\varphi^2) + \mathcal{O}(\theta^3, \varepsilon^4)$$

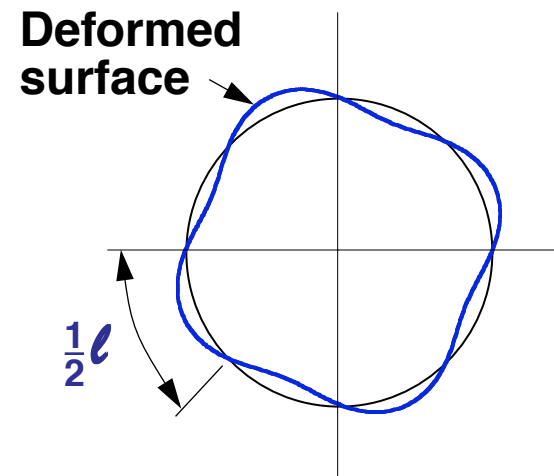
$$\begin{aligned} 2\mathbf{K}_{12}^{\circ} = & 2\boldsymbol{\chi}_{12}^{\circ} - (\mathbf{e}_{12}^{\circ} + \varphi_1\varphi_2) \left(\frac{1}{\mathbf{r}_{11}} + \frac{1}{\mathbf{r}_{22}} \right) + \frac{\mathbf{e}_{11}^{\circ} + \mathbf{e}_{22}^{\circ}}{\mathbf{r}_{12}} + \frac{2\varphi^2}{\mathbf{r}_{12}} \\ & + \varphi_1 \left(\frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{e}_{12}^{\circ}}{\partial \xi_1} - \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{e}_{11}^{\circ}}{\partial \xi_2} \right) + \varphi_2 \left(\frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{e}_{12}^{\circ}}{\partial \xi_2} - \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{e}_{22}^{\circ}}{\partial \xi_1} \right) + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Let h denote the maximum value of the shell thickness $h(\xi_1, \xi_2)$, and let $\mathcal{R}$ denote the smallest magnitude of the reference-surface curvatures and torsions $r_{\alpha\beta}(\xi_1, \xi_2)$
- Similarly, let ℓ denote the smallest wavelength of the deformation pattern exhibited by the shell reference surface, as illustrated below



Deformed cylindrical-shell
reference surface



Cross-sectional view

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Recalling that the purpose of a shell theory is to circumvent the use of three-dimensional elasticity theory for curved structures with one characteristic dimension significantly smaller than the other two, it

follows that for practical applications of shell theory, $\frac{h}{R} < 1$

- Next, noting that the changes in reference-surface curvatures and torsions only contribute to the shell strains at points off the reference

surface, it is seen that terms such as $\frac{\varphi_\alpha \varphi_\beta}{r_{\gamma\rho}}$, $\frac{\varphi_\alpha \varphi}{r_{\gamma\rho}}$, $\frac{\varphi^2}{r_{\gamma\rho}}$, and $\frac{e_{\alpha\beta}^o}{r_{\gamma\rho}}$ are

associated with contributions of $O\left(\theta^2 \frac{h}{R}\right)$ to the shell strains

- In addition, legitimate applications of most practical shell theories presume the smallest wavelength of the deformation pattern exhibited by the reference surface is large compared to the maximum shell

thickness; that is, $\frac{h}{\ell} \ll 1$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- The requirement $\frac{\ell}{\ell} \ll 1$ implies that $\left| \frac{\xi_3}{\mathbf{A}_{(\gamma)}} \frac{\partial \mathbf{e}_{\alpha\beta}^o}{\partial \xi_\gamma} \right| \ll 1$; that is, the strain gradients are at most second-order terms
- Thus, all nonlinear terms appearing in the changes in reference-surface curvatures and torsions are associated with third-order contributions to the shell strains and can be neglected for the case of “small” strains and “moderate” rotations
- Enforcing these conditions yields

$$\xi_3 \mathbf{K}_{11}^o = \xi_3 \left[\boldsymbol{\chi}_{11}^o + \frac{1}{\mathbf{r}_{12}} \left(\mathbf{e}_{11}^o \cot \theta_{12} - \mathbf{e}_{12}^o \csc \theta_{12} \right) - \frac{\mathbf{e}_{11}^o}{\mathbf{r}_{11}} \right] + \mathcal{O} \left(\theta \left(\frac{\ell}{\ell} \right)^2, \theta^2 \frac{\ell}{\mathcal{R}}, \theta^3, \varepsilon^4 \right)$$

$$\xi_3 \mathbf{K}_{22}^o = \xi_3 \left[\boldsymbol{\chi}_{22}^o + \frac{1}{\mathbf{r}_{21}} \left(\mathbf{e}_{12}^o \csc \theta_{12} - \mathbf{e}_{22}^o \cot \theta_{12} \right) - \frac{\mathbf{e}_{22}^o}{\mathbf{r}_{22}} \right] + \mathcal{O} \left(\theta \left(\frac{\ell}{\ell} \right)^2, \theta^2 \frac{\ell}{\mathcal{R}}, \theta^3, \varepsilon^4 \right)$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$2\xi_3 K_{12}^{\circ} = \xi_3 \left[2\chi_{12}^{\circ} \csc\theta_{12} - (\chi_{11}^{\circ} + \chi_{22}^{\circ}) \cot\theta_{12} - \frac{e_{12}^{\circ} \csc\theta_{12} - e_{11}^{\circ} \cot\theta_{12}}{r_{11}} + \frac{e_{11}^{\circ}}{r_{12}} - \frac{e_{22}^{\circ}}{r_{21}} + \frac{e_{22}^{\circ} \cot\theta_{12} - e_{12}^{\circ} \csc\theta_{12}}{r_{22}} \right] + \mathcal{O}\left(\theta\left(\frac{\ell}{\ell}\right)^2, \theta^2 \frac{\ell}{\mathcal{R}}, \theta^3, \varepsilon^4\right)$$

- It is important to point out that the linear terms involving $e_{\alpha\beta}^{\circ}$ are also third-order terms and can also be neglected; however, because these terms are linear they are often retained in “moderate” rotation shell theories
- Thus, two common sets of strains that are based on “small” strains and “moderate” rotations appear in the literature that are referred to herein as the **Equations of Pietraszkiewicz** and **Sanders’ Equations**
 - These equations are listed subsequently

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

- Finally, the general form of the displacement-vector field for points of the shell is given by

$$\vec{U} = \vec{u} + \xi_3(\hat{\boldsymbol{a}} - \hat{\boldsymbol{n}}) + F_1(\xi_3)\gamma_1^\circ \hat{\boldsymbol{a}}_1 + F_2(\xi_3)\gamma_2^\circ \hat{\boldsymbol{a}}_2 + \overrightarrow{\mathcal{O}(\varepsilon^4)}$$

- Substituting the expressions for the convected basis, for “small” strains and “moderate” rotations, into the previous expression and simplifying yields

$$\vec{U} = \vec{u} + \xi_3(\hat{\boldsymbol{a}} - \hat{\boldsymbol{n}}) + F_1(\xi_3)\gamma_1^\circ \hat{\boldsymbol{a}}_1 + F_2(\xi_3)\gamma_2^\circ \hat{\boldsymbol{a}}_2 + \overrightarrow{\mathcal{O}(\theta\varepsilon^2, \varepsilon^4)}$$

- Next; $\vec{U} = U_1\hat{\boldsymbol{g}}_1 + U_2\hat{\boldsymbol{g}}_2 + U_3\hat{\boldsymbol{g}}_3$, $\hat{\boldsymbol{g}}_1 = \mu_{11}\hat{\boldsymbol{a}}_1 + \mu_{12}\hat{\boldsymbol{a}}_2$, $\hat{\boldsymbol{g}}_2 = \mu_{21}\hat{\boldsymbol{a}}_1 + \mu_{22}\hat{\boldsymbol{a}}_2$,

$$\vec{u} = u_1\hat{\boldsymbol{a}}_1 + u_2\hat{\boldsymbol{a}}_2 + w\hat{\boldsymbol{n}}, \text{ and } \hat{\boldsymbol{a}} = \alpha_1\hat{\boldsymbol{a}}_1 + \alpha_2\hat{\boldsymbol{a}}_2 + \alpha_3\hat{\boldsymbol{n}}; \text{ with}$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONTINUED

$$u_1 = \varphi_1 - \varphi(\varphi_2 + \varphi_1 \cos\theta_{12}) \csc\theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$u_2 = \varphi_2 + \varphi(\varphi_1 + \varphi_2 \cos\theta_{12}) \csc\theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

$$u_3 = 1 - \frac{1}{2}(\varphi_1^2 + \varphi_2^2) - \varphi_1\varphi_2 \cos\theta_{12} + \mathcal{O}(\theta^3, \varepsilon^4)$$

are used to get $U_1 = \frac{\mu_{22}U_1^0 - \mu_{21}U_2^0}{\mu_{11}\mu_{22} - \mu_{12}\mu_{21}}$, $U_2 = \frac{\mu_{11}U_2^0 - \mu_{12}U_1^0}{\mu_{11}\mu_{22} - \mu_{12}\mu_{21}}$, and $U_3 = U_3^0$;

where

$$U_1^0 = u_1 + \xi_3 \left[\varphi_1 - \varphi(\varphi_2 + \varphi_1 \cos\theta_{12}) \csc\theta_{12} \right] + F_1(\xi_3) \gamma_1^0 + \mathcal{O}(\theta\varepsilon^2, \theta^3, \varepsilon^4)$$

$$U_2^0 = u_2 + \xi_3 \left[\varphi_2 + \varphi(\varphi_1 + \varphi_2 \cos\theta_{12}) \csc\theta_{12} \right] + F_2(\xi_3) \gamma_2^0 + \mathcal{O}(\theta\varepsilon^2, \theta^3, \varepsilon^4)$$

$$U_3^0 = w - \xi_3 \left[\frac{1}{2}(\varphi_1^2 + \varphi_2^2) + \varphi_1\varphi_2 \cos\theta_{12} \right] + \mathcal{O}(\theta\varepsilon^2, \theta^3, \varepsilon^4)$$

“SMALL” STRAINS AND “MODERATE” ROTATIONS CONCLUDED

$$\mu_{11} = \frac{1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{12}}\right)^2}}$$

$$\mu_{12} = \frac{-\frac{\xi_3 \csc\theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{12}}\right)^2}}$$

$$\mu_{21} = \frac{\frac{\xi_3 \csc\theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}}$$

$$\mu_{22} = \frac{1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}}$$

EQUATIONS OF PIETRASZKIEWICZ

- The “small” strain and “moderate” rotation equations of Pietraszkiewicz are given by

$$\varepsilon_{11}^{\circ} = \mathbf{e}_{11}^{\circ} + \frac{1}{2}(\varphi_1 + \varphi_2 \cos\theta_{12})^2 + \frac{1}{2}\varphi^2 + \varphi(\mathbf{e}_{11}^{\circ}\cot\theta_{12} + \mathbf{e}_{12}^{\circ}\csc\theta_{12})$$

$$\varepsilon_{22}^{\circ} = \mathbf{e}_{22}^{\circ} + \frac{1}{2}(\varphi_2 + \varphi_1 \cos\theta_{12})^2 + \frac{1}{2}\varphi^2 + \varphi(\mathbf{e}_{22}^{\circ}\cot\theta_{12} - \mathbf{e}_{12}^{\circ}\csc\theta_{12})$$

$$2\varepsilon_{12}^{\circ} = 2\mathbf{e}_{12}^{\circ} + (\varphi_1 + \varphi_2 \cos\theta_{12})(\varphi_2 + \varphi_1 \cos\theta_{12}) + \varphi^2 \cos\theta_{12} + \varphi(\mathbf{e}_{22}^{\circ} - \mathbf{e}_{11}^{\circ})\csc\theta_{12}$$

$$\mathbf{K}_{11}^{\circ} = \chi_{11}^{\circ} + \frac{1}{r_{12}}(\mathbf{e}_{11}^{\circ}\cot\theta_{12} - \mathbf{e}_{12}^{\circ}\csc\theta_{12}) - \frac{\mathbf{e}_{11}^{\circ}}{r_{11}}$$

$$\mathbf{K}_{22}^{\circ} = \chi_{22}^{\circ} + \frac{1}{r_{21}}(\mathbf{e}_{12}^{\circ}\csc\theta_{12} - \mathbf{e}_{22}^{\circ}\cot\theta_{12}) - \frac{\mathbf{e}_{22}^{\circ}}{r_{22}}$$

EQUATIONS OF PIETRASZKIEWICZ

CONCLUDED

$$2\kappa_{12}^{\circ} = 2\chi_{12}^{\circ} \csc\theta_{12} - (\chi_{11}^{\circ} + \chi_{22}^{\circ}) \cot\theta_{12} + \frac{\mathbf{e}_{11}^{\circ}}{r_{12}} - \frac{\mathbf{e}_{22}^{\circ}}{r_{21}} - \frac{\mathbf{e}_{12}^{\circ} \csc\theta_{12} - \mathbf{e}_{11}^{\circ} \cot\theta_{12}}{r_{11}} + \frac{\mathbf{e}_{22}^{\circ} \cot\theta_{12} - \mathbf{e}_{12}^{\circ} \csc\theta_{12}}{r_{22}}$$

- For **orthogonal reference-surface Gaussian coordinates**, the equations reduce to

$$\varepsilon_{11}^{\circ} = \mathbf{e}_{11}^{\circ} + \frac{1}{2}(\varphi_1^2 + \varphi^2 + 2\mathbf{e}_{12}^{\circ}\varphi)$$

$$\varepsilon_{22}^{\circ} = \mathbf{e}_{22}^{\circ} + \frac{1}{2}(\varphi_2^2 + \varphi^2 - 2\mathbf{e}_{12}^{\circ}\varphi)$$

$$2\varepsilon_{12}^{\circ} = 2\mathbf{e}_{12}^{\circ} + \varphi_1\varphi_2 + \varphi(\mathbf{e}_{22}^{\circ} - \mathbf{e}_{11}^{\circ})$$

$$\kappa_{11}^{\circ} = \chi_{11}^{\circ} - \frac{\mathbf{e}_{11}^{\circ}}{r_{11}} - \frac{\mathbf{e}_{12}^{\circ}}{r_{12}}$$

$$\kappa_{22}^{\circ} = \chi_{22}^{\circ} - \frac{\mathbf{e}_{22}^{\circ}}{r_{22}} - \frac{\mathbf{e}_{12}^{\circ}}{r_{12}}$$

$$\kappa_{12}^{\circ} = \chi_{12}^{\circ} + \frac{\mathbf{e}_{11}^{\circ} + \mathbf{e}_{22}^{\circ}}{2r_{12}} - \frac{\mathbf{e}_{12}^{\circ}}{2} \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right)$$

SANDERS' EQUATIONS

- The “small” strain and “moderately small” rotation equations of Sanders are given by

$$\epsilon_{11}^{\circ} = e_{11}^{\circ} + \frac{1}{2} \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)^2 + \frac{1}{2} \varphi^2$$

$$\epsilon_{22}^{\circ} = e_{22}^{\circ} + \frac{1}{2} \left(\varphi_2 + \varphi_1 \cos \theta_{12} \right)^2 + \frac{1}{2} \varphi^2$$

$$2\epsilon_{12}^{\circ} = 2e_{12}^{\circ} + \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \left(\varphi_2 + \varphi_1 \cos \theta_{12} \right) + \varphi^2 \cos \theta_{12}$$

$$\kappa_{11}^{\circ} = \chi_{11}^{\circ} = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) - \frac{\varphi_2 \sin \theta_{12}}{\rho_{11}} - \frac{\varphi}{r_{12}}$$

$$\kappa_{22}^{\circ} = \chi_{22}^{\circ} = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} \left(\varphi_2 + \varphi_1 \cos \theta_{12} \right) + \frac{\varphi_1 \sin \theta_{12}}{\rho_{22}} - \frac{\varphi}{r_{21}}$$

$$2\kappa_{12}^{\circ} = 2\chi_{12}^{\circ} = \frac{1}{A_2} \frac{\partial \varphi_1}{\partial \xi_2} + \frac{\cos \theta_{12}}{A_2} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\cos \theta_{12}}{A_1} \frac{\partial \varphi_1}{\partial \xi_1} + \frac{1}{A_1} \frac{\partial \varphi_2}{\partial \xi_1} + \left(\frac{\varphi_1}{\rho_{11}} - \frac{\varphi_2}{\rho_{22}} \right) \sin \theta_{12} - \varphi \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc \theta_{12}$$

SANDERS' EQUATIONS

CONCLUDED

- For **orthogonal reference-surface Gaussian coordinates**, the equations reduce to

$$\varepsilon_{11}^{\circ} = \mathbf{e}_{11}^{\circ} + \frac{1}{2}(\varphi_1^2 + \varphi^2)$$

$$\varepsilon_{22}^{\circ} = \mathbf{e}_{22}^{\circ} + \frac{1}{2}(\varphi_2^2 + \varphi^2)$$

$$2\varepsilon_{12}^{\circ} = 2\mathbf{e}_{12}^{\circ} + \varphi_1\varphi_2$$

$$\mathbf{K}_{11}^{\circ} = \chi_{11}^{\circ} = \frac{1}{\mathbf{A}_1} \frac{\partial \varphi_1}{\partial \xi_1} - \frac{\varphi_2}{\rho_{11}} - \frac{\varphi}{r_{12}}$$

$$\mathbf{K}_{22}^{\circ} = \chi_{22}^{\circ} = \frac{1}{\mathbf{A}_2} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\varphi_1}{\rho_{22}} + \frac{\varphi}{r_{12}}$$

$$2\mathbf{K}_{12}^{\circ} = 2\chi_{12}^{\circ} = \frac{1}{\mathbf{A}_2} \frac{\partial \varphi_1}{\partial \xi_2} + \frac{1}{\mathbf{A}_1} \frac{\partial \varphi_2}{\partial \xi_1} + \left(\frac{\varphi_1}{\rho_{11}} - \frac{\varphi_2}{\rho_{22}} \right) - \varphi \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$$

**RESUME' OF EQUATIONS FOR LINEARIZED
STRAINS AND ROTATIONS -
GENERAL GAUSSIAN COORDINATES**

LINEAR EQUATIONS FOR GENERAL COORDINATES

- For the classical case of completely linearized deformation, the magnitude of the linear rotation and strain parameters are presumed to be the same order as the membrane strains
- The Green-Lagrange shell strains $\left\{ \varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23} \right\}$ are given by

$$\begin{aligned} \varepsilon_{11} \left(\frac{H_1}{A_1} \right)^2 = & \varepsilon_{11}^o \left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right] + \xi_3 K_{11}^o \left(1 + \frac{\xi_3}{r_{11}} \right) \\ & + \frac{1}{2} \left(\frac{\xi_3}{r_{12}} \right) \left[2\varepsilon_{12}^o \left(\frac{\xi_3}{r_{11}} - \frac{\xi_3}{r_{22}} \right) \csc \theta_{12} - \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right) \right] \csc^2 \theta_{12} \\ & - \xi_3 \left(\frac{\xi_3}{r_{12}} \right) \left[K_{12}^o - \frac{1}{2} \cot \theta_{12} \left(K_{11}^o - K_{22}^o \right) \right] + \Gamma_{11} \end{aligned}$$

where

$$\left(\frac{H_1}{A_1} \right)^2 = \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}} \right)^2$$

LINEAR EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned} \epsilon_{22} \left(\frac{H_2}{A_2} \right)^2 &= \epsilon_{22}^o \left[\left(1 + \frac{\xi_3}{r_{22}} \right)^2 + \left(\frac{\xi_3}{r_{21}} \right)^2 \right] + \xi_3 K_{22}^o \left(1 + \frac{\xi_3}{r_{22}} \right) \\ &+ \frac{1}{2} \left(\frac{\xi_3}{r_{21}} \right) \left[2\epsilon_{12}^o \left(\frac{\xi_3}{r_{11}} - \frac{\xi_3}{r_{22}} \right) \csc\theta_{12} - \left(\epsilon_{11}^o + \epsilon_{22}^o \right) \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right) \right] \csc^2\theta_{12} \\ &+ \xi_3 \left(\frac{\xi_3}{r_{21}} \right) \left[K_{12}^o + \frac{1}{2} \cot\theta_{12} \left(K_{11}^o - K_{22}^o \right) \right] + \Gamma_{22} \end{aligned}$$

where

$$\left(\frac{H_2}{A_2} \right)^2 = \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}} \right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}} \right)^2$$

LINEAR EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned}
 2\varepsilon_{12} \frac{\mathbf{H}_1 \mathbf{H}_2}{\mathbf{A}_1 \mathbf{A}_2} = & \varepsilon_{12}^o \left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 - \left(\frac{\xi_3}{r_{12}} \right)^2 + \left(1 + \frac{\xi_3}{r_{22}} \right)^2 - \left(\frac{\xi_3}{r_{21}} \right)^2 \right] \\
 & + \varepsilon_{12}^o \left[\left(\frac{\xi_3}{r_{12}} - \frac{\xi_3}{r_{21}} \right) \left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cot \theta_{12} + \left(\frac{\xi_3}{r_{11}} - \frac{\xi_3}{r_{22}} \right)^2 \left(\csc^4 \theta_{12} - 1 \right) \right] \\
 & - \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \left[\left(\frac{\xi_3}{r_{12}} - \frac{\xi_3}{r_{21}} \right) \left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) + \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right)^2 \cot \theta_{12} \right] \frac{\csc \theta_{12}}{2} \\
 & + \xi_3 \frac{K_{11}^o}{2} \left[\left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cot \theta_{12} + \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right) \csc^2 \theta_{12} - \frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right] \sin \theta_{12} \\
 & + \xi_3 \frac{K_{22}^o}{2} \left[\left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \cot \theta_{12} - \left(\frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right) \csc^2 \theta_{12} - \frac{\xi_3}{r_{12}} + \frac{\xi_3}{r_{21}} \right] \sin \theta_{12} \\
 & + \xi_3 K_{12}^o \left[2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} + \left(\frac{\xi_3}{r_{12}} - \frac{\xi_3}{r_{21}} \right) \cot \theta_{12} \right] \sin \theta_{12} + 2\Gamma_{12}
 \end{aligned}$$

LINEAR EQUATIONS FOR GENERAL COORDINATES

CONTINUED

where

$$\frac{H_1 H_2}{A_1 A_2} = \left[\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}} \right)^2 \right]^{\frac{1}{2}} \left[\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}} \right)^2 \right]^{\frac{1}{2}}$$

$$\begin{aligned} \Gamma_{11} = & F_2(\xi_3) \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(\frac{1}{A_1} \frac{\partial}{\partial \xi_1} (\gamma_2^\circ \cos \theta_{12}) - \frac{\gamma_2^\circ \sin \theta_{12}}{\rho_{11}} \right) \\ & - F_2(\xi_3) \frac{\xi_3}{r_{12}} \left(\frac{\csc \theta_{12}}{A_1} \frac{\partial \gamma_2^\circ}{\partial \xi_1} \right) + F_1(\xi_3) \left[\left(1 + \frac{\xi_3}{r_{11}} \right) \frac{1}{A_1} \frac{\partial \gamma_1^\circ}{\partial \xi_1} - \frac{\xi_3}{r_{12}} \frac{\gamma_1^\circ}{\rho_{11}} \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\begin{aligned} \Gamma_{22} = & F_1(\xi_3) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \left(\frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\gamma_1^\circ \cos \theta_{12}) + \frac{\gamma_1^\circ \sin \theta_{12}}{\rho_{22}} \right) \\ & + F_1(\xi_3) \frac{\xi_3}{r_{21}} \left(\frac{\csc \theta_{12}}{A_2} \frac{\partial \gamma_1^\circ}{\partial \xi_2} \right) + F_2(\xi_3) \left[\left(1 + \frac{\xi_3}{r_{22}} \right) \frac{1}{A_2} \frac{\partial \gamma_2^\circ}{\partial \xi_2} - \frac{\xi_3}{r_{21}} \frac{\gamma_2^\circ}{\rho_{22}} \right] + \mathcal{O}(\varepsilon^4) \end{aligned}$$

LINEAR EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\begin{aligned}
 2\Gamma_{12} = & F_1(\xi_3) \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \frac{1}{A_2} \frac{\partial \gamma_1^o}{\partial \xi_2} + F_2(\xi_3) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \frac{1}{A_1} \frac{\partial \gamma_2^o}{\partial \xi_1} \\
 & + F_1(\xi_3) \left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}} \right) \left(\frac{\cos \theta_{12}}{A_1} \frac{\partial \gamma_1^o}{\partial \xi_1} + \frac{\gamma_1^o \sin \theta_{12}}{\rho_{11}} \right) \\
 & + F_2(\xi_3) \left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}} \right) \left(\frac{\cos \theta_{12}}{A_2} \frac{\partial \gamma_2^o}{\partial \xi_2} - \frac{\gamma_2^o \sin \theta_{12}}{\rho_{22}} \right) \\
 & + F_1(\xi_3) \left[\frac{\xi_3}{r_{21}} \left(\frac{\csc \theta_{12}}{A_1} \frac{\partial \gamma_1^o}{\partial \xi} \right) - \frac{\xi_3}{r_{12}} \left(\frac{\csc \theta_{12}}{A_2} \frac{\partial}{\partial \xi_2} (\gamma_1^o \cos \theta_{12}) + \frac{\gamma_1^o}{\rho_{22}} \right) \right] \\
 & + F_2(\xi_3) \left[\frac{\xi_3}{r_{21}} \left(\frac{\csc \theta_{12}}{A_1} \frac{\partial}{\partial \xi_1} (\gamma_2^o \cos \theta_{12}) - \frac{\gamma_2^o}{\rho_{11}} \right) - \frac{\xi_3}{r_{12}} \left(\frac{\csc \theta_{12}}{A_2} \frac{\partial \gamma_2^o}{\partial \xi_2} \right) \right] + \mathcal{O}(\varepsilon^4)
 \end{aligned}$$

LINEAR EQUATIONS FOR GENERAL COORDINATES

CONTINUED

$$\gamma_1^{\circ} = \csc^2 \theta_{12} \left(2\varepsilon_{13}^{\circ} - 2\varepsilon_{23}^{\circ} \cos \theta_{12} \right)$$

$$\gamma_2^{\circ} = \csc^2 \theta_{12} \left(2\varepsilon_{23}^{\circ} - 2\varepsilon_{13}^{\circ} \cos \theta_{12} \right)$$

where it is noted that $2\varepsilon_{13}^{\circ}$ and $2\varepsilon_{23}^{\circ}$ are fundamental unknowns and the geodesic curvatures are given by

$$\frac{1}{\rho_{11}} = \frac{\csc \theta_{12}}{A_1 A_2} \left(\frac{\partial}{\partial \xi_1} [A_2 \cos \theta_{12}] - \frac{\partial A_1}{\partial \xi_2} \right)$$

$$\frac{1}{\rho_{22}} = - \frac{\csc \theta_{12}}{A_1 A_2} \left(\frac{\partial}{\partial \xi_2} [A_1 \cos \theta_{12}] - \frac{\partial A_2}{\partial \xi_1} \right)$$

- The transverse shearing strains are given by

$$2\varepsilon_{13} \frac{H_1}{A_1} = \left[F_1'(\xi_3) + \frac{P_1(\xi_3)}{r_{11}} \right] \gamma_1^{\circ} + \left[\left(F_2'(\xi_3) + \frac{P_2(\xi_3)}{r_{11}} \right) \cos \theta_{12} - \frac{P_2(\xi_3) \sin \theta_{12}}{r_{12}} \right] \gamma_2^{\circ}$$

$$2\varepsilon_{23} \frac{H_2}{A_2} = \left[F_2'(\xi_3) + \frac{P_2(\xi_3)}{r_{22}} \right] \gamma_2^{\circ} + \left[\left(F_1'(\xi_3) + \frac{P_1(\xi_3)}{r_{22}} \right) \cos \theta_{12} + \frac{P_1(\xi_3) \sin \theta_{12}}{r_{21}} \right] \gamma_1^{\circ}$$

LINEAR EQUATIONS FOR GENERAL COORDINATES

CONTINUED

where $P_1(\xi_3) \equiv \xi_3 F_1'(\xi_3) - F_1(\xi_3)$ and $P_2(\xi_3) \equiv \xi_3 F_2'(\xi_3) - F_2(\xi_3)$

- The reference-surface membrane strains are given by

$$\epsilon_{11}^o = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (u_1 + u_2 \cos \theta_{12}) - \frac{u_2 \sin \theta_{12}}{\rho_{11}} + \frac{w}{r_{11}}$$

$$\epsilon_{22}^o = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (u_2 + u_1 \cos \theta_{12}) + \frac{u_1 \sin \theta_{12}}{\rho_{22}} + \frac{w}{r_{22}}$$

$$\begin{aligned} 2\epsilon_{12}^o = & \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} + \left(\frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} + \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} \right) \cos \theta_{12} \\ & + \left(\frac{u_1}{\rho_{11}} - \frac{u_2}{\rho_{22}} \right) \sin \theta_{12} + w \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) \cos \theta_{12} + w \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \sin \theta_{12} \end{aligned}$$

LINEAR EQUATIONS FOR GENERAL COORDINATES

CONTINUED

- The reference-surface bending and twisting strains are given by

$$\kappa_{11}^{\circ} = \chi_{11}^{\circ} - \frac{\epsilon_{11}^{\circ}}{r_{11}} + \frac{\epsilon_{11}^{\circ} \cot \theta_{12} - \epsilon_{12}^{\circ} \csc \theta_{12}}{r_{12}}$$

$$\kappa_{22}^{\circ} = \chi_{22}^{\circ} - \frac{\epsilon_{22}^{\circ}}{r_{22}} + \frac{\epsilon_{12}^{\circ} \csc \theta_{12} - \epsilon_{22}^{\circ} \cot \theta_{12}}{r_{21}}$$

$$2\kappa_{12}^{\circ} = \left[2\chi_{12}^{\circ} - \epsilon_{12}^{\circ} \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) \right] \csc \theta_{12} - \cot \theta_{12} \left[\chi_{11}^{\circ} - \frac{\epsilon_{11}^{\circ}}{r_{11}} + \chi_{22}^{\circ} - \frac{\epsilon_{22}^{\circ}}{r_{22}} \right] + \frac{\epsilon_{11}^{\circ}}{r_{12}} - \frac{\epsilon_{22}^{\circ}}{r_{21}}$$

where

LINEAR EQUATIONS FOR GENERAL COORDINATES CONTINUED

$$\chi_{11}^{\circ} = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (\varphi_1 + \varphi_2 \cos \theta_{12}) - \frac{\varphi_2 \sin \theta_{12}}{\rho_{11}} - \frac{\varphi}{r_{12}}$$

$$\chi_{22}^{\circ} = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\varphi_2 + \varphi_1 \cos \theta_{12}) + \frac{\varphi_1 \sin \theta_{12}}{\rho_{22}} - \frac{\varphi}{r_{21}}$$

$$2\chi_{12}^{\circ} = \frac{1}{A_2} \frac{\partial \varphi_1}{\partial \xi_2} + \frac{\cos \theta_{12}}{A_2} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\cos \theta_{12}}{A_1} \frac{\partial \varphi_1}{\partial \xi_1} + \frac{1}{A_1} \frac{\partial \varphi_2}{\partial \xi_1} + \left(\frac{\varphi_1}{\rho_{11}} - \frac{\varphi_2}{\rho_{22}} \right) \sin \theta_{12} - \varphi \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc \theta_{12}$$

- It is worth noting that no shell thinness approximations have been made to obtain these equations, and the bending-torsion strain parameters defined by χ_{11}° , χ_{22}° , and χ_{12}° are identical to those of the Sanders-Koiter “best” first approximation thin-shell theory, which vanish under the action of rigid-body rotations, in addition to the linear membrane strains

LINEAR EQUATIONS FOR GENERAL COORDINATES

CONTINUED

- The linear rotation parameters are given by

$$\varphi_1 = \left(\frac{u_1}{r_{11}} - \frac{1}{A_1} \frac{\partial w}{\partial \xi_1} \right) \csc^2 \theta_{12} - u_1 \left(\frac{1}{r_{21}} + \frac{\cot \theta_{12}}{r_{22}} \right) \cot \theta_{12} + \left(\frac{\cot \theta_{12}}{A_2} \frac{\partial w}{\partial \xi_2} + \frac{u_2}{r_{21}} \right) \csc \theta_{12}$$

$$\varphi_2 = \left(\frac{u_2}{r_{22}} - \frac{1}{A_2} \frac{\partial w}{\partial \xi_2} \right) \csc^2 \theta_{12} + u_2 \left(\frac{1}{r_{12}} - \frac{\cot \theta_{12}}{r_{11}} \right) \cot \theta_{12} + \left(\frac{\cot \theta_{12}}{A_1} \frac{\partial w}{\partial \xi_1} - \frac{u_1}{r_{12}} \right) \csc \theta_{12}$$

$$2\varphi = \left(\frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} \right) \cot \theta_{12} + \left(\frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} \right) \csc \theta_{12} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}}$$

LINEAR EQUATIONS FOR GENERAL COORDINATES

CONTINUED

- The components of the displacement vector field

$\vec{U} = U_1 \hat{g}_1 + U_2 \hat{g}_2 + U_3 \hat{g}_3$ are given by

$$U_1 = \frac{\mu_{22} U_1^0 - \mu_{21} U_2^0}{\mu_{11} \mu_{22} - \mu_{12} \mu_{21}}, \quad U_2 = \frac{\mu_{11} U_2^0 - \mu_{12} U_1^0}{\mu_{11} \mu_{22} - \mu_{12} \mu_{21}}, \quad \text{and} \quad U_3 = U_3^0; \quad \text{where}$$

$$U_1^0 = u_1 + \xi_3 \varphi_1 + \frac{F_1(\xi_3)}{\sin^2 \theta_{12}} \left(2\varepsilon_{13}^0 - 2\varepsilon_{23}^0 \cos \theta_{12} \right)$$

$$U_2^0 = u_2 + \xi_3 \varphi_2 + \frac{F_2(\xi_3)}{\sin^2 \theta_{12}} \left(2\varepsilon_{23}^0 - 2\varepsilon_{13}^0 \cos \theta_{12} \right)$$

$$U_3^0 = w$$

and

LINEAR EQUATIONS FOR GENERAL COORDINATES

CONCLUDED

$$\mu_{11} = \frac{1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{12}}\right)^2}}$$

$$\mu_{12} = \frac{-\frac{\xi_3 \csc\theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot\theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{12}}\right)^2}}$$

$$\mu_{21} = \frac{\frac{\xi_3 \csc\theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}}$$

$$\mu_{22} = \frac{1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}}$$

**RESUME' OF EQUATIONS FOR LINEARIZED
STRAINS AND ROTATIONS -
ORTHOGONAL GAUSSIAN COORDINATES**

LINEAR EQUATIONS FOR ORTHOGONAL COORDINATES

- For **orthogonal reference-surface Gaussian coordinates**, the Green-Lagrange shell strains $\left\{ \epsilon_{11}, \epsilon_{22}, 2\epsilon_{12}, 2\epsilon_{13}, 2\epsilon_{23} \right\}$ are given by

$$\begin{aligned} \epsilon_{11} \left(\frac{\mathbf{H}_1}{\mathbf{A}_1} \right)^2 = & \epsilon_{11}^o \left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right] + \xi_3 K_{11}^o \left(1 + \frac{\xi_3}{r_{11}} \right) \\ & + \epsilon_{12}^o \left(\frac{\xi_3}{r_{12}} \right) \left(\frac{\xi_3}{r_{11}} - \frac{\xi_3}{r_{22}} \right) - \xi_3 K_{12}^o \left(\frac{\xi_3}{r_{12}} \right) + \Gamma_{11} \end{aligned}$$

$$\begin{aligned} \epsilon_{22} \left(\frac{\mathbf{H}_2}{\mathbf{A}_2} \right)^2 = & \epsilon_{22}^o \left[\left(1 + \frac{\xi_3}{r_{22}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right] + \xi_3 K_{22}^o \left(1 + \frac{\xi_3}{r_{22}} \right) \\ & - \epsilon_{12}^o \left(\frac{\xi_3}{r_{12}} \right) \left(\frac{\xi_3}{r_{11}} - \frac{\xi_3}{r_{22}} \right) - \xi_3 K_{12}^o \left(\frac{\xi_3}{r_{12}} \right) + \Gamma_{22} \end{aligned}$$

LINEAR EQUATIONS FOR ORTHOGONAL COORDINATES - CONTINUED

$$2\varepsilon_{12} \frac{H_1 H_2}{A_1 A_2} = \varepsilon_{12}^o \left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(1 + \frac{\xi_3}{r_{22}} \right)^2 - 2 \left(\frac{\xi_3}{r_{12}} \right)^2 \right] - \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \left(\frac{\xi_3}{r_{12}} \right) \left(2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right) \\ - \xi_3 \left(K_{11}^o + K_{22}^o \right) \left(\frac{\xi_3}{r_{12}} \right) + \xi_3 K_{12}^o \left[2 + \frac{\xi_3}{r_{11}} + \frac{\xi_3}{r_{22}} \right] + 2\Gamma_{12}$$

with

$$\left(\frac{H_1}{A_1} \right)^2 = \left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2$$

$$\left(\frac{H_2}{A_2} \right)^2 = \left(1 + \frac{\xi_3}{r_{22}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2$$

$$\frac{H_1 H_2}{A_1 A_2} = \left[\left(1 + \frac{\xi_3}{r_{11}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right]^{\frac{1}{2}} \left[\left(1 + \frac{\xi_3}{r_{22}} \right)^2 + \left(\frac{\xi_3}{r_{12}} \right)^2 \right]^{\frac{1}{2}}$$

LINEAR EQUATIONS FOR ORTHOGONAL COORDINATES - CONTINUED

where

$$\Gamma_{11} = \left(1 + \frac{\xi_3}{r_{11}}\right) \left[F_1(\xi_3) \frac{1}{A_1} \frac{\partial 2\epsilon_{13}^o}{\partial \xi_1} - F_2(\xi_3) \frac{2\epsilon_{23}^o}{\rho_{11}} \right] - \frac{\xi_3}{r_{12}} \left[F_1(\xi_3) \frac{2\epsilon_{13}^o}{\rho_{11}} + F_2(\xi_3) \frac{1}{A_1} \frac{\partial 2\epsilon_{23}^o}{\partial \xi_1} \right]$$

$$\Gamma_{22} = \left(1 + \frac{\xi_3}{r_{22}}\right) \left[F_1(\xi_3) \frac{2\epsilon_{13}^o}{\rho_{22}} + F_2(\xi_3) \frac{1}{A_2} \frac{\partial 2\epsilon_{23}^o}{\partial \xi_2} \right] - \frac{\xi_3}{r_{12}} \left[F_1(\xi_3) \frac{1}{A_2} \frac{\partial 2\epsilon_{13}^o}{\partial \xi_2} - F_2(\xi_3) \frac{2\epsilon_{23}^o}{\rho_{22}} \right]$$

$$2\Gamma_{12} = \left(1 + \frac{\xi_3}{r_{11}}\right) \left[F_1(\xi_3) \frac{1}{A_2} \frac{\partial 2\epsilon_{13}^o}{\partial \xi_2} - F_2(\xi_3) \frac{2\epsilon_{23}^o}{\rho_{22}} \right] + \left(1 + \frac{\xi_3}{r_{22}}\right) \left[F_2(\xi_3) \frac{1}{A_1} \frac{\partial 2\epsilon_{23}^o}{\partial \xi_1} + F_1(\xi_3) \frac{2\epsilon_{13}^o}{\rho_{11}} \right] - \frac{\xi_3}{r_{12}} \left[F_1(\xi_3) \left(\frac{1}{A_1} \frac{\partial 2\epsilon_{13}^o}{\partial \xi_1} + \frac{2\epsilon_{13}^o}{\rho_{22}} \right) + F_2(\xi_3) \left(\frac{1}{A_2} \frac{\partial 2\epsilon_{23}^o}{\partial \xi_2} - \frac{2\epsilon_{23}^o}{\rho_{11}} \right) \right]$$

and where it is noted that $2\epsilon_{13}^o$ and $2\epsilon_{23}^o$ are fundamental unknowns

LINEAR EQUATIONS FOR ORTHOGONAL COORDINATES - CONTINUED

- The transverse shearing strains are given by

$$2\epsilon_{13} \frac{H_1}{A_1} = \left[F_1'(\xi_3) \left(1 + \frac{\xi_3}{r_{11}} \right) - \frac{F_1(\xi_3)}{r_{11}} \right] 2\epsilon_{13}^o - \left[\frac{\xi_3 F_2'(\xi_3) - F_2(\xi_3)}{r_{12}} \right] 2\epsilon_{23}^o$$

$$2\epsilon_{23} \frac{H_2}{A_2} = \left[F_2'(\xi_3) \left(1 + \frac{\xi_3}{r_{22}} \right) - \frac{F_2(\xi_3)}{r_{22}} \right] 2\epsilon_{23}^o - \left[\frac{\xi_3 F_1'(\xi_3) - F_1(\xi_3)}{r_{12}} \right] 2\epsilon_{13}^o$$

- The reference-surface membrane strains are given by

$$\epsilon_{11}^o = \frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} - \frac{u_2}{\rho_{11}} + \frac{w}{r_{11}} \quad \epsilon_{22}^o = \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{\rho_{22}} + \frac{w}{r_{22}}$$

$$2\epsilon_{12}^o = \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} + \frac{u_1}{\rho_{11}} - \frac{u_2}{\rho_{22}} - \frac{2w}{r_{12}}$$

LINEAR EQUATIONS FOR ORTHOGONAL COORDINATES - CONTINUED

- The reference-surface bending and twisting strains are given by

$$\kappa_{11}^{\circ} = \chi_{11}^{\circ} - \frac{\epsilon_{11}^{\circ}}{r_{11}} - \frac{\epsilon_{12}^{\circ}}{r_{12}}$$

$$\kappa_{22}^{\circ} = \chi_{22}^{\circ} - \frac{\epsilon_{22}^{\circ}}{r_{22}} - \frac{\epsilon_{12}^{\circ}}{r_{12}}$$

$$2\kappa_{12}^{\circ} = 2\chi_{12}^{\circ} - \epsilon_{12}^{\circ} \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) + \frac{\epsilon_{11}^{\circ} + \epsilon_{22}^{\circ}}{r_{12}} \quad \text{where}$$

$$\chi_{11}^{\circ} = \frac{1}{A_1} \frac{\partial \varphi_1}{\partial \xi_1} - \frac{\varphi_2}{\rho_{11}} - \frac{\varphi}{r_{12}}$$

$$\chi_{22}^{\circ} = \frac{1}{A_2} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\varphi_1}{\rho_{22}} + \frac{\varphi}{r_{12}}$$

$$2\chi_{12}^{\circ} = \frac{1}{A_2} \frac{\partial \varphi_1}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial \varphi_2}{\partial \xi_1} + \frac{\varphi_1}{\rho_{11}} - \frac{\varphi_2}{\rho_{22}} - \varphi \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$$

- The linear rotation parameters are given by $\varphi_1 = \frac{u_1}{r_{11}} - \frac{u_2}{r_{12}} - \frac{1}{A_1} \frac{\partial w}{\partial \xi_1}$,

$$\varphi_2 = -\frac{u_1}{r_{12}} + \frac{u_2}{r_{22}} - \frac{1}{A_2} \frac{\partial w}{\partial \xi_2}, \quad \text{and} \quad 2\varphi = \frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}}$$

LINEAR EQUATIONS FOR ORTHOGONAL COORDINATES - CONCLUDED

- The components of the displacement vector $\vec{U} = U_1 \hat{g}_1 + U_2 \hat{g}_2 + U_3 \hat{g}_3$ are given by

$$U_1 = \frac{\mu_{22} U_1^0 - \mu_{21} U_2^0}{\mu_{11} \mu_{22} - \mu_{12} \mu_{21}}, \quad U_2 = \frac{\mu_{11} U_2^0 - \mu_{12} U_1^0}{\mu_{11} \mu_{22} - \mu_{12} \mu_{21}}, \quad \text{and} \quad U_3 = U_3^0; \quad \text{where}$$

$$U_1^0 = u_1 + \xi_3 \varphi_1 + 2\varepsilon_{13}^0 F_1(\xi_3)$$

$$U_2^0 = u_2 + \xi_3 \varphi_2 + 2\varepsilon_{23}^0 F_2(\xi_3)$$

$$U_3^0 = w$$

$$\mu_{11} = \frac{1 + \frac{\xi_3}{r_{11}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

$$\mu_{12} = \frac{-\frac{\xi_3}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

$$\mu_{21} = \frac{-\frac{\xi_3}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

$$\mu_{22} = \frac{1 + \frac{\xi_3}{r_{22}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}}\right)^2 + \left(\frac{\xi_3}{r_{12}}\right)^2}}$$

SPECIAL CASES OF THE LINEARIZED SHELL STRAINS

SPECIAL CASES OF THE LINEARIZED SHELL STRAINS

- The linearized shell-strain equations presented herein contain several special cases of historical interest as proper subsets
- First, consider the specialization of the shell strains for Gaussian reference-surface coordinates that are **principal-curvature coordinates**
 - For this case; $r_{11} \rightarrow R_1$, $r_{22} \rightarrow R_2$, and $r_{12} = -r_{21} \rightarrow \infty$
 - R_1 and R_2 are the principal values of r_{11} and r_{22} , respectively
 - In addition, the Gaussian coordinates of the reference surface are orthogonal; that is, $\theta_{12} = \frac{\pi}{2}$
- The linear rotations reduce to

$$\varphi_1 = \frac{u_1}{R_1} - \frac{1}{A_1} \frac{\partial w}{\partial \xi_1}$$

$$\varphi_2 = \frac{u_2}{R_2} - \frac{1}{A_2} \frac{\partial w}{\partial \xi_2}$$

$$\varphi = \frac{1}{2} \left(\frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}} \right)$$

SPECIAL CASES OF THE LINEARIZED SHELL STRAINS

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where the geodesic curvatures are given by

$$\frac{1}{\rho_{11}} = -\frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2} \quad \text{and} \quad \frac{1}{\rho_{22}} = \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \xi_1}$$

- Likewise, the linear membrane strains reduce to

$$\epsilon_{11}^{\circ} = \frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} - \frac{u_2}{\rho_{11}} + \frac{w}{R_1} \quad \epsilon_{22}^{\circ} = \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{\rho_{22}} + \frac{w}{R_2}$$

$$2\epsilon_{12}^{\circ} = \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} + \frac{u_1}{\rho_{11}} - \frac{u_2}{\rho_{22}}$$

- The **linear-deformation** parameters χ_{11}° , χ_{22}° , and χ_{12}° are given by

$$\chi_{11}^{\circ} = \frac{1}{A_1} \frac{\partial \varphi_1}{\partial \xi_1} - \frac{\varphi_2}{\rho_{11}} \quad \chi_{22}^{\circ} = \frac{1}{A_2} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\varphi_1}{\rho_{22}}$$

SPECIAL CASES OF THE LINEARIZED SHELL STRAINS

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$$2\chi_{12}^{\circ} = \frac{1}{A_1} \frac{\partial \varphi_2}{\partial \xi_1} + \frac{\varphi_1}{\rho_{11}} + \frac{1}{A_2} \frac{\partial \varphi_1}{\partial \xi_2} - \frac{\varphi_2}{\rho_{22}} - \varphi \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

- In addition, the changes in reference-surface curvatures and torsion

become $\kappa_{11}^{\circ} = \chi_{11}^{\circ} - \frac{\varepsilon_{11}^{\circ}}{R_1}$, $\kappa_{22}^{\circ} = \chi_{22}^{\circ} - \frac{\varepsilon_{22}^{\circ}}{R_2}$, and $2\kappa_{12}^{\circ} = 2\chi_{12}^{\circ} - \varepsilon_{12}^{\circ} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$

- For this special class of Gaussian reference-surface coordinates, the normal strains in the shell become

$$\varepsilon_{11} = \frac{1}{\left(1 + \frac{\xi_3}{R_1}\right)} \left[\varepsilon_{11}^{\circ} + \xi_3 \chi_{11}^{\circ} + F_1(\xi_3) \frac{1}{A_1} \frac{\partial 2\varepsilon_{13}^{\circ}}{\partial \xi_1} - F_2(\xi_3) \frac{2\varepsilon_{23}^{\circ}}{\rho_{11}} \right]$$

SPECIAL CASES OF THE LINEARIZED SHELL STRAINS

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$$\epsilon_{22} = \frac{1}{\left(1 + \frac{\xi_3}{R_2}\right)} \left[\epsilon_{22}^{\circ} + \xi_3 \chi_{22}^{\circ} + F_1(\xi_3) \frac{2\epsilon_{13}^{\circ}}{\rho_{22}} + F_2(\xi_3) \frac{1}{A_2} \frac{\partial 2\epsilon_{23}^{\circ}}{\partial \xi_2} \right]$$

- Likewise, the shearing strains reduce to

$$2\epsilon_{12} = \frac{2\epsilon_{12}^{\circ} \left[1 - \frac{(\xi_3)^2}{R_1 R_2} \right] + \left[\chi_{12}^{\circ} + \frac{1}{2} \epsilon_{12}^{\circ} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right] \left\{ 2\xi_3 + (\xi_3)^2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right\} + 2\Gamma_{12}}{\left(1 + \frac{\xi_3}{R_1} \right) \left(1 + \frac{\xi_3}{R_2} \right)}$$

with

$$2\Gamma_{12} = \left(1 + \frac{\xi_3}{R_1} \right) \left[F_1(\xi_3) \frac{1}{A_2} \frac{\partial 2\epsilon_{13}^{\circ}}{\partial \xi_2} - F_2(\xi_3) \frac{2\epsilon_{23}^{\circ}}{\rho_{22}} \right] + \left(1 + \frac{\xi_3}{R_2} \right) \left[F_2(\xi_3) \frac{1}{A_1} \frac{\partial 2\epsilon_{23}^{\circ}}{\partial \xi_1} + F_1(\xi_3) \frac{2\epsilon_{13}^{\circ}}{\rho_{11}} \right]$$

SPECIAL CASES OF THE LINEARIZED SHELL STRAINS

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$$2\varepsilon_{13} = \frac{1}{\left(1 + \frac{\xi_3}{R_1}\right)} \left[2\varepsilon_{13}^{\circ} F_1'(\xi_3) + \left[\xi_3 F_1'(\xi_3) - F_1(\xi_3) \right] \frac{2\varepsilon_{13}^{\circ}}{R_1} \right]$$

$$2\varepsilon_{23} = \frac{1}{\left(1 + \frac{\xi_3}{R_2}\right)} \left[2\varepsilon_{23}^{\circ} F_2'(\xi_3) + \left[\xi_3 F_2'(\xi_3) - F_2(\xi_3) \right] \frac{2\varepsilon_{23}^{\circ}}{R_2} \right]$$

- The shifters are given in terms of the Kronecker Delta symbol by $\mu_{\alpha\beta} = \delta_{\alpha\beta}$; thus, $\hat{g}_{\alpha} = \hat{a}_{\alpha}$
- The components of the displacement vector field for points of the shell, $\vec{U} = U_1 \hat{a}_1 + U_2 \hat{a}_2 + U_3 \hat{n}$, are given by $U_3 = w$ and

$$U_1 = \left(1 + \frac{\xi_3}{R_1}\right) \left[u_1 + \xi_3 \varphi_1 + 2\varepsilon_{13}^{\circ} F_1(\xi_3) \right]$$

$$U_2 = \left(1 + \frac{\xi_3}{R_2}\right) \left[u_2 + \xi_3 \varphi_2 + 2\varepsilon_{23}^{\circ} F_2(\xi_3) \right]$$

SPECIAL CASES OF THE LINEARIZED SHELL STRAINS

CONTINUED

- By neglecting the transverse shearing strains and noting that

$$2\chi_{12}^o + \epsilon_{12}^o \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = \frac{1}{A_1} \frac{\partial \varphi_2}{\partial \xi_1} + \frac{\varphi_1}{\rho_{11}} + \frac{1}{A_2} \frac{\partial \varphi_1}{\partial \xi_2} - \frac{\varphi_2}{\rho_{22}} \\ + \frac{1}{R_1} \left(\frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} - \frac{u_2}{\rho_{22}} \right) + \frac{1}{R_2} \left(\frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} + \frac{u_1}{\rho_{11}} \right)$$

the strain equations of the present study reduce exactly to the linear strains attributed to Flügge, Lur'e, Byrne, Goldenveizer, and Novozhilov; as given by Leissa and Kraus in the following references

- Leissa, A. W.: *Vibration of Shells*. NASA SP-288, 1973.
- Kraus, H.: *Thin Elastic Shells-An Introduction to the Theoretical Foundations and the Analysis of Their Static and Dynamic Behavior*. John Wiley & Sons, Inc., 1967.

SPECIAL CASES OF THE LINEARIZED SHELL STRAINS

CONTINUED

- It is important to reiterate that no shell thickness approximations have been made to obtain these equations
- In addition, the bending-torsion strain parameters defined by χ_{11}^o , χ_{22}^o , and χ_{12}^o are identical to those of the Sanders-Koiter “best” first approximation thin-shell theory
- These strain parameters vanish under the action of rigid-body rotations, in addition to the linear membrane strains
- Now consider the case of a thin shell, for which $\left| \frac{\xi_3}{R_\alpha} \right| \ll 1$

SPECIAL CASES OF THE LINEARIZED SHELL STRAINS

CONTINUED

- Expanding $\left(1 + \frac{\xi_3}{R_1}\right)^{-1}$ and $\left(1 + \frac{\xi_3}{R_2}\right)^{-1}$ into power series; substituting the results into the strain equations attributed to Flügge, Lur'e, Byrne, Goldenveizer, and Novozhilov; and neglecting products of $\frac{\xi_3}{R_\alpha}$ and products of $\frac{\xi_3}{R_\alpha}$ and linear strain parameters yields

$$\epsilon_{11} = \epsilon_{11}^o + \xi_3 \chi_{11}^o + F_1(\xi_3) \frac{1}{A_1} \frac{\partial 2\epsilon_{13}^o}{\partial \xi_1} - F_2(\xi_3) \frac{2\epsilon_{23}^o}{\rho_{11}}$$

$$\epsilon_{22} = \epsilon_{22}^o + \xi_3 \chi_{22}^o + F_1(\xi_3) \frac{2\epsilon_{13}^o}{\rho_{22}} + F_2(\xi_3) \frac{1}{A_2} \frac{\partial 2\epsilon_{23}^o}{\partial \xi_2}$$

SPECIAL CASES OF THE LINEARIZED SHELL STRAINS

CONTINUED

- Similarly, one can obtain

$$2\varepsilon_{12} = 2\varepsilon_{12}^{\circ} + \xi_3 \left[2\chi_{12}^{\circ} + \varepsilon_{12}^{\circ} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right] \\ + F_1(\xi_3) \left[\frac{1}{A_2} \frac{\partial 2\varepsilon_{13}^{\circ}}{\partial \xi_2} + \frac{2\varepsilon_{13}^{\circ}}{\rho_{11}} \right] + F_2(\xi_3) \left[\frac{1}{A_1} \frac{\partial 2\varepsilon_{23}^{\circ}}{\partial \xi_1} - \frac{2\varepsilon_{23}^{\circ}}{\rho_{22}} \right]$$

where the negligible products of $\frac{\xi_3}{R_\alpha}$ and ε_{12}° have been retained

SPECIAL CASES OF THE LINEARIZED SHELL STRAINS

CONTINUED

- Furthermore, the transverse shearing strains reduce to

$$2\varepsilon_{13} = 2\varepsilon_{13}^{\circ} F_1'(\xi_3) + \left[\xi_3 F_1'(\xi_3) - F_1(\xi_3) \right] \frac{2\varepsilon_{13}^{\circ}}{R_1} \quad \text{and}$$

$$2\varepsilon_{23} = 2\varepsilon_{23}^{\circ} F_2'(\xi_3) + \left[\xi_3 F_2'(\xi_3) - F_2(\xi_3) \right] \frac{2\varepsilon_{23}^{\circ}}{R_2}$$

- When the transverse shearing strains are neglected in these equations, they reduce to the linear strains attributed to **Love and Timoshenko** in NASA SP-288
- It is important to note that these strains also vanish for rigid-body motions

SPECIAL CASES OF THE LINEARIZED SHELL STRAINS

CONTINUED

- If the products of $\frac{\xi_3}{R_\alpha}$ and ϵ_{12}^o are neglected in

$$2\epsilon_{12} = 2\epsilon_{12}^o + \xi_3 \left[2\chi_{12}^o + \epsilon_{12}^o \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right] \quad \text{such that} \quad 2\epsilon_{12} = 2\epsilon_{12}^o + \xi_3 [2\chi_{12}^o]$$

then the Love-Timoshenko strains reduce to those given by Sanders in

- Sanders, J. L.: *An Improved First Approximation Theory for Thin Shells*. NASA TR R-24, 1959
- Another well-known derivation of first-approximation, thin-shell strains, based on the work of A. E. H. Love, was given by E. Reissner in
 - Reissner, E.: A New Derivation of the Equations for the Deformation of Elastic Shells. *American Journal of Mathematics*, vol. 63, 1941, pp. 177-184

SPECIAL CASES OF THE LINEARIZED SHELL STRAINS

CONCLUDED

- It is important to point out that in Reissner's derivation, the strains are identical to the Love-Timoshenko strains except for $2\varepsilon_{12}$
- More specifically, in Reissner's derivation, the quantity

$$\chi_{12}^{\circ} + \frac{e_{12}^{\circ}}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

appearing in $2\varepsilon_{12} = 2\varepsilon_{12}^{\circ} + \xi_3 \left[2\chi_{12}^{\circ} + \varepsilon_{12}^{\circ} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right]$ is replaced with

$$\chi_{12}^{\circ} + \varphi \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = \frac{1}{A_1} \frac{\partial \varphi_2}{\partial \xi_1} + \frac{\varphi_1}{\rho_{11}} + \frac{1}{A_2} \frac{\partial \varphi_1}{\partial \xi_2} - \frac{\varphi_2}{\rho_{22}},$$

which does not vanish for rigid-body motions, but is adequate for many practical applications

STRAINS FOR SHELLS WITH “SMALL” INITIAL GEOMETRIC IMPERFECTIONS

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS

- Over the past several decades, many studies have been conducted that show relatively small geometric deviations from the idealized geometry of a thin-walled shell can produce rather significant changes in the structural behavior, compared to that of the corresponding idealized shell
 - Thus, accounting for relatively small geometric deviations in the kinematic equations is important, particularly when buckling must be addressed
- Following the present custom, relatively “small” geometric deviations from the idealized geometry of a thin-walled shell are referred to herein as **“small” initial geometric imperfections**
- In general, the geometry of an imperfect shell is characterized by a set of properties $\{ \mathbf{A}_1, \mathbf{A}_2, \theta_{12}, \mathbf{r}_{11}, \mathbf{r}_{12}, \mathbf{r}_{21}, \mathbf{r}_{22}, \rho_{11}, \rho_{22} \}$ that are slightly different from those of the corresponding idealized, perfect shell

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- Once these geometric quantities are known, the shell strains can be obtained from the strain equations presented herein
- Typically, the exact expressions for $\{A_1, A_2, \theta_{12}, r_{11}, r_{12}, r_{21}, r_{22}, \rho_{11}, \rho_{22}\}$ are very complicated for imperfect shells and involve the use of Fourier series to represent the deviation from an idealized geometry
- Moreover, design engineers are interested in establishing behavioral trends that are based on idealized geometries
- Thus, a simpler approach is used in the present study to account for “small” initial geometric imperfections that follows the approach presented by **Donnell** on p. 349 of the book:
 - **Donnell, L. H.: *Beams, Plates, and Shells*. McGraw-Hill Book Co., 1976.**

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- In particular, the effects of initial geometric imperfections are modeled by representing the deviations from a given idealized geometry by a **normal displacement field** $w_i(\xi_1, \xi_2)$
- That is, the normal displacement field of the shell reference surface is represented by the sum of a component $w_i(\xi_1, \xi_2)$ associated with the strain-free unloaded state and a corresponding component $w(\xi_1, \xi_2, t)$ that is produced by applied mechanical and thermal loads and by applied displacements
 - The total **“normal displacement”** from the geometrically perfect shell reference surface is given by $w_i(\xi_1, \xi_2) + w(\xi_1, \xi_2, t)$
- Next $w_i(\xi_1, \xi_2) + w(\xi_1, \xi_2, t)$ is substituted for $w(\xi_1, \xi_2, t)$ in the strain equations, and $u_\alpha(\xi_1, \xi_2, t)$ and $w(\xi_1, \xi_2, t)$ are set equal to zero, consistent with setting the applied loads and displacements to zero

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- This step yields residual strains associated with the initial imperfections that must be removed to obtain an unloaded-shell state
- Thus, strain expressions that include the effects of initial geometric imperfections are obtained by taking the strains obtained by replacing $w(\xi_1, \xi_2, \epsilon)$ with $w_i(\xi_1, \xi_2) + w(\xi_1, \xi_2, \epsilon)$ and then subtracting the corresponding “no-load” residual strains
- With the strains for geometrically imperfect shells known, the geometric parameters $\left\{ \mathcal{A}_1, \mathcal{A}_2, \theta_{12}, \kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22}, \rho_{11}, \rho_{22} \right\}$ that define the deformed-shell reference surface in terms of the strains are also known
 - Moreover, the imperfect-shell strains must satisfy the compatibility equations presented herein previously
- Details of this approach are presented subsequently for shells that undergo “small strains” and “finite rotations”

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- By observing that the strain fields for the shell depend on the strain fields for the idealized reference surface, it follows that the influence of the initial geometric imperfections enters into the analysis through the reference-surface strains
- Consider the reference-surface strains for a geometrically perfect shell undergoing “small strains” and “finite rotations” that are given by

$$\varepsilon_{11}^{\circ} = \Delta_{11} + \Delta_{12} \cos\theta_{12} + \frac{1}{2} \left(\Delta_{11}^2 + 2\Delta_{11}\Delta_{12} \cos\theta_{12} + \Delta_{12}^2 + \Delta_{13}^2 \right) + \mathcal{O}(\varepsilon^4)$$

$$\varepsilon_{22}^{\circ} = \Delta_{21} \cos\theta_{12} + \Delta_{22} + \frac{1}{2} \left(\Delta_{21}^2 + 2\Delta_{21}\Delta_{22} \cos\theta_{12} + \Delta_{22}^2 + \Delta_{23}^2 \right) + \mathcal{O}(\varepsilon^4)$$

$$\begin{aligned} 2\varepsilon_{12}^{\circ} = & \Delta_{12} + \Delta_{21} + (\Delta_{11} + \Delta_{22})\cos\theta_{12} \\ & + \Delta_{11}\Delta_{21} + \Delta_{12}\Delta_{22} + \Delta_{13}\Delta_{23} + (\Delta_{11}\Delta_{22} + \Delta_{12}\Delta_{21})\cos\theta_{12} + \mathcal{O}(\varepsilon^4) \end{aligned}$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

where

$$\Delta_{11} = \mathbf{e}_{11}^{\circ} \csc^2 \theta_{12} - \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} + \varphi \right) \cot \theta_{12}$$

$$\Delta_{12} = \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} + \varphi - \mathbf{e}_{11}^{\circ} \cot \theta_{12} \right) \csc \theta_{12}$$

$$\Delta_{13} = - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right)$$

$$\Delta_{21} = \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \varphi - \mathbf{e}_{22}^{\circ} \cot \theta_{12} \right) \csc \theta_{12}$$

$$\Delta_{22} = \mathbf{e}_{22}^{\circ} \csc^2 \theta_{12} - \left(\mathbf{e}_{12}^{\circ} \csc \theta_{12} - \varphi \right) \cot \theta_{12}$$

$$\Delta_{23} = - \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right)$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- These equations show that the reference-surface strains are quadratic in the following terms

$$\varphi_1 = \left(\frac{u_1}{r_{11}} - \frac{1}{A_1} \frac{\partial w}{\partial \xi_1} \right) \csc^2 \theta_{12} - u_1 \left(\frac{1}{r_{21}} + \frac{\cot \theta_{12}}{r_{22}} \right) \cot \theta_{12} + \left(\frac{\cot \theta_{12}}{A_2} \frac{\partial w}{\partial \xi_2} + \frac{u_2}{r_{21}} \right) \csc \theta_{12}$$

$$\varphi_2 = \left(\frac{u_2}{r_{22}} - \frac{1}{A_2} \frac{\partial w}{\partial \xi_2} \right) \csc^2 \theta_{12} + u_2 \left(\frac{1}{r_{12}} - \frac{\cot \theta_{12}}{r_{11}} \right) \cot \theta_{12} + \left(\frac{\cot \theta_{12}}{A_1} \frac{\partial w}{\partial \xi_1} - \frac{u_1}{r_{12}} \right) \csc \theta_{12}$$

$$2\varphi = \left(\frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} \right) \cot \theta_{12} + \left(\frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} \right) \csc \theta_{12} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}}$$

$$e_{11}^o = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (u_1 + u_2 \cos \theta_{12}) - \frac{u_2 \sin \theta_{12}}{\rho_{11}} + \frac{w}{r_{11}}$$

$$e_{22}^o = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (u_2 + u_1 \cos \theta_{12}) + \frac{u_1 \sin \theta_{12}}{\rho_{22}} + \frac{w}{r_{22}}$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

$$2e_{12}^o = \frac{1}{A_2} \frac{\partial u_1}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial u_2}{\partial \xi_1} + \left(\frac{1}{A_1} \frac{\partial u_1}{\partial \xi_1} + \frac{1}{A_2} \frac{\partial u_2}{\partial \xi_2} \right) \cos \theta_{12} + \left(\frac{u_1}{\rho_{11}} - \frac{u_2}{\rho_{22}} \right) \sin \theta_{12} + w \left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) \cos \theta_{12} + w \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \sin \theta_{12}$$

- By substituting $w \rightarrow w + w_i$ in the previous equations, it follows that a given equation is converted into itself plus an increment associated with the initial geometric imperfection

- For example substituting $w \rightarrow w + w_i$ into the previous equation

for φ_1 produces φ_1 plus the term $\varphi_1^i \equiv \frac{\csc \theta_{12} \cot \theta_{12}}{A_2} \frac{\partial w_i}{\partial \xi_2} - \frac{\csc^2 \theta_{12}}{A_1} \frac{\partial w_i}{\partial \xi_1}$

- This process is indicated by the notation $\varphi_1 \rightarrow \varphi_1 + \varphi_1^i$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- Similarly, $\varphi_2 \rightarrow \varphi_2 + \varphi_2^i$ where $\varphi_2^i \equiv \frac{\csc\theta_{12}\cot\theta_{12}}{A_1} \frac{\partial w_i}{\partial \xi_1} - \frac{\csc^2\theta_{12}}{A_2} \frac{\partial w_i}{\partial \xi_2}$
- In contrast, the rotation φ remains unchanged
- In addition,

$$\mathbf{e}_{11}^o \rightarrow \mathbf{e}_{11}^o + \frac{\mathbf{w}_i}{r_{11}}$$

$$\mathbf{e}_{22}^o \rightarrow \mathbf{e}_{22}^o + \frac{\mathbf{w}_i}{r_{22}}$$

and

$$\mathbf{e}_{12}^o \rightarrow \mathbf{e}_{12}^o + \frac{w_i \sin\theta_{12}}{2} \left[\left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) \cot\theta_{12} + \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \right]$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- Moreover, by using the identity $\frac{1}{r_{12}} + \frac{1}{r_{21}} = \cot\theta_{12} \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right)$, it follows that

$$\Delta_{11} \rightarrow \Delta_{11} + w_i \left(\frac{1}{r_{11}} + \frac{\cot\theta_{12}}{r_{12}} \right)$$

$$\Delta_{12} \rightarrow \Delta_{12} - \frac{w_i \csc\theta_{12}}{r_{12}}$$

$$\Delta_{13} \rightarrow \Delta_{13} + \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1}$$

$$\Delta_{21} \rightarrow \Delta_{21} + \frac{w_i \csc\theta_{12}}{r_{21}}$$

$$\Delta_{22} \rightarrow \Delta_{22} + w_i \left(\frac{1}{r_{22}} - \frac{\cot\theta_{12}}{r_{21}} \right)$$

$$\Delta_{23} \rightarrow \Delta_{23} + \frac{1}{A_2} \frac{\partial w_i}{\partial \xi_2}$$

- Substituting these expressions into the reference-surface strain

$$\epsilon_{11}^o = \Delta_{11} + \Delta_{12} \cos\theta_{12} + \frac{1}{2} \left(\Delta_{11}^2 + 2\Delta_{11}\Delta_{12} \cos\theta_{12} + \Delta_{12}^2 + \Delta_{13}^2 \right) + \mathcal{O}(\epsilon^4)$$

gives

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

$$\begin{aligned} \epsilon_{11}^o \rightarrow \epsilon_{11}^o + \frac{w_i}{r_{11}} + \left(\Delta_{11} + \Delta_{12} \cos \theta_{12} \right) \frac{w_i}{r_{11}} - \Delta_{12} \left(\frac{w_i \sin \theta_{12}}{r_{12}} \right) + \Delta_{13} \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1} \\ + \frac{1}{2} \left[\left(\frac{w_i}{r_{11}} \right)^2 + \left(\frac{w_i}{r_{12}} \right)^2 + \left(\frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1} \right)^2 \right] + \mathcal{O}(\epsilon^4) \end{aligned}$$

- When the shell is not subjected to loads ($u_1 = u_2 = w = 0$), this expression reduces to

$$\frac{w_i}{r_{11}} + \frac{1}{2} \left[\left(\frac{w_i}{r_{11}} \right)^2 + \left(\frac{w_i}{r_{12}} \right)^2 + \left(\frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1} \right)^2 \right]$$

- For the shell to be strain free in the unloaded state, these terms must be subtracted from the strain expression given above

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- This process yields the strain

$$\begin{aligned}\epsilon_{11}^o = & \left(\Delta_{11} + \Delta_{12} \cos\theta_{12} \right) \left(1 + \frac{w_i}{r_{11}} \right) - \Delta_{12} \left(\frac{w_i \sin\theta_{12}}{r_{12}} \right) + \Delta_{13} \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1} \\ & + \frac{1}{2} \left(\Delta_{11}^2 + 2\Delta_{11}\Delta_{12} \cos\theta_{12} + \Delta_{12}^2 + \Delta_{13}^2 \right) + \mathcal{O}(\epsilon^4)\end{aligned}$$

which is the same as

$$\begin{aligned}\epsilon_{11}^o = & \mathbf{e}_{11}^o \left(1 + \frac{w_i}{r_{11}} - \cot\theta_{12} \frac{w_i}{r_{12}} \right) - \left(\mathbf{e}_{12}^o \csc\theta_{12} + \varphi \right) \left(\mathbf{e}_{11}^o \cot\theta_{12} + \frac{w_i}{r_{12}} \right) \\ & - \left(\varphi_1 + \varphi_2 \cos\theta_{12} \right) \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1} + \frac{1}{2} \left(\mathbf{e}_{11}^o \csc\theta_{12} \right)^2 + \frac{1}{2} \left(\mathbf{e}_{12}^o \csc\theta_{12} + \varphi \right)^2 \\ & + \frac{1}{2} \left(\varphi_1 + \varphi_2 \cos\theta_{12} \right)^2 + \mathcal{O}(\epsilon^4)\end{aligned}$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- To simplify the presentation and enhance clarity, let the reference-surface strains of a geometrically perfect shell be denoted by

$$\begin{aligned}\bar{\epsilon}_{11}^{\circ} = \mathbf{e}_{11}^{\circ} - & \left(\mathbf{e}_{12}^{\circ} \csc\theta_{12} + \varphi \right) \mathbf{e}_{11}^{\circ} \cot\theta_{12} + \frac{1}{2} \left(\mathbf{e}_{11}^{\circ} \csc\theta_{12} \right)^2 \\ & + \frac{1}{2} \left(\mathbf{e}_{12}^{\circ} \csc\theta_{12} + \varphi \right)^2 + \frac{1}{2} \left(\varphi_1 + \varphi_2 \cos\theta_{12} \right)^2 + \mathcal{O}(\epsilon^4)\end{aligned}$$

$$\begin{aligned}\bar{\epsilon}_{22}^{\circ} = \mathbf{e}_{22}^{\circ} - & \left(\mathbf{e}_{12}^{\circ} \csc\theta_{12} - \varphi \right) \mathbf{e}_{22}^{\circ} \cot\theta_{12} + \frac{1}{2} \left(\mathbf{e}_{22}^{\circ} \csc\theta_{12} \right)^2 \\ & + \frac{1}{2} \left(\mathbf{e}_{12}^{\circ} \csc\theta_{12} - \varphi \right)^2 + \frac{1}{2} \left(\varphi_1 \cos\theta_{12} + \varphi_2 \right)^2 + \mathcal{O}(\epsilon^4)\end{aligned}$$

$$\begin{aligned}2\bar{\epsilon}_{12}^{\circ} = 2\mathbf{e}_{12}^{\circ} + & \left[\mathbf{e}_{11}^{\circ} \left(\mathbf{e}_{12}^{\circ} \csc\theta_{12} - \varphi \right) + \mathbf{e}_{22}^{\circ} \left(\mathbf{e}_{12}^{\circ} \csc\theta_{12} + \varphi \right) \right] \csc\theta_{12} \\ & - \left[\left(\mathbf{e}_{12}^{\circ} + \varphi \sin\theta_{12} \right) \left(\mathbf{e}_{12}^{\circ} - \varphi \sin\theta_{12} \right) + \mathbf{e}_{11}^{\circ} \mathbf{e}_{22}^{\circ} \right] \cot\theta_{12} \csc\theta_{12} \\ & + \left(\varphi_1 + \varphi_2 \cos\theta_{12} \right) \left(\varphi_1 \cos\theta_{12} + \varphi_2 \right) + \mathcal{O}(\epsilon^4)\end{aligned}$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- Then, ϵ_{11}° for the corresponding imperfect shell is given by

$$\epsilon_{11}^{\circ} = \bar{\epsilon}_{11}^{\circ} + \bar{\bar{\epsilon}}_{11}^{\circ}, \text{ where}$$

$$\bar{\bar{\epsilon}}_{11}^{\circ} = \mathbf{e}_{11}^{\circ} \left(\frac{w_i}{r_{11}} - \cot\theta_{12} \frac{w_i}{r_{12}} \right) - \left(\mathbf{e}_{12}^{\circ} \csc\theta_{12} + \varphi \right) \frac{w_i}{r_{12}} - \left(\varphi_1 + \varphi_2 \cos\theta_{12} \right) \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1}$$

- Likewise, the other two strains are given by

$$\epsilon_{22}^{\circ} = \bar{\epsilon}_{22}^{\circ} + \bar{\bar{\epsilon}}_{22}^{\circ} \quad \text{and} \quad 2\epsilon_{12}^{\circ} = 2\bar{\epsilon}_{12}^{\circ} + 2\bar{\bar{\epsilon}}_{12}^{\circ}, \text{ where}$$

$$\bar{\bar{\epsilon}}_{22}^{\circ} = \mathbf{e}_{22}^{\circ} \left(\frac{w_i}{r_{22}} - \cot\theta_{12} \frac{w_i}{r_{21}} \right) + \left(\mathbf{e}_{12}^{\circ} \csc\theta_{12} - \varphi \right) \frac{w_i}{r_{21}} - \left(\varphi_1 \cos\theta_{12} + \varphi_2 \right) \frac{1}{A_2} \frac{\partial w_i}{\partial \xi_2}$$

and

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

$$\begin{aligned}
 2\bar{\bar{\epsilon}}_{12}^{\circ} = & \left(\mathbf{e}_{12}^{\circ} + \varphi \sin\theta_{12} \right) \left(\frac{\mathbf{w}_i}{\mathbf{r}_{22}} - \frac{\mathbf{w}_i \cot\theta_{12}}{\mathbf{r}_{21}} \right) + \left(\mathbf{e}_{12}^{\circ} - \varphi \sin\theta_{12} \right) \left(\frac{\mathbf{w}_i}{\mathbf{r}_{11}} + \frac{\mathbf{w}_i \cot\theta_{12}}{\mathbf{r}_{12}} \right) \\
 & + \left(\mathbf{e}_{11}^{\circ} \frac{\mathbf{w}_i}{\mathbf{r}_{21}} - \mathbf{e}_{22}^{\circ} \frac{\mathbf{w}_i}{\mathbf{r}_{12}} \right) \csc\theta_{12} - \left(\varphi_1 + \varphi_2 \cos\theta_{12} \right) \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{w}_i}{\partial \xi_2} - \left(\varphi_1 \cos\theta_{12} + \varphi_2 \right) \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{w}_i}{\partial \xi_1}
 \end{aligned}$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- Now, consider the previously derived expressions for the changes in reference-surface curvatures and torsion, for a geometrically perfect shell, given by

$$K_{11}^{\circ} = \frac{1}{\bar{r}_{11}} - \frac{1 + 3\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}}{r_{11}} + \mathcal{O}(\varepsilon^4)$$

$$K_{22}^{\circ} = \frac{1}{\bar{r}_{22}} - \frac{1 + \varepsilon_{11}^{\circ} + 3\varepsilon_{22}^{\circ}}{r_{22}} + \mathcal{O}(\varepsilon^4)$$

$$2K_{12}^{\circ} = - \left[\frac{1}{\bar{r}_{12}} - \frac{1}{\bar{r}_{21}} - \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) \right] + \frac{2}{r_{12}} (2\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}) - \frac{2}{r_{21}} (\varepsilon_{11}^{\circ} + 2\varepsilon_{22}^{\circ}) + \mathcal{O}(\varepsilon^4)$$

where

$$\begin{aligned} \frac{1}{\bar{r}_{11}} = & \left(1 + e_{11}^{\circ} \right) \mathbf{m}^{(1)} \Big|_{\langle 1 \rangle} + \left(\cos \theta_{12} + e_{12}^{\circ} + \varphi \sin \theta_{12} \right) \mathbf{m}^{(2)} \Big|_{\langle 1 \rangle} \\ & - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \mathbf{m}^{(3)} \Big|_{\langle 1 \rangle} \end{aligned}$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

$$\frac{1}{\bar{r}_{22}} = (1 + \Delta_{22}) \left[m^{(1)} \Big|_{\langle 2 \rangle} \cos \theta_{12} + m^{(2)} \Big|_{\langle 2 \rangle} \right] \\ + \Delta_{21} \left[m^{(1)} \Big|_{\langle 2 \rangle} + m^{(2)} \Big|_{\langle 2 \rangle} \cos \theta_{12} \right] + \Delta_{23} \left[m^{(3)} \Big|_{\langle 2 \rangle} \right]$$

$$\frac{1}{\bar{r}_{12}} = \left[m_2 \Delta_{13} - m_3 \Delta_{12} \right] m^{(1)} \Big|_{\langle 1 \rangle} \sin \theta_{12} + \left[m_3 (1 + \Delta_{11}) - m_1 \Delta_{13} \right] m^{(2)} \Big|_{\langle 1 \rangle} \sin \theta_{12} \\ + \left[m_1 \Delta_{12} - m_2 (1 + \Delta_{11}) \right] m^{(3)} \Big|_{\langle 1 \rangle} \sin \theta_{12}$$

$$\frac{1}{\bar{r}_{21}} = \left[m_3 (1 + \Delta_{22}) - m_2 \Delta_{23} \right] m^{(1)} \Big|_{\langle 2 \rangle} \sin \theta_{12} + \left[m_1 \Delta_{23} - m_3 \Delta_{21} \right] m^{(2)} \Big|_{\langle 2 \rangle} \sin \theta_{12} \\ + \left[m_2 \Delta_{21} - m_1 (1 + \Delta_{22}) \right] m^{(3)} \Big|_{\langle 2 \rangle} \sin \theta_{12}$$

with

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

$$\begin{aligned} m_1 = \varphi_1 - \left(e_{12}^o \csc \theta_{12} + \varphi \right) \left(\varphi_1 \cot \theta_{12} + \varphi_2 \csc \theta_{12} \right) \\ + e_{22}^o \csc \theta_{12} \left(\varphi_1 \csc \theta_{12} + \varphi_2 \cot \theta_{12} \right) \end{aligned}$$

$$\begin{aligned} m_2 = \varphi_2 - \left(e_{12}^o \csc \theta_{12} - \varphi \right) \left(\varphi_1 \csc \theta_{12} + \varphi_2 \cot \theta_{12} \right) \\ + e_{11}^o \csc \theta_{12} \left(\varphi_1 \cot \theta_{12} + \varphi_2 \csc \theta_{12} \right) \end{aligned}$$

$$\begin{aligned} m_3 = 1 + \varphi^2 + \left(e_{11}^o + e_{22}^o + e_{11}^o e_{22}^o - \left(e_{12}^o \right)^2 \right) \csc^2 \theta_{12} \\ - 2 e_{12}^o \cot \theta_{12} \csc \theta_{12} \end{aligned}$$

$$m^{(1)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_1}{\partial \xi_1} - \frac{m_2 \csc \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\csc \theta_{12}}{\rho_{11}} \left(m_1 \cos \theta_{12} + m_2 \right) + m_3 \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right)$$

$$m^{(2)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_2}{\partial \xi_1} + \frac{\csc \theta_{12}}{\rho_{11}} \left(m_1 + m_2 \cos \theta_{12} \right) + \frac{m_2 \cot \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - m_3 \frac{\csc \theta_{12}}{r_{12}}$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

$$m^{(3)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_3}{\partial \xi_1} - \frac{m_1}{r_{11}} + m_2 \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right)$$

$$m^{(1)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_1}{\partial \xi_2} + \frac{m_1 \cot \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} - \frac{\csc \theta_{12}}{\rho_{22}} \left(m_1 \cos \theta_{12} + m_2 \right) + m_3 \frac{\csc \theta_{12}}{r_{21}}$$

$$m^{(2)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_2}{\partial \xi_2} - \frac{m_1 \csc \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} + \frac{\csc \theta_{12}}{\rho_{22}} \left(m_1 + m_2 \cos \theta_{12} \right) + m_3 \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right)$$

$$m^{(3)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_3}{\partial \xi_2} - m_1 \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) - \frac{m_2}{r_{22}}$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- Next, substituting $\varphi_1 \rightarrow \varphi_1 + \varphi_1^i$, $\varphi_2 \rightarrow \varphi_2 + \varphi_2^i$, $\mathbf{e}_{11}^o \rightarrow \mathbf{e}_{11}^o + \frac{w_i}{r_{11}}$,

$$\mathbf{e}_{22}^o \rightarrow \mathbf{e}_{22}^o + \frac{w_i}{r_{22}}, \text{ and } \mathbf{e}_{12}^o \rightarrow \mathbf{e}_{12}^o + \frac{w_i \sin \theta_{12}}{2} \left[\left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) \cot \theta_{12} + \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \right]$$

into the components of $\vec{m}$ and eliminating terms that produce strains

in the absence of applied loads yields $m_1 \rightarrow m_1 + m_1^i$, $m_2 \rightarrow m_2 + m_2^i$,

and $m_3 \rightarrow m_3 + m_3^i$; where

$$m_1^i = \varphi_1^i + \left(\mathbf{e}_{12}^o \csc \theta_{12} + \varphi \right) \csc \theta_{12} \frac{1}{A_2} \frac{\partial w_i}{\partial \xi_2} - \mathbf{e}_{22}^o \csc^2 \theta_{12} \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1} + \left(\frac{w_i}{r_{22}} - \frac{w_i \cot \theta_{12}}{r_{21}} \right) \varphi_1 - \frac{w_i \csc \theta_{12}}{r_{21}} \varphi_2$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

$$\begin{aligned} m_2^i = \varphi_2^i + \left(e_{12}^o \csc \theta_{12} - \varphi \right) \csc \theta_{12} \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1} - e_{11}^o \csc^2 \theta_{12} \frac{1}{A_2} \frac{\partial w_i}{\partial \xi_2} \\ + \frac{w_i \csc \theta_{12}}{r_{12}} \varphi_1 + \left(\frac{w_i}{r_{11}} + \frac{w_i \cot \theta_{12}}{r_{12}} \right) \varphi_2 \end{aligned}$$

$$\begin{aligned} m_3^i = \left(e_{22}^o \csc \theta_{12} - e_{12}^o \cot \theta_{12} \right) \frac{w_i}{r_{11}} \csc \theta_{12} + \left(e_{11}^o \csc \theta_{12} - e_{12}^o \cot \theta_{12} \right) \frac{w_i}{r_{22}} \csc \theta_{12} \\ - \left(\frac{w_i}{r_{21}} - \frac{w_i}{r_{12}} \right) e_{12}^o \csc \theta_{12} \end{aligned}$$

- In addition, $m^{(k)} \Big|_{\langle \alpha \rangle} \rightarrow m^{(k)} \Big|_{\langle \alpha \rangle} + m_i^{(k)} \Big|_{\langle \alpha \rangle}$ for $k \in \{1, 2, 3\}$ and $\alpha \in \{1, 2\}$, where

$$m_i^{(1)} \Big|_{\langle 1 \rangle} = \frac{1}{A_1} \frac{\partial m_1^i}{\partial \xi_1} - \frac{m_2^i \csc \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - \frac{\csc \theta_{12}}{\rho_{11}} \left(m_1^i \cos \theta_{12} + m_2^i \right) + m_3^i \left(\frac{1}{r_{11}} + \frac{\cot \theta_{12}}{r_{12}} \right)$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

$$m_i^{(2)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_2^i}{\partial \xi_1} + \frac{\csc \theta_{12}}{\rho_{11}} \left(m_1^i + m_2^i \cos \theta_{12} \right) + \frac{m_2^i \cot \theta_{12}}{A_1} \frac{\partial \theta_{12}}{\partial \xi_1} - m_3^i \frac{\csc \theta_{12}}{r_{12}}$$

$$m_i^{(3)} \Big|_{(1)} = \frac{1}{A_1} \frac{\partial m_3^i}{\partial \xi_1} - \frac{m_1^i}{r_{11}} + m_2^i \left(\frac{\sin \theta_{12}}{r_{12}} - \frac{\cos \theta_{12}}{r_{11}} \right)$$

$$m^{(1)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_1}{\partial \xi_2} + \frac{m_1 \cot \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} - \frac{\csc \theta_{12}}{\rho_{22}} \left(m_1 \cos \theta_{12} + m_2 \right) + m_3 \frac{\csc \theta_{12}}{r_{21}}$$

$$m^{(2)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_2}{\partial \xi_2} - \frac{m_1 \csc \theta_{12}}{A_2} \frac{\partial \theta_{12}}{\partial \xi_2} + \frac{\csc \theta_{12}}{\rho_{22}} \left(m_1 + m_2 \cos \theta_{12} \right) + m_3 \left(\frac{1}{r_{22}} - \frac{\cot \theta_{12}}{r_{21}} \right)$$

$$m^{(3)} \Big|_{(2)} = \frac{1}{A_2} \frac{\partial m_3}{\partial \xi_2} - m_1 \left(\frac{\sin \theta_{12}}{r_{21}} + \frac{\cos \theta_{12}}{r_{22}} \right) - \frac{m_2}{r_{22}}$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- Substituting these results, and the corresponding previously obtained results, into the expression for $\bar{\bar{\tau}}_{\alpha\beta}^{-1}$ and eliminating terms that produce

strains in the absence of applied loads, yields

$$\frac{1}{\bar{\bar{\tau}}_{\alpha\beta}} \rightarrow \frac{1}{\bar{\tau}_{\alpha\beta}} + \frac{1}{\bar{\tau}_{\alpha\beta}^i}$$

- The expressions for $\frac{1}{\bar{\tau}_{\alpha\beta}^i}$ are very complicated nonlinear functions; for example,

$$\begin{aligned} \frac{1}{\bar{\tau}_{11}^i} = & \mathbf{e}_{11}^o \mathbf{m}_i^{(1)} \Big|_{\langle 1 \rangle} + \left(\mathbf{e}_{12}^o + \varphi \sin \theta_{12} \right) \mathbf{m}_i^{(2)} \Big|_{\langle 1 \rangle} - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \mathbf{m}_i^{(3)} \Big|_{\langle 1 \rangle} + \left(\frac{\mathbf{w}_i}{\mathbf{r}_{11}} \right) \mathbf{m}^{(1)} \Big|_{\langle 1 \rangle} \\ & + \frac{\mathbf{w}_i \sin \theta_{12}}{2} \left[\left(\frac{1}{\mathbf{r}_{11}} + \frac{1}{\mathbf{r}_{22}} \right) \cot \theta_{12} + \left(\frac{1}{\mathbf{r}_{21}} - \frac{1}{\mathbf{r}_{12}} \right) \right] \mathbf{m}^{(2)} \Big|_{\langle 1 \rangle} + \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{w}_i}{\partial \xi_1} \mathbf{m}^{(3)} \Big|_{\langle 1 \rangle} \end{aligned}$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- The changes in reference-surface curvatures and torsions are expressed as $\bar{\kappa}_{\alpha\beta}^{\circ} = \bar{\kappa}_{\alpha\beta}^{\circ} + \bar{\bar{\kappa}}_{\alpha\beta}^{\circ}$, where the single and double overbars denote the quantity for the geometrically perfect and imperfect shells, respectively
- In particular,

$$\bar{\kappa}_{11}^{\circ} = \frac{1}{\bar{r}_{11}} - \frac{1 + 3\bar{\epsilon}_{11}^{\circ} + \bar{\epsilon}_{22}^{\circ}}{r_{11}} + \mathcal{O}(\epsilon^4)$$

$$\bar{\kappa}_{22}^{\circ} = \frac{1}{\bar{r}_{22}} - \frac{1 + \bar{\epsilon}_{11}^{\circ} + 3\bar{\epsilon}_{22}^{\circ}}{r_{22}} + \mathcal{O}(\epsilon^4)$$

$$2\bar{\kappa}_{12}^{\circ} = - \left[\frac{1}{\bar{r}_{12}} - \frac{1}{\bar{r}_{21}} - \left(\frac{1}{r_{12}} - \frac{1}{r_{21}} \right) \right] + \frac{2}{r_{12}} (2\bar{\epsilon}_{11}^{\circ} + \bar{\epsilon}_{22}^{\circ}) - \frac{2}{r_{21}} (\bar{\epsilon}_{11}^{\circ} + 2\bar{\epsilon}_{22}^{\circ}) + \mathcal{O}(\epsilon^4)$$

for the geometrically perfect shell

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- Moreover, the contributions of the initial geometric imperfection to the changes in curvatures and torsions are given by

$$\bar{\bar{K}}_{11}^{\circ} = \frac{1}{\bar{\bar{\mathbf{z}}}_i} - \frac{w_i}{r_{11}} \left(\frac{3}{r_{11}} + \frac{1}{r_{22}} \right)$$

$$\bar{\bar{K}}_{22}^{\circ} = \frac{1}{\bar{\bar{\mathbf{z}}}_i} - \frac{w_i}{r_{22}} \left(\frac{1}{r_{11}} + \frac{3}{r_{22}} \right)$$

$$2\bar{\bar{K}}_{12}^{\circ} = -\frac{1}{\bar{\bar{\mathbf{z}}}_{12}} + \frac{1}{\bar{\bar{\mathbf{z}}}_{21}} + \frac{2w_i}{r_{12}} \left(\frac{2}{r_{11}} + \frac{1}{r_{22}} \right) - \frac{2w_i}{r_{21}} \left(\frac{1}{r_{11}} + \frac{2}{r_{22}} \right)$$

- Inspection of these equations for the general case of “**small strains**” and “**finite rotations**” reveals that modelling the effects of initial geometric imperfections is a difficult process

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

- The displacement fields for points of the shell are given by

$$\vec{U} = U_1 \hat{g}_1 + U_2 \hat{g}_2 + U_3 \hat{g}_3 \quad \text{with}$$

$$U_1 = \frac{\mu_{22} U_1^0 - \mu_{21} U_2^0}{\mu_{11} \mu_{22} - \mu_{12} \mu_{21}}, \quad U_2 = \frac{\mu_{11} U_2^0 - \mu_{12} U_1^0}{\mu_{11} \mu_{22} - \mu_{12} \mu_{21}}, \quad \text{and} \quad U_3 = U_3^0; \quad \text{and where}$$

$$\mu_{11} = \frac{1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}}\right)^2}}$$

$$\mu_{12} = \frac{-\frac{\xi_3 \csc \theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}}\right)^2}}$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

$$\mu_{21} = \frac{\frac{\xi_3 \csc\theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}}$$

$$\mu_{22} = \frac{1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot\theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc\theta_{12}}{r_{21}}\right)^2}}$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONTINUED

$$\begin{aligned} \mathbf{U}_1^0 = & \mathbf{u}_1 + \xi_3 \left(\mathbf{m}_1 + \mathbf{m}_1^i \right) \left(1 + 2\varepsilon_{12}^o \cot\theta_{12} \csc\theta_{12} - \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \csc^2\theta_{12} + \mathcal{O}(\varepsilon^4) \right) \\ & + \left(1 + \Delta_{11} + \frac{\mathbf{w}_i}{\mathbf{r}_{11}} + \frac{\mathbf{w}_i \cot\theta_{12}}{\mathbf{r}_{12}} \right) \left(\mathbf{F}_1(\xi_3) \boldsymbol{\gamma}_1^o + \mathcal{O}(\varepsilon^4) \right) \\ & + \left[\Delta_{21} + \frac{\mathbf{w}_i \csc\theta_{12}}{\mathbf{r}_{21}} \right] \left(\mathbf{F}_2(\xi_3) \boldsymbol{\gamma}_2^o + \mathcal{O}(\varepsilon^4) \right) \end{aligned}$$

$$\begin{aligned} \mathbf{U}_2^0 = & \mathbf{u}_2 + \xi_3 \left(\mathbf{m}_2 + \mathbf{m}_2^i \right) \left(1 + 2\varepsilon_{12}^o \cot\theta_{12} \csc\theta_{12} - \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \csc^2\theta_{12} + \mathcal{O}(\varepsilon^4) \right) \\ & + \left[\Delta_{12} - \frac{\mathbf{w}_i \csc\theta_{12}}{\mathbf{r}_{12}} \right] \left(\mathbf{F}_1(\xi_3) \boldsymbol{\gamma}_1^o + \mathcal{O}(\varepsilon^4) \right) \\ & + \left(1 + \Delta_{22} + \frac{\mathbf{w}_i}{\mathbf{r}_{22}} - \frac{\mathbf{w}_i \cot\theta_{12}}{\mathbf{r}_{21}} \right) \left(\mathbf{F}_2(\xi_3) \boldsymbol{\gamma}_2^o + \mathcal{O}(\varepsilon^4) \right) \end{aligned}$$

STRAINS FOR SHELLS WITH INITIAL GEOMETRIC IMPERFECTIONS - CONCLUDED

$$\begin{aligned} U_3^0 = w + w_i + \xi_3 & \left[\left(m_3 + m_3^i \right) \left(1 + 2\varepsilon_{12}^o \cot\theta_{12} \csc\theta_{12} - \left(\varepsilon_{11}^o + \varepsilon_{22}^o \right) \csc^2\theta_{12} + \mathcal{O}(\varepsilon^4) \right) - 1 \right] \\ & + \left(\Delta_{13} + \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1} \right) \left(F_1(\xi_3) \gamma_1^o + \mathcal{O}(\varepsilon^4) \right) + \left(\Delta_{23} + \frac{1}{A_2} \frac{\partial w_i}{\partial \xi_2} \right) \left(F_2(\xi_3) \gamma_2^o + \mathcal{O}(\varepsilon^4) \right) \end{aligned}$$

**“SMALL” STRAINS, “MODERATE” ROTATIONS,
AND “SMALL” INITIAL
GEOMETRIC IMPERFECTIONS**

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS

- The previous section of the present study shows that including the effects of “small” initial geometric imperfections in the equations for shell undergoing “small” strains and “finite” rotations is a rather difficult process
- Fortunately, many engineering problems of practical importance undergo **“small” strain and “moderate” rotations**
- For this class of deformations, including the effects of “small” initial geometric imperfections is much simpler
- Subsequently, the strains for imperfect shells are derived by using a **“small” strain and “moderate” rotation** theory, like that of Pietraszkiewicz, that includes nonlinear bending action

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS CONTINUED

- For a “small” strain and “moderate” rotation theory like that of Pietraszkiewicz (1980), that includes nonlinear bending action, the magnitudes of the linear deformation parameters are restricted by

$$|\varphi_\alpha| \leq \mathcal{O}(\theta), \quad |\varphi| \leq \mathcal{O}(\theta), \quad \text{and} \quad |\mathbf{e}_{\beta\gamma}^\circ| \leq \mathcal{O}(\theta^2); \quad \text{where} \quad 0 < \theta < 1$$

- For this case, the reference-surface Green-Lagrange strains are given in terms of the linear deformation parameters by

$$\boldsymbol{\varepsilon}_{11}^\circ = \mathbf{e}_{11}^\circ + \frac{1}{2}(\varphi_1 + \varphi_2 \cos\theta_{12})^2 + \frac{1}{2}\varphi^2 + \varphi(\mathbf{e}_{12}^\circ \csc\theta_{12} - \mathbf{e}_{11}^\circ \cot\theta_{12}) + \mathcal{O}(\theta^4, \varepsilon^4)$$

$$\boldsymbol{\varepsilon}_{22}^\circ = \mathbf{e}_{22}^\circ + \frac{1}{2}(\varphi_2 + \varphi_1 \cos\theta_{12})^2 + \frac{1}{2}\varphi^2 + \varphi(\mathbf{e}_{22}^\circ \cot\theta_{12} - \mathbf{e}_{12}^\circ \csc\theta_{12}) + \mathcal{O}(\theta^4, \varepsilon^4)$$

$$\begin{aligned} 2\boldsymbol{\varepsilon}_{12}^\circ = 2\mathbf{e}_{12}^\circ + & (\varphi_1 + \varphi_2 \cos\theta_{12})(\varphi_2 + \varphi_1 \cos\theta_{12}) \\ & - \varphi^2 \cos\theta_{12} + \varphi(\mathbf{e}_{22}^\circ - \mathbf{e}_{11}^\circ) \csc\theta_{12} + \mathcal{O}(\theta^4, \varepsilon^4) \end{aligned}$$

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS CONTINUED

- In these strain expressions, $\mathcal{O}(\theta^4, \varepsilon^4)$ indicates that terms fourth order in the rotations and fourth order in the strains are neglected
- The **changes in reference-surface curvatures**, $K_{11}^o(\xi_1, \xi_2, t)$ and $K_{22}^o(\xi_1, \xi_2, t)$, and the **change in reference-surface torsion**, $K_{12}^o(\xi_1, \xi_2, t)$, for a geometrically perfect shell that are caused by deformation are given by

$$K_{11}^o = \chi_{11}^o + \frac{1}{r_{12}} \left[\mathbf{e}_{11}^o \cot \theta_{12} - \mathbf{e}_{12}^o \csc \theta_{12} - (\varphi_1 \varphi_2 + \varphi_2^2 \cos \theta_{12}) \sin \theta_{12} \right] \\ - \frac{1}{2r_{11}} \left(2\mathbf{e}_{11}^o + \varphi_1^2 + \varphi_2^2 + 2\varphi^2 + 2\varphi_1 \varphi_2 \cos \theta_{12} \right) - \frac{\varphi_2 \sin \theta_{12}}{\mathbf{A}_1} \frac{\partial \varphi}{\partial \xi_1} + \mathcal{O}(\theta^3, \varepsilon^4)$$

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS CONTINUED

$$\begin{aligned} K_{22}^{\circ} = & \chi_{22}^{\circ} + \frac{1}{r_{21}} \left[e_{12}^{\circ} \csc \theta_{12} - e_{22}^{\circ} \cot \theta_{12} + (\varphi_1 \varphi_2 + \varphi_1^2 \cos \theta_{12}) \sin \theta_{12} \right] \\ & - \frac{1}{2r_{22}} \left[2e_{22}^{\circ} + \varphi_1^2 + \varphi_2^2 + 2\varphi^2 + 2\varphi_1 \varphi_2 \cos \theta_{12} \right] + \frac{\varphi_1 \sin \theta_{12}}{A_2} \frac{\partial \varphi}{\partial \xi_2} + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

$$\begin{aligned} 2K_{12}^{\circ} = & 2\chi_{12}^{\circ} \csc \theta_{12} - (\chi_{11}^{\circ} + \chi_{22}^{\circ}) \cot \theta_{12} + \frac{e_{11}^{\circ}}{r_{12}} - \frac{e_{22}^{\circ}}{r_{21}} \\ & - \frac{e_{12}^{\circ} \csc \theta_{12} - e_{11}^{\circ} \cot \theta_{12}}{r_{11}} + \frac{e_{22}^{\circ} \cot \theta_{12} - e_{12}^{\circ} \csc \theta_{12}}{r_{22}} \\ & + \frac{1}{r_{12}} \left[(\varphi_1 + \varphi_2 \cos \theta_{12})^2 + \varphi^2 \right] - \frac{1}{r_{21}} \left[(\varphi_2 + \varphi_1 \cos \theta_{12})^2 + \varphi^2 \right] \\ & + (\varphi_1 + \varphi_2 \cos \theta_{12}) \frac{1}{A_1} \frac{\partial \varphi}{\partial \xi_1} - (\varphi_2 + \varphi_1 \cos \theta_{12}) \frac{1}{A_2} \frac{\partial \varphi}{\partial \xi_2} + \mathcal{O}(\theta^3, \varepsilon^4) \end{aligned}$$

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS CONTINUED

where the additional linear deformation parameters associated with bending and twisting of the reference surface are given by

$$\chi_{11}^{\circ} = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (\varphi_1 + \varphi_2 \cos \theta_{12}) - \frac{\varphi_2 \sin \theta_{12}}{\rho_{11}} - \frac{\varphi}{r_{12}}$$

$$\chi_{22}^{\circ} = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\varphi_2 + \varphi_1 \cos \theta_{12}) + \frac{\varphi_1 \sin \theta_{12}}{\rho_{22}} - \frac{\varphi}{r_{21}}$$

$$2\chi_{12}^{\circ} = \frac{1}{A_2} \frac{\partial \varphi_1}{\partial \xi_2} + \frac{\cos \theta_{12}}{A_2} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\cos \theta_{12}}{A_1} \frac{\partial \varphi_1}{\partial \xi_1} + \frac{1}{A_1} \frac{\partial \varphi_2}{\partial \xi_1} \\ + \left(\frac{\varphi_1}{\rho_{11}} - \frac{\varphi_2}{\rho_{22}} \right) \sin \theta_{12} - \varphi \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc \theta_{12}$$

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS CONTINUED

- Making the substitutions $\varphi_1 \rightarrow \varphi_1 + \varphi_1^i$, $\varphi_2 \rightarrow \varphi_2 + \varphi_2^i$, $\mathbf{e}_{11}^o \rightarrow \mathbf{e}_{11}^o + \frac{\mathbf{w}_i}{r_{11}}$,

$$\mathbf{e}_{22}^o \rightarrow \mathbf{e}_{22}^o + \frac{\mathbf{w}_i}{r_{22}}, \text{ and } \mathbf{e}_{12}^o \rightarrow \mathbf{e}_{12}^o + \frac{\mathbf{w}_i \sin \theta_{12}}{2} \left[\left(\frac{1}{r_{11}} + \frac{1}{r_{22}} \right) \cot \theta_{12} + \left(\frac{1}{r_{21}} - \frac{1}{r_{12}} \right) \right]$$

and then enforcing the requirement that the strains vanish in the

unloaded state yields $\boldsymbol{\varepsilon}_{\alpha\beta}^o = \bar{\boldsymbol{\varepsilon}}_{\alpha\beta}^o + \bar{\bar{\boldsymbol{\varepsilon}}}_{\alpha\beta}^o$ and $\mathbf{K}_{\alpha\beta}^o = \bar{\mathbf{K}}_{\alpha\beta}^o + \bar{\bar{\mathbf{K}}}_{\alpha\beta}^o$, where the single and double overbars denote the quantity for the geometrically perfect and imperfect shells, respectively

- Specifically, the contributions of the initial geometric imperfection to the reference-surface strain measures are given by

$$\bar{\bar{\boldsymbol{\varepsilon}}}_{11}^o = - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \frac{1}{\mathbf{A}_1} \frac{\partial \mathbf{w}_i}{\partial \xi_1} - \varphi \frac{\mathbf{w}_i}{r_{12}}$$

$$\bar{\bar{\boldsymbol{\varepsilon}}}_{22}^o = - \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) \frac{1}{\mathbf{A}_2} \frac{\partial \mathbf{w}_i}{\partial \xi_2} - \varphi \frac{\mathbf{w}_i}{r_{21}}$$

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS CONTINUED

$$2\bar{\bar{\epsilon}}_{12}^o = \varphi \csc\theta_{12} \left(\frac{w_i}{r_{22}} - \frac{w_i}{r_{11}} \right) - \left(\varphi_1 + \varphi_2 \cos\theta_{12} \right) \frac{1}{A_2} \frac{\partial w_i}{\partial \xi_2} - \left(\varphi_1 \cos\theta_{12} + \varphi_2 \right) \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1}$$

$$\begin{aligned} \bar{K}_{11}^o = - & \left[\frac{\varphi_1 + \varphi_2 \cos\theta_{12}}{r_{12}} + \frac{1}{A_1} \frac{\partial \varphi}{\partial \xi_1} \right] \left(\frac{\cot\theta_{12}}{A_1} \frac{\partial w_i}{\partial \xi_1} - \frac{\csc\theta_{12}}{A_2} \frac{\partial w_i}{\partial \xi_2} \right) \\ & + \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1} \frac{\varphi_2 \sin\theta_{12}}{r_{12}} + \frac{1}{r_{11}} \left(\frac{\varphi_1}{A_1} \frac{\partial w_i}{\partial \xi_1} + \frac{\varphi_2}{A_2} \frac{\partial w_i}{\partial \xi_2} \right) \end{aligned}$$

$$\begin{aligned} \bar{K}_{22}^o = - & \left[\frac{\varphi_2 + \varphi_1 \cos\theta_{12}}{r_{21}} + \frac{1}{A_2} \frac{\partial \varphi}{\partial \xi_2} \right] \left(\frac{\csc\theta_{12}}{A_1} \frac{\partial w_i}{\partial \xi_1} - \frac{\cot\theta_{12}}{A_2} \frac{\partial w_i}{\partial \xi_2} \right) \\ & - \frac{1}{A_2} \frac{\partial w_i}{\partial \xi_2} \frac{\varphi_1 \sin\theta_{12}}{r_{21}} + \frac{1}{r_{22}} \left(\frac{\varphi_1}{A_1} \frac{\partial w_i}{\partial \xi_1} + \frac{\varphi_2}{A_2} \frac{\partial w_i}{\partial \xi_2} \right) \end{aligned}$$

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS CONTINUED

$$2\bar{\bar{K}}_{12}^0 = - \left[2 \frac{\varphi_1 + \varphi_2 \cos \theta_{12}}{r_{12}} + \frac{1}{A_1} \frac{\partial \varphi}{\partial \xi_1} \right] \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1} + \left[2 \frac{\varphi_2 + \varphi_1 \cos \theta_{12}}{r_{21}} + \frac{1}{A_2} \frac{\partial \varphi}{\partial \xi_2} \right] \frac{1}{A_2} \frac{\partial w_i}{\partial \xi_2}$$

- The displacement fields for points of the shell are given by

$$\vec{U} = U_1 \hat{g}_1 + U_2 \hat{g}_2 + U_3 \hat{g}_3 \quad \text{with}$$

$$U_1 = \frac{\mu_{22} U_1^0 - \mu_{21} U_2^0}{\mu_{11} \mu_{22} - \mu_{12} \mu_{21}}, \quad U_2 = \frac{\mu_{11} U_2^0 - \mu_{12} U_1^0}{\mu_{11} \mu_{22} - \mu_{12} \mu_{21}}, \quad \text{and} \quad U_3 = U_3^0; \quad \text{and where}$$

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS CONTINUED

$$\mu_{11} = \frac{1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}}\right)^2}}$$

$$\mu_{12} = \frac{-\frac{\xi_3 \csc \theta_{12}}{r_{12}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{11}} + \frac{\xi_3 \cot \theta_{12}}{r_{12}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{12}}\right)^2}}$$

$$\mu_{21} = \frac{\frac{\xi_3 \csc \theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}}\right)^2}}$$

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS CONTINUED

$$\mu_{22} = \frac{1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}}{\sqrt{\left(1 + \frac{\xi_3}{r_{22}} - \frac{\xi_3 \cot \theta_{12}}{r_{21}}\right)^2 + \left(\frac{\xi_3 \csc \theta_{12}}{r_{21}}\right)^2}}$$

$$\mathbf{U}_1^0 = \mathbf{u}_1 + \xi_3 \left[\varphi_1 + \varphi_1^i - \varphi \left(\varphi_2 + \varphi_1 \cos \theta_{12} - \frac{1}{A_2} \frac{\partial w_i}{\partial \xi_2} \right) \csc \theta_{12} \right] + \mathbf{F}_1(\xi_3) \gamma_1^0 + \mathcal{O}(\theta \varepsilon^2, \theta^3, \varepsilon^4)$$

$$\mathbf{U}_2^0 = \mathbf{u}_2 + \xi_3 \left[\varphi_2 + \varphi_2^i + \varphi \left(\varphi_1 + \varphi_2 \cos \theta_{12} - \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1} \right) \csc \theta_{12} \right] + \mathbf{F}_2(\xi_3) \gamma_2^0 + \mathcal{O}(\theta \varepsilon^2, \theta^3, \varepsilon^4)$$

$$\mathbf{U}_3^0 = \mathbf{w} + \mathbf{w}_i - \xi_3 \left[\frac{1}{2} (\varphi_1 + \varphi_1^i)^2 + \frac{1}{2} (\varphi_2 + \varphi_2^i)^2 + (\varphi_1 + \varphi_1^i) (\varphi_2 + \varphi_2^i) \cos \theta_{12} \right] + \mathcal{O}(\theta \varepsilon^2, \theta^3, \varepsilon^4)$$

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS CONTINUED

- Limitation estimates on the magnitude of the initial geometric imperfections are obtained by recalling that “small” strains are

characterized by $\left| \boldsymbol{\varepsilon}_{\beta\gamma}^o \right| \leq \mathcal{O}(\varepsilon^2)$

- As a result, it follows that $\left| \bar{\bar{\boldsymbol{\varepsilon}}}_{\beta\gamma}^o \right| \leq \mathcal{O}(\varepsilon^2)$

- Enforcing $\left| \varphi_\alpha \right| \leq \mathcal{O}(\theta)$, $\left| \varphi \right| \leq \mathcal{O}(\theta)$, and $\left| \bar{\bar{\boldsymbol{\varepsilon}}}_{\beta\gamma}^o \right| \leq \mathcal{O}(\theta^2)$ in the previous equations for $\bar{\bar{\boldsymbol{\varepsilon}}}_{\beta\gamma}^o$ yields the conditions

$$\left| \frac{1}{\mathbf{A}_{(\alpha)}} \frac{\partial w_i}{\partial \xi_\alpha} \right| \leq \mathcal{O}(\theta) \quad \text{and} \quad \left| \frac{w_i}{\mathbf{r}_{\alpha\beta}} \right| \leq \mathcal{O}(\theta)$$

- In addition, it follows that $\left| \bar{\bar{\mathbf{K}}}_{\beta\gamma}^o \right| \leq \mathcal{O}(\theta^2)$

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS CONTINUED

- For the “small” strain and “moderate” rotation theory of Pietraszkiewicz, that neglects nonlinear bending action,

$$\bar{\bar{\epsilon}}_{11}^o = - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1} - \varphi \frac{w_i}{r_{12}}$$

$$\bar{\bar{\epsilon}}_{22}^o = - \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) \frac{1}{A_2} \frac{\partial w_i}{\partial \xi_2} - \varphi \frac{w_i}{r_{21}}$$

$$2\bar{\bar{\epsilon}}_{12}^o = \varphi \csc \theta_{12} \left(\frac{w_i}{r_{22}} - \frac{w_i}{r_{11}} \right) - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \frac{1}{A_2} \frac{\partial w_i}{\partial \xi_2} - \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1}$$

- However, because the changes in reference-surface curvatures and torsion are linear, it follows that $\bar{\bar{\kappa}}_{11}^o = \bar{\bar{\kappa}}_{22}^o = 2\bar{\bar{\kappa}}_{12}^o = 0$

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS CONTINUED

- For the “small” strain and “moderately small” rotation theory of **Sanders**, the reference-surface strains for a geometrically perfect shell are given by

$$\epsilon_{11}^{\circ} = e_{11}^{\circ} + \frac{1}{2}(\varphi_1 + \varphi_2 \cos\theta_{12})^2 + \frac{1}{2}\varphi^2$$

$$\epsilon_{22}^{\circ} = e_{22}^{\circ} + \frac{1}{2}(\varphi_2 + \varphi_1 \cos\theta_{12})^2 + \frac{1}{2}\varphi^2$$

$$2\epsilon_{12}^{\circ} = 2e_{12}^{\circ} + (\varphi_1 + \varphi_2 \cos\theta_{12})(\varphi_2 + \varphi_1 \cos\theta_{12}) + \varphi^2 \cos\theta_{12}$$

- In addition,

$$\kappa_{11}^{\circ} = \chi_{11}^{\circ} = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} (\varphi_1 + \varphi_2 \cos\theta_{12}) - \frac{\varphi_2 \sin\theta_{12}}{\rho_{11}} - \frac{\varphi}{r_{12}}$$

$$\kappa_{22}^{\circ} = \chi_{22}^{\circ} = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} (\varphi_2 + \varphi_1 \cos\theta_{12}) + \frac{\varphi_1 \sin\theta_{12}}{\rho_{22}} - \frac{\varphi}{r_{21}} \quad \text{and}$$

“SMALL” STRAINS, “MODERATE” ROTATIONS, AND “SMALL” INITIAL GEOMETRIC IMPERFECTIONS CONCLUDED

$$2\kappa_{12}^{\circ} = 2\chi_{12}^{\circ} = \frac{1}{A_2} \frac{\partial \varphi_1}{\partial \xi_2} + \frac{\cos \theta_{12}}{A_2} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\cos \theta_{12}}{A_1} \frac{\partial \varphi_1}{\partial \xi_1} + \frac{1}{A_1} \frac{\partial \varphi_2}{\partial \xi_1} + \left(\frac{\varphi_1}{\rho_{11}} - \frac{\varphi_2}{\rho_{22}} \right) \sin \theta_{12} - \varphi \left(\frac{1}{r_{11}} - \frac{1}{r_{22}} \right) \csc \theta_{12}$$

- For these equations,

$$\bar{\bar{\epsilon}}_{11}^{\circ} = - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1}$$

$$\bar{\bar{\epsilon}}_{22}^{\circ} = - \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) \frac{1}{A_2} \frac{\partial w_i}{\partial \xi_2}$$

$$2\bar{\bar{\epsilon}}_{12}^{\circ} = - \left(\varphi_1 + \varphi_2 \cos \theta_{12} \right) \frac{1}{A_2} \frac{\partial w_i}{\partial \xi_2} - \left(\varphi_1 \cos \theta_{12} + \varphi_2 \right) \frac{1}{A_1} \frac{\partial w_i}{\partial \xi_1}$$

- Because the changes in reference-surface curvatures and torsion are linear, it follows that $\bar{\bar{\kappa}}_{11}^{\circ} = \bar{\bar{\kappa}}_{22}^{\circ} = 2\bar{\bar{\kappa}}_{12}^{\circ} = 0$

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14. ABSTRACT An in-depth exposition on the nonlinear deformations of shells with "small" initial geometric imperfections, is presented without the use of tensors. First, the mathematical descriptions of an undeformed-shell reference surface, and its deformed image, are given in general nonorthogonal coordinates. The two-dimensional Green-Lagrange strains of the reference surface derived and simplified for the case of "small" strains. Linearized reference-surface strains, rotations, curvatures, and torsions are then derived and used to obtain the "small" Green-Lagrange strains in terms of linear deformation measures. Next, the geometry of the deformed shell is described mathematically and the "small" three-dimensional Green-Lagrange strains are given. The deformations of the shell and its reference surface are related by introducing a kinematic hypothesis that includes transverse shearing deformations and contains the classical Love-Kirchhoff kinematic hypothesis as a proper, explicit subset. Lastly, summaries of the essential equations are given for general nonorthogonal and orthogonal coordinates, and the basis for further simplification of the equations is discussed.						
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